Joint optimization of input covariance and relay precoder for full-duplex amplify-and-forward relay channels

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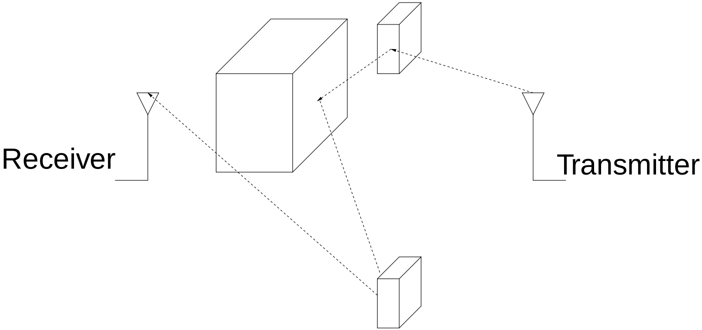
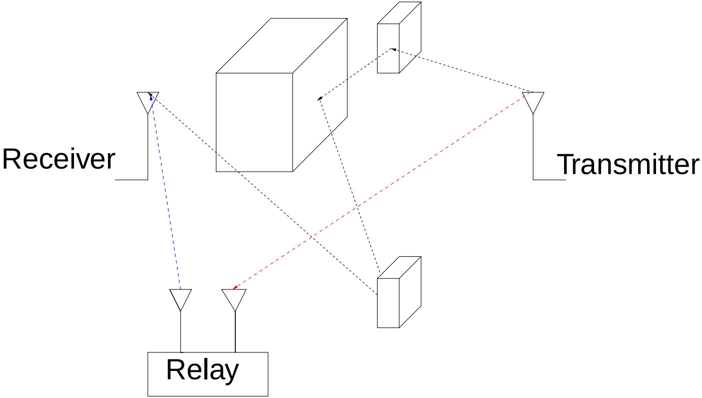
Introduction

The wireless channel:

- Advantage: Flexible communication between transmitter-receiver pairs.
- Challenges: Channel variations and weak received signal.



Introduction



- The challenges of the wireless channel can be mitigated using relays.

What is a relay?

- Intermediate node to assist communication between transmitter and receiver.
- Capable of transmitting, receiving and processing signals.
- Fixed or mobile.
- Relay has power budget and operates over certain frequency band.
- Relay may be full or half duplex.
- Popular relay design objectives:
  - Minimize communication errors.
  - Maximize rate at which data can be reliably communicated; i.e., achieve channel capacity.

Background

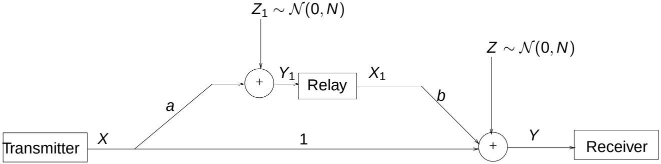
- Relay channel introduced by van der Meulen in 1968 and first published in 1971.
- The capacity of general, even scalar, relay channel is an open problem.
- Only the capacity of particular channels is known; e.g.,
  - Degraded relay channels using block Markov signalling. (Cover and El-Gamal 1979)
  - Reversely degraded relay channels, relay remains silent. (Cover and El-Gamal 1979)
- More recent work: (Kramer, Gastpar and Gupta 2005)

Background

- Other signalling techniques:
  - Compress and forward using block Markov signalling (Cover and El-Gamal 1979)
  - Amplify-and-forward relaying.
- One of the least computationally demanding relaying techniques.
- Under certain conditions was shown to outperform sophisticated decode-and-forward and compress-and-forward (El-Gamal, Mohseni and Zahedi, 2006).

System model

- We follow the general AWGN model in (El-Gamal, Mohseni and Zahedi, 2006):



- Transmitter sends Gaussian vectors with potentially correlated entries.

System model

- Transmitter sends data in the form of length-k vectors, X .
- Relay processes its observed signal, Y_1 by a strictly lower triangular matrix; e.g.,

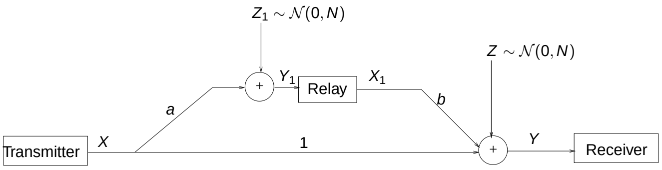
$$DX = \begin{bmatrix} 0 & 0 & 0 & 0 \\ d_{21} & 0 & 0 & 0 \\ d_{31} & d_{32} & 0 & 0 \\ d_{41} & d_{42} & d_{43} & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix};$$

- Relay does not decode the received signal.

System model

- Covariance of transmitted Gaussian vectors is denoted by $\Sigma = \mathrm{E}\{XX^T\}$.
- Relay uses a lower triangular $k \times k$ matrix, D, to process and forward received signal.
- Relay signal contaminated by Gaussian noise, $Z_1 \sim \mathcal{N}(0, NI_k)$
- Receiver signal contaminated by Gaussian noise, $Z \sim \mathcal{N}(0, NI_k)$
- What is the maximum rate achieved by the amplify-and-forward scheme?

The design problem



- Relay transmitted signal:

$$X_1 = DY_1 = D(aX + Z_1) = aDX + DZ_1.$$

- Received signal:

$$Y = X + bX_1 + Z = (I + abD)X + bDZ_1 + Z.$$

- Signal covariance:
$\mathrm{E}\{(I + abD)XX^T(I + abD)^T\} = (I + abD)\Sigma(I + abD)^T$
- Noise covariance:
$\mathrm{E}\{(bDZ_1 + Z)(bDZ_1 + Z)^T\} = N(I + b^2DD^T)$

The design problem

- Power constraints:
  - For transmitter: $\mathrm{Tr}\big(\mathrm{E}\{XX^T\}\big) = \mathrm{Tr}(\Sigma) \leq c_1$.
  - For relay:
    $\mathrm{Tr}\big(\mathrm{E}\{X_1X_1^T\}\big) = a^2 N\,\mathrm{Tr}(D\Sigma D^T) + N\,\mathrm{Tr}(DD^T) \leq c_2$.
- How to find jointly optimal Σ and D?

Optimizing Σ and D

- Signal covariance:
  $\mathrm{E}\{(I + abD)XX^T(I + abD)^T\} = (I + abD)\Sigma(I + abD)^T$
- Noise covariance:
  $\mathrm{E}\{(bDZ_1 + Z)(bDZ_1 + Z)^T\} = N(I + b^2DD^T)$
- Design problem:

$$\max_{\Sigma, D} \quad \log \frac{\det\Big((I + abD)\Sigma(I + abD^T) + N(I + b^2DD^T)\Big)}{\det\Big(N(I + b^2DD^T)\Big)},$$

$$\text{subject to} \quad \Sigma \succeq 0, \qquad \mathrm{Tr}(\Sigma) \leq c_1,$$

$$\mathrm{Tr}(a^2D\Sigma D^T + NDD^T) \leq c_2,$$

$$D_{ij} = 0, \quad j \geq i,$$

Optimizing Σ for fixed D

- For fixed D, original optimization becomes equivalent to

$$
\begin{aligned}
\max_{\Sigma} \quad & \log\det\left(H\Sigma H^T + I\right), \\
\text{subject to} \quad & \Sigma \succeq 0, \qquad \operatorname{Tr}(\Sigma) \leq c_1, \\
& \operatorname{Tr}(D\Sigma D^T) \leq c_3,
\end{aligned}
$$

where $H = \frac{1}{\sqrt{N}}(I + b^2 DD^T)^{-1/2}(I + abD)$ and $c_3 = \frac{1}{a^2}\left(c_2 - N\operatorname{Tr}(DD^T)\right)$.

- Problem is convex in Σ.
- For strictly positive c_1 and c_3, the relative interior of the feasible set not empty.
- KKT conditions are necessary and sufficient for optimality.

Optimizing Σ for fixed D

$$\begin{aligned} \max_{\Sigma} \quad & \log\det\left(H\Sigma H^T + I\right), \\ \text{subject to} \quad & \Sigma \succeq 0, \qquad \mathrm{Tr}(\Sigma) \leq c_1, \\ & \mathrm{Tr}(D\Sigma D^T) \leq c_3, \end{aligned}$$

- The Lagrangian corresponding to the optimization problem is

$$\begin{aligned} L(\Sigma, \mu_1, \mu_2, \Phi) = -\log\det(H\Sigma H^T + I) + \mu_1\left(\mathrm{Tr}(\Sigma) - c_1\right) \\ + \mu_2\left(\mathrm{Tr}(D\Sigma D^T) - c_3\right) - \mathrm{Tr}(\Phi\Sigma). \end{aligned}$$

The KKT conditions

- Gradient of Lagrangian:

$$\nabla_\Sigma L(\Sigma, \mu_1, \mu_2, \Phi) = -H^T H(I + \Sigma H^T H)^{-1} + \mu_1 I + \mu_2 D^T D - \Phi = 0$$

- Primal feasibility:

$$\Sigma \succeq 0, \quad \mathrm{Tr}(\Sigma) \leq c_1, \quad \mathrm{Tr}(D\Sigma D^T) \leq c_3$$

- Dual feasibility:

$$\Phi \succeq 0, \quad \mu_1 \geq 0, \quad \mu_2 \geq 0$$

- Complementarity slackness:

$$\mathrm{Tr}(\Phi\Sigma) = 0, \quad \mu_1\big(\mathrm{Tr}(\Sigma) - c_1\big) = 0, \quad \mu_2\big(\mathrm{Tr}(D\Sigma D^T) - c_3\big) = 0.$$

Solving the KKT conditions

- Solving $\nabla_\Sigma L = 0$ yields

$$\Sigma = (\mu_1 I + \mu_2 D^T D - \Phi)^{-1} - (H^T H)^{-1}.$$

- Since $H^T H \succ 0$, $\Phi \succeq 0$ and $\Sigma \succeq 0$, we have

$$\mu_1 I + \mu_2 D^T D = \left(\Sigma + (H^T H)^{-1}\right) + \Phi \succ 0.$$

- Since $D^T D$ is rank deficient, $\mu_1 > 0$.
- Hence, $\mu_1 \big(\mathrm{Tr}(\Sigma) - c_1\big) = 0$ yields

$$\mathrm{Tr}(\Sigma) = c_1.$$

- Transmitter must use all available power.

Solving the KKT conditions

- From complementarity slackness, we have

$$\Phi\Sigma = \Sigma\Phi = 0.$$

- Using,

$$\Sigma = (\mu_1 I + \mu_2 D^T D - \Phi)^{-1} - (H^T H)^{-1}.$$

- it follows that

$$\Big((\mu_1 I + \mu_2 D^T D - \Phi)^{-1} - (H^T H)^{-1}\Big)\Phi = 0.$$

- What choices of μ_1, μ_2, Φ solve this equation?

Solving the KKT conditions

- What choices of μ_1, μ_2, Φ solve

$$\Big((\mu_1 I + \mu_2 D^T D - \Phi)^{-1} - (H^T H)^{-1}\Big)\Phi = 0?$$

- Choosing $(\mu_1 I + \mu_2 D^T D - \Phi)^{-1} = (H^T H)^{-1}$ does not work, because in this case $\Sigma = 0$ and $\mathrm{Tr}(\Sigma) \neq c_1$.
- We have two possibilities:
  - Case I:
  $$\Phi = 0$$
  - Case II:
  $$\Phi \neq 0,$$
  and thus
  $$\Phi \in \mathcal{N}\Big((\mu_1 I + \mu_2 D^T D - \Phi)^{-1} - (H^T H)^{-1}\Big).$$
- We will study these cases separately.

Solving the KKT conditions—Case I: $\Phi = 0$

- For any Φ,

$$\Sigma = (\mu_1 I + \mu_2 D^T D - \Phi)^{-1} - (H^T H)^{-1}.$$

- When $\Phi = 0$: ($\nu_1 = 1/\mu_1$ and $\nu_2 = \mu_2/\mu_1$, $\mu_1 > 0$.)

$$\Sigma = \nu_1 (I + \nu_2 D^T D)^{-1} - (H^T H)^{-1}$$

- Since $\mathrm{Tr}(\Sigma) = c_1$,

$$\nu_1 = \frac{c_1 + \mathrm{Tr}\big((H^T H)^{-1}\big)}{\mathrm{Tr}\Big(\big(I + \nu_2 D^T D\big)^{-1}\Big)}.$$

- Is there a solution of the KKT system with $\Phi = 0$?
- Study ν_2 to solve KKT system.

Solving the KKT conditions—Case I: $\Phi = 0$

- Consider the conditions:

$$\mathrm{Tr}(D\Sigma D^T) \leq c_3 \quad \textit{and} \quad \mu_2\big(\mathrm{Tr}(D\Sigma D^T) - c_3\big) = 0.$$

- When $\Phi = 0$, first condition becomes

$$\frac{\mathrm{Tr}\big(D^T D(I + \nu_2 D^T D)^{-1}\big)}{\mathrm{Tr}\Big(\big(I + \nu_2 D^T D\big)^{-1}\Big)} \leq \frac{c_3 + \mathrm{Tr}\big(D^T D(H^T H)^{-1}\big)}{c_1 + \mathrm{Tr}\Big((H^T H)^{-1}\Big)}.$$

- What is the effect of ν_2 on this constraint?
- Lemma: Left hand side monotonically decreasing in ν_2.

Algorithm for solving the KKT system when $\Phi = 0$.

Algorithm:

- Set $\nu_2 = 0$.
- If $\frac{\mathrm{Tr}\, D^T D}{k} \leq \frac{c_3 + \mathrm{Tr}\left(D^T D (H^T H)^{-1}\right)}{c_1 + \mathrm{Tr}\left((H^T H)^{-1}\right)}$
  - Is $\Sigma = \frac{c_1 + \mathrm{Tr}\left((H^T H)^{-1}\right)}{k} I - (H^T H)^{-1} \succeq 0$?
  - Yes: KKT system solved with $\Phi = 0$.
  - No: no solution of KKT system exists with $\Phi = 0$.
- If not, increase ν_2 until

$$\frac{\mathrm{Tr}\left(D^T D (I + \nu_2 D^T D)^{-1}\right)}{\mathrm{Tr}\left(\left(I + \nu_2 D^T D\right)^{-1}\right)} = \frac{c_3 + \mathrm{Tr}\left(D^T D (H^T H)^{-1}\right)}{c_1 + \mathrm{Tr}\left((H^T H)^{-1}\right)}.$$

- Check resulting Σ for positive definiteness.

Solving the KKT conditions—Case II: $\Phi \neq 0$

- In this case $\Phi \in \mathcal{N}\Big((\mu_1 I + \mu_2 D^T D - \Phi)^{-1} - (H^T H)^{-1}\Big)$.
- Let $\Phi = U_\Phi \Lambda_\Phi U_\Phi^T$.
- Let $U_\Phi^\perp$ span the null space of Φ.
- If Λ_Σ denotes the eigenvalues of Σ, then

$$\Sigma = (\mu_1 I + \mu_2 D^T D - \Phi)^{-1} - (H^T H)^{-1} = U_\Phi^\perp \Lambda_\Sigma (U_\phi^\perp)^T.$$

- Let $\text{rank}(\Sigma) = k_\Sigma$.

Case II: $\Phi \neq 0$

- Manipulating

$$(\mu_1 I + \mu_2 D^T D - \Phi)^{-1} - (H^T H)^{-1} = U_\Phi^\perp \Lambda_\Sigma (U_\phi^\perp)^T,$$

we obtain

$$(U_\Phi^\perp)^T(\mu_1 I + \mu_2 D^T D)(H^T H)^{-1} U_\Phi^\perp \Lambda_\Sigma^{-1} = \Lambda_\Sigma^{-1} - (U_\Phi^\perp)^T(\mu_1 I + \mu_2 D^T D) U_\Phi^\perp.$$

- Hence, either
  - $k_\Sigma > 1$ and

    $$(U_\Phi^\perp)^T(\mu_1 I + \mu_2 D^T D)(H^T H)^{-1} U_\Phi^\perp \Lambda_\Sigma^{-1} \quad \text{symmetric, or}$$

  - $k_\Sigma = 1$.
- Cases of $k_\Sigma = 1$ and $k_\Sigma > 1$ considered separately.

Case II: $\Phi \neq 0$

The case of $k_\Sigma > 1$:

- The matrix D is strictly lower triangular.
- Conclude that $D^T D$ and $H^T H$ do not commute.
- Hence, to satisfy

$$\Lambda_\Sigma^{-1} - (U_\Phi^\perp)^T(\mu_1 I + \mu_2 D^T D)(H^T H)^{-1} U_\Phi^\perp \Lambda_\Sigma^{-1} = (U_\Phi^\perp)^T(\mu_1 I + \mu_2 D^T D) U_\Phi^\perp,$$

μ_2 must be equal to zero.

Case II: $\Phi \neq 0$—Implications

When $\Phi \neq 0$ and $k_\Sigma > 1$:

- Conclusion 1: $\mu_2 = 0$.
- Conclusion 2:

$$\Lambda_\Sigma^{-1} - \mu_1 (U_\Phi^\perp)^T (H^T H)^{-1} U_\Phi^\perp \Lambda_\Sigma^{-1} = \mu_1 I_{k_\Sigma}. \qquad (1)$$

  Hence, $(U_\Phi^\perp)^T (H^T H)^{-1} U_\Phi^\perp$ must be diagonal. Therefore:

$$U_\Phi^\perp = U_{H_\Phi}.$$

- Conclusion 3: Eigenvectors of Σ and Φ form complementary subsets of eigenvectors of $H^T H$.
- From (1)

$$\Lambda_\Sigma = \frac{1}{\mu_1} I_{k_\Sigma} - \Lambda_{H_\Sigma}^{-1}; \textit{and}$$

$$\Lambda_\Phi = \mu_1 I_{k_\Phi} - \Lambda_{H_\Phi}.$$

Case II: $\Phi \neq 0$—Implications

When $\Phi \neq 0$ and $k_\Sigma > 1$:

- Conclusion 4:

$$\Lambda_{H_\Sigma} \succ \mu_1 I_{k_\Sigma} \quad \text{and} \quad \mu_1 I_{k_\Phi} \succ \Lambda_{H_\Phi} \succ 0.$$

- That is, eigenvalues values of Φ and Σ must be distinct.
- The Lagrange multiplier

$$\mu_1 = \frac{k_\Sigma}{c_1 + \mathrm{Tr}(\Lambda_{H_\Sigma}^{-1})}.$$

Algorithm for solving the KKT system when $\Phi \neq 0$ and $k_\Sigma > 1$

Algorithm:

- Arrange eigenvalues of $H^T H$ in descending order.
- Set $k_\Sigma = 2$,
  - Assign first k_Σ eigenvalues of $H^T H$ to Λ_{H_Σ}
  - Assign remaining $k_\Phi = k - k_\Sigma$ eigenvalues of $H^T H$ to Λ_{H_Φ}
  - Compute $\mu_1 = \frac{k_\Sigma}{c_1 + \text{Tr}(\Lambda_{H_\Sigma}^{-1})}$
  - If $\Lambda_{H_\Sigma} \succ \mu_1 I_{k_\Sigma}$ and $\mu_1 I_{k_\Phi} \succ \Lambda_{H_\Phi} \succ 0$, construct Σ

$$\Lambda_\Sigma = \frac{1}{\mu_1} I_{k_\Sigma} - \Lambda_{H_\Sigma}^{-1}; \textit{and}$$

    the eigenvectors of $H^T H$.
- If, for this k_Σ, μ_1 does not satisfy the constraint, $k_\Sigma \leftarrow k_\Sigma + 1$, repeat.

Case II: $\Phi \neq 0$ and $k_\Sigma = 1$

- The input covariance

$$\Sigma = c_1 w w^T, \qquad w^T w = 1. \tag{2}$$

- When $\nu_2 > 0$, we must have

$$w^T D^T D w = c_3 / c_1.$$

- Notice that $c_3/c_1 \leq \lambda_{\max}(D^T D)$.
- The general expression of Σ is

$$\Sigma = (\mu_1 I + \mu_2 D^T D - \Phi)^{-1} - (H^T H)^{-1}.$$

- Using (2), repeated application of the matrix inversion lemma yields

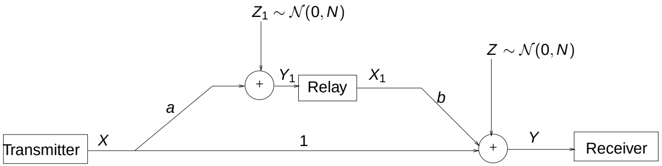
$$w^T H^T H (\mu_1 I + \mu_2 D^T D)^{-1} = \gamma w^T.$$

- γ and w are generalized eigenvalue-eigenvector pair of $(\mu_1 I + \mu_2 D^T D)^{-1} H^T H$.

Algorithm for solving the KKT system when $\Phi \neq 0$ and $k_\Sigma = 1$

- Search for a pair $(\mu_1, \mu_2) \in (0, 1/c_1) \times [0, 1/c_3 - \mu_1 c_1/c_3)$.
- For each pair, find w to be a generalized eigenvectors of $(\mu_1 I + \mu_2 D^T D)^{-1} H^T H$.
- If $c_3/c_1 - w^T D^T D w > \epsilon$ or $c_3/c_1 - w^T D^T D w < 0$, repeat.

Numerical Example 1

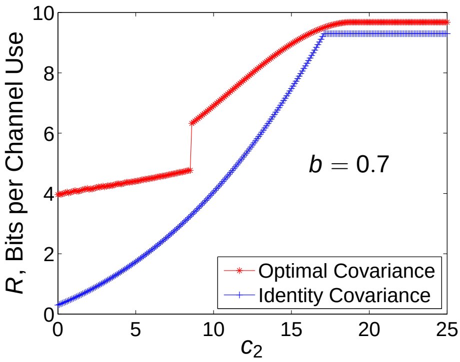


- $a = 4.5$, $N = 0.7$,
- $c_1 = 10$, $k = 4$, $c_3 = \frac{1}{a^2}\left(c_2 - N\,\mathrm{Tr}(D^T D)\right)$, $D = \frac{\tilde{D}}{\mathrm{Tr}(\tilde{D})}$,

$$\tilde{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

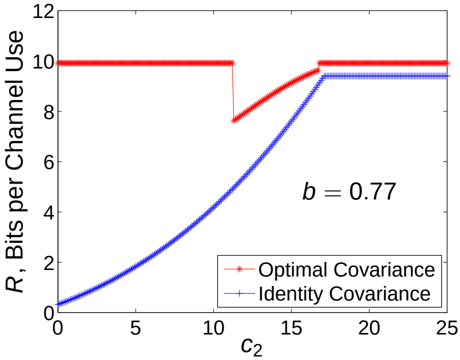
- For comparison, we use $\Sigma_T = \gamma_0 I$.
- γ_0 is chosen to satisfy one of $\mathrm{Tr}(\Sigma_T) \leq c_1$ and $\mathrm{Tr}(D\Sigma_T D^T) \leq c_3$ with equality.
- Rate expression: $R = \log_2 \det(H\Sigma H^T + I)$.

Numerical Example 1



- For low c_2, optimal Σ is rank one.
- For high c_2, optimal Σ is full rank.

Numerical Example 2



- For low and high c_2, optimal Σ is rank three. Relay power constraint not active.
- For intermediate c_2, optimal Σ is full rank.

Summary of covariance optimization

- The simplicity of amplify-and-forward relaying makes it appealing for industrial applications.
- Input covariance and relay precoder design is difficult to design jointly.
- For fixed relay precoder, the design of input covariance is convex optimization problem.
- Analytical solution of the problem of designing input covariance enables efficient search for jointly optimal relay precoders.

Summary of covariance optimization

How to find analytical solution for input covariance?

- Solve the KKT optimality conditions.
- Show that there are two possibilities: $\Phi = 0$ or $\Phi \neq 0$.
- Study each possibility separately.

What about the relay precoder D?

Optimizing the relay precoder

- Finding optimal relay precoder is still difficult.
- Non-convex problem with intricate structure.
- Finding a globally optimal solution may be too ambitious.
- Can we find close to optimal relay precoders?
- Our approach: use branch-and-bound-type technique. Consider the cases:
  - $\Phi = 0$ with $\nu_2 = 0$ and $\nu_2 > 0$.
  - $\Phi \neq 0$ and $\nu_2 = 0$ ($k_\Sigma > 1$).
  - $\Phi \neq 0$ and $\nu_2 \geq 0$ ($k_\Sigma = 1$).

Case I: $\Phi = 0$

The design problem for fixed ν_1 and ν_2:

$$\begin{aligned}
\min_{D} \quad & \log\det\Big((I + b^2 D^T D)(I + \nu_2 D^T D)\Big), \\
\text{subject to} \quad & \nu_1 (I + \nu_2 D^T D)^{-1} \succeq (H^T H)^{-1}, \\
& \nu_1 \operatorname{Tr}\big((I + \nu_2 D^T D)^{-1}\big) = \operatorname{Tr}\big((H^T H)^{-1}\big) + c_1, \\
& \operatorname{Tr}\Big(D^T D\big(a^2 \nu_1 (I + \nu_2 D^T D)^{-1}\Big) \\
& \qquad - a^2 \operatorname{Tr}\big(D^T D (H^T H)^{-1}\big) + N \operatorname{Tr}(D^T D) = c_2, \\
& D_{ij} = 0, \quad j \geq i,
\end{aligned}$$

where

$$H^T H = \frac{1}{N}(I + abD^T)(I + b^2 DD^T)^{-1}(I + abD).$$

Optimal Relay Precoder—Case I: $\Phi = 0$

- Proposition: For any $\nu_1 > 0$ and $\nu_2 \geq 0$, optimal relay precoder is at most rank-1.
- Implication: When Σ is full rank, optimal relay precoder is rank-1.
- Efficient algorithm for finding optimal rank-1 D and the corresponding ν_1 and ν_2.

Case II: $\Phi \neq 0$

When $\Phi \neq 0$, we consider the following cases:

- $k_\Sigma > 1$ and $\nu_2 = 0$.
- $k_\Sigma = 1$ and $\nu_2 \geq 0$.

Case II: $\Phi \neq 0$, $k_\Sigma > 1$

Design problem:

$$\begin{aligned} \min_{D} \quad & \det\big(\Lambda_{H_\Sigma}^{-1})\big) \\ \text{subject to} \quad & \mathrm{Tr}(\Lambda_{H_\Sigma}^{-1}) = \nu_1 k_\Sigma - c_1 \\ & N\,\mathrm{Tr}(D^T D) + a^2 \nu_1\,\mathrm{Tr}(U_{H_\Sigma}^T D^T D U_{H_\Sigma}) \\ & \quad - a^2\,\mathrm{Tr}\big(U_{H_\Sigma}^T D^T D U_{H_\Sigma} \Lambda_{H_\Sigma}^{-1}\big) \leq c_2 \\ & D_{ij} = 0, \quad \forall j \geq i. \end{aligned}$$

- Λ_{H_Σ} denotes the diagonal matrix of largest k_Σ eigenvalues of $H^T H$.
- $H^T H = \frac{1}{N}(I + abD^T)(I + b^2 DD^T)^{-1}(I + abD)$

Case II: $\Phi \neq 0$, $k_\Sigma > 1$

In this case, we develop necessary conditions that an optimal precoder must satisfy:

- $\Lambda_{H_\Sigma} \succ \mu_1 I_{k_\Sigma}$ and $\mu_1 I_{k_\Phi} \succ \Lambda_{H_\Phi} \succ 0$.
- $\Lambda_{H_\Sigma} = \gamma_1 I_{k_1} \oplus \gamma_1 I_{k_2}$, $k_1 + k_2 = k_\Sigma$.
- Can we find precoders that satisfy these conditions?

Optimal Rank-1 relay precoders—Case II: $\Phi \neq 0$, $k_\Sigma > 1$

- To satisfy necessary conditions, $k_\Sigma = k - 1$.
- Explicit expression for optimal rank-1 D parametrized by one parameter, ν_1
- Efficient algorithm for finding optimal ν_1
- Design of optimal rank-1 relay precoder does not depend on particular left and right singular vectors.

Special precoders with rank$(D) > 1$ for $\Phi \neq 0, k_\Sigma > 1$

- Can we find precoders that satisfy necessary optimality conditions with rank$(D) > 1$?
- Use

$$D = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \sigma_1 & 0 & \cdots & 0 & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{k-1} & 0 \end{bmatrix}.$$

- Choice yields tridiagonal $H^T H$.
- Lemma: Eigenvalues of tridiagonal matrices are distinct unless some off-diagonal entries are zero.

Special D with rank$(D) > 1$ for $\Phi \neq 0, k_\Sigma > 1$

- Using lemma, we show that setting

$$\begin{aligned} \sigma_{2i} &= 0, \qquad i = 1, \ldots, \frac{k}{2} - 1, \\ \sigma_{2i-1} &= 0, \qquad i = 1, \ldots, r, \\ \sigma_{2i-1} &= \sigma, \qquad i = r+1, \ldots, \frac{k}{2}. \end{aligned}$$

  yields D that satisfy necessary optimality conditions.
- Corresponding ranks of input covariances:

$$k_\Sigma = \frac{k}{2} \pm r, \qquad r = 1, \ldots, \frac{k}{2} - 2.$$

- Efficient algorithm to determine optimal k_Σ and D.
- Global optimality not guaranteed. (Ongoing investigation)

Optimal Precoder Design Problem

- When $k_\Sigma = 1$, $\Sigma = c_1 w w^T$.
- w generalized eigenvector of $(H^T H)^{-1}(\mu_1 I + \mu_2 D^T D)$.
- Design problem:

$$
\begin{aligned}
\max \quad & w^T H^T H w \\
\text{subject to} \quad & a^2 w^T D^T D w + N\,\mathrm{Tr}(D^T D) \leq c_2 \\
& (H^T H)^{-1}(\mu_1 I + \mu_2 D^T D) w = \gamma w, \\
& D_{ij} = 0, \quad j \geq i.
\end{aligned}
$$

- Optimization over w, D, μ_1 and μ_2.
- Power constraint satisfied with equality because

$$
w^T H^T H w = \frac{1}{\gamma}(\mu_1 + \mu_2 w^T D^T D w).
$$

Insight into optimal D when $k_\Sigma = 1$

- Objective $w^T H^T H w$ upper bounded by $\lambda_{\max}(H^T H)$.
- $\lambda_{\max}(H^T H)$ is upper bounded by monotonically increasing function of max singular value of D.
- For D with given norm, upper bound is maximized when D is rank-1.

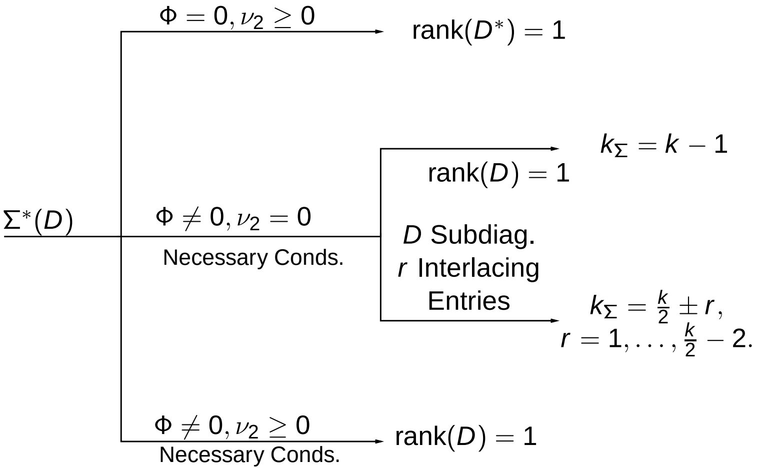
Optimal rank-1 precoder when $k_\Sigma = 1$

- Restrict D to be rank-1; i.e., $D = \sigma u v^T$.
- Not necessarily optimal.
- We show that

$$\frac{c_2}{a^2 + N} \leq \sigma^2 \leq \frac{c_2}{N}$$

- Find explicit expressions for optimal μ_1, μ_2 and w.

Summary of Relay Precoder Optimization



Conclusion

- Amplify-and-forward is an attractive relaying technique.
- Joint design of input covariance, Σ, and relay precoder, D, difficult.
- For fixed D, design problem convex in Σ.
- Obtain closed form solution of the KKT optimality conditions.
- For each solution, study the corresponding relay precoder design problem.
- For some cases, the optimal expression for D can be derived.
- For other cases, precoders that satisfy necessary optimality conditions are available.