Elements of Geometric Measure Theory in the Wiener space[1]

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[1] Dedicated to Herbert Federer (1920-2010)

Introduction

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This theory, developed in the '50 by Caccioppoli, De Giorgi, Federer, leads to general notions of surface area, to a deeper understanding of the Gauss-Green formula, and marks the beginning of modern Geometric Measure Theory.

In Wiener spaces the finite-codimension theory for "smooth" surfaces was developed in '88 by Airault-Malliavin. In more recent years the *BV* theory has been extended to the Wiener space by Fukushima, motivated by infinite-dimensional diffusion processes in nonsmooth domains.

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The Wiener space

X separable Banach space, $\gamma \in \mathscr{P}(X)$ with $\int_X x\,d\gamma = 0$, not supported in a proper subspace of X. We say that γ is Gaussian if $x \mapsto \langle x^*, x\rangle$ has a Gaussian law (in $\mathbb{R}$) for all $x \in X^* \setminus \{0\}$.

The Cameron-Martin subspace $H \subset X$ is defined by

$$H := \{h \in X :\ (\tau_h)_\sharp \gamma \ll \gamma\}.$$

It turns out that H is dense in X, but $\gamma(H) = 0$!

There is a natural way to extract from the density β_h of $(\tau_h)_\sharp\gamma$ w.r.t. γ an Hilbert norm $|\cdot|_H$ which makes the inclusion of H in X compact. In finite dimensions, with the standard Gaussian,

$$\beta_h(x) = e^{-|h|^2/2 + \langle x, h\rangle}.$$

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The Wiener space

Another way to introduce H is via the formula

$$H = \left\{ \int_X f(x)x \, d\gamma(x) : \; f \in L^2(X,\gamma) \right\}$$

and the integration by parts formula

$$\int_X \partial_h \phi \, d\gamma = - \int_X \phi \hat{h} \, d\gamma \qquad (\phi \text{ smooth})$$

that makes $h \in H \mapsto \hat{h} \in L^2(X,\gamma)$ an isometry.

When

$$h = \int_X \langle x^*, x \rangle x \, d\gamma(x) \qquad \text{for some } x^* \in X^*$$

then $\hat{h}(x)$ is precisely $\langle x^*, x \rangle$ and this class of vectors is dense in H.

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Cylindrical projection and factorization

If $x_1^*, \ldots, x_n^*$ are such that $\langle x_i^*, \cdot \rangle$ are an orthonormal basis in $L^2(X, \gamma)$ the corresponding vectors h_i are orthonormal in H and we can define "orthogonal projections"

$$\Pi_n(x) := \sum_{i=1}^{n} \langle x_i^*, x \rangle h_i$$

onto the space H_n spanned by $h_1, \ldots, h_n$.

This induces a factorization $X = Y \oplus H_n$ of X and a factorization of $\gamma = \gamma_n^\perp \otimes \gamma_n$, with $\gamma_n^\perp$ Gaussian and γ_n standard Gaussian in H_n. In addition $H_n^\perp$ is the Cameron-Martin space of $(Y, \gamma_n^\perp)$.

Many facts of the theory can be proved via cylindrical projection and passage to the limit, but things are not always that easy.

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Connections with optimal transport

Brenier's theorem can be extended to the Wiener space, considering the cost function:

$$c(x,y) := \begin{cases} |x-y|_H^2 & \text{if } x-y \in H; \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem. (Feyel-Ustünel) *For all μ_0, $\mu_1 \ll \gamma$, if the transport cost is finite there exists a unique optimal transport map T. The displacement map $T - \mathrm{Id}$ is H-valued and, if $\mu_0 = \gamma$ and $\mu_1 = f\gamma$, we have*

$$(*) \qquad \frac{1}{2} W_2^2(f\gamma, \gamma) \leq \int_X f \ln f \, d\gamma.$$

The inequality (*), ensuring that the transport cost is finite whenever the entropy is finite, is the limiting case of Talagrand's inequality

$$\frac{1}{2} W_2^2(f\gamma_n, \gamma_n) \leq \int_{\mathbb{R}^n} f \ln f \, d\gamma_n.$$

On the other hand, the existence of optimal maps is more subtle and it *does not* rely on optimal Kantorovich potentials (see also Bogachev-Kolesnikov).

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Connections with optimal transport

As in the finite-dimensional theory, the $L^2(X, \gamma)$ gradient flow of the "Dirichlet" energy

$$\int_X |\nabla u|_H^2 \, d\gamma$$

and the Wasserstein gradient flow of the relative entropy $\int_X f \ln f \, d\gamma$ coincide (Fang-Shao-Sturm).

The first "heat" flow is classical and known as Ornstein-Uhlenbeck semigroup. It has a nice explicit expression, known as Mehler's formula:

$$u_t(x) = \int_X u_0(e^{-t}x + \sqrt{1 - e^{-2t}}y) \, d\gamma(y).$$

When $X = \mathbb{R}^n$ and $\gamma_n = G_n \mathscr{L}^n$ is the standard Gaussian, the density $\rho_t := u_t G_n$ w.r.t. $\mathscr{L}^n$ solves the Fokker-Planck equation

$$\frac{d}{dt}\rho_t = \nabla \cdot (\nabla \rho_t + x\rho_t).$$

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Classical Geometric Measure Theory

We say that $E \in \mathscr{B}(\mathbb{R}^n)$ has finite perimeter if there exists a vector-valued measure with finite total variation

$$D\chi_E = \big(D_1\chi_E, \ldots, D_n\chi_E\big)$$

representing the distributional derivative of χ_E, i.e.

$$\int_E \frac{\partial \phi}{\partial x_i}\,dx = -\int_{\mathbb{R}^n} \phi\,dD_i\chi_E \qquad \forall \phi \in C^1_c(\mathbb{R}^n),\ i = 1, \ldots, n.$$

When E has a sufficiently nice boundary, the Gauss-Green theorem gives

$$D\chi_E = \nu_E \mathscr{H}^{n-1} \llcorner \partial E \qquad \text{with } \nu_E \text{ inner unit normal.}$$

For this reason we may define *perimeter* of E the quantity

$$P(E) := |D\chi_E|(\mathbb{R}^n),$$

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One more definition of perimeter

De Giorgi gave another definition of perimeter, whose relevance has been overlooked until recent times. He noticed that Jensen's inequality and the semigroup property yield

$$t \mapsto \int_{\mathbb{R}^n} |\nabla_x u(t,x)|\, dx \quad \text{is nonincreasing in } (0,+\infty)$$

along solutions $u(t,x)$ to the heat equation.

Then, taking χ_E as initial condition, he defined

$$P(E) := \lim_{t \downarrow 0} \int_{\mathbb{R}^n} |\nabla_x u(t,x)|\, dx \in [0,\infty]$$

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De Giorgi gave another definition of perimeter, whose relevance has been overlooked until recent times. He noticed that Jensen's inequality and the semigroup property yield

$$t \mapsto \int_{\mathbb{R}^n} |\nabla_x u(t,x)|\,dx \quad \text{is nonincreasing in } (0,+\infty)$$

along solutions $u(t,x)$ to the heat equation.

Then, taking χ_E as initial condition, he defined

$$P(E) := \lim_{t\downarrow 0} \int_{\mathbb{R}^n} |\nabla_x u(t,x)|\,dx \in [0,\infty]$$

and he proved that this definition is consistent with the "distributional" one.

SCUOLA NORMALE SUPERIORE

Measure-theoretic boundaries

Simple examples shows that $D\chi_E$ is concentrated on sets much smaller than ∂E. Hence, in order to represent properly $D\chi_E$, finer and measure-theoretic notions of boundary are needed:

Federer's essential boundary $\partial^* E$:

$$\partial^* E := \left\{ x : \limsup_{r\downarrow 0} \frac{\mathscr{L}^n(B_r(x)\cap E)}{\mathscr{L}^n(B_r(x))} > 0,\ \limsup_{r\downarrow 0} \frac{\mathscr{L}^n(B_r(x)\setminus E)}{\mathscr{L}^n(B_r(x))} > 0 \right\}.$$

It is at least $\mathscr{L}^n$-negligible, by Lebesgue's theorem. We have $\partial^* E = \mathbb{R}^n \setminus (E^0 \cup E^1)$.

De Giorgi's reduced boundary:

$$\mathscr{F} E := \left\{ x \in \mathrm{spt}|D\chi_E| :\ \exists \nu_E(x) := \lim_{r\downarrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \text{ and } |\nu_E(x)| = 1 \right\}.$$

By Besicovitch's differentiation theorem $D\chi_E$ is concentrated on $\mathscr{F} E$ and $D\chi_E = \nu_E |D\chi_E|$.

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SCUOLA NORMALE SUPERIORE

Structure of $D\chi_E$

Theorem. (De Giorgi-Federer) *For any set of finite perimeter E we have:*

(a) $|D\chi_E|(B) = \mathscr{H}^{m-1}(B \cap \mathscr{F}E)$ *for all* $B \in \mathscr{B}(\mathbb{R}^m)$;

(b) $\mathscr{F}E \subset E^{1/2} \subset \partial^* E$, *but* $\mathscr{H}^{m-1}(\partial^* E \setminus \mathscr{F}E) = 0$;

(c) $\mathscr{F}E$ *is contained in the union of countably many Lipschitz hypersurfaces.*

These results, of central importance for the development of modern GMT, reduce somehow the gap between the weak and the classical Gauss-Green formulas.

The proof of these statements is mostly based on a blow-up analysis, and in particular in the proof of the convergence

$$\frac{1}{r}(E - x) \to \text{halfspace as } r \downarrow 0 \text{ for all } x \in \mathscr{F}E.$$

This procedure will not be applicable in infinite-dimensional spaces.

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SCUOLA NORMALE SUPERIORE

Gaussian theory: finite dimensions

$X = \mathbb{R}^m$, $G_m(x) = (2\pi)^{-m/2} e^{-|x|^2/2}$, $\gamma = G_m \mathscr{L}^m$ standard Gaussian.

Since $\partial_h \gamma = -\langle x, h\rangle \gamma$ we have the integration by parts formula

$$\int_X f \partial_h \phi \, d\gamma = -\int_X \phi \partial_h f \, d\gamma + \int_X \langle x, h\rangle f \phi \, d\gamma \qquad h \in X$$

It can be used, with $f = \chi_E$, to define a weak derivative $D_\gamma \chi_E$.

Obviously $D_\gamma \chi_E = G_m D\chi_E$ and all "local" regularity properties remain true.

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Infinite dimensions: good news

• Still the integration by parts formula along directions in H makes sense, and this leads to a Sobolev (and BV) theory (Gross, Malliavin, Fukushima).

• The Ornstein-Uhlenbeck semigroup, given by Mehler's formula

$$\rho_t(x) = \int_X \rho_0(e^{-t}x + \sqrt{1-e^{-2t}}y)\, d\gamma(y)$$

provides a nice smoothing operator (along directions of H).

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Infinite dimensions: bad news

• Preiss-Tiser showed that Lebesgue theorem holds if the covariance operator of γ decays sufficiently fact (quite fast, indeed). Preiss provided also an example of a Gaussian measure γ in a Hilbert space X and $f \in L^\infty(X,\gamma)$ such that

$$\limsup_{r \downarrow 0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} f \, d\gamma > f(x) \quad \text{in a set of } \gamma\text{-positive measure.}$$

So, no Lebesgue theorem can be expected in general and the definition of essential boundary becomes problematic.

• Of course also no Besicovitch theorem can be expected, so there is no hope to define the reduced boundary in the traditional way.

A possible way out is based on the understanding that the norm of X is somehow not natural, and not related to γ as closely as H and its norm.

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SCUOLA NORMALE SUPERIORE

Questions

Let E be a set of finite perimeter in (X,γ) and let $D_\gamma\chi_E$ be the corresponding H-valued measure.

- How "large" is the (measure theoretic) support of $D_\gamma\chi_E$?
- Can we define suitable notions of essential and reduced boundary?
- Can we extend De Giorgi's representation theorem of $|D\chi_E|$ to this context? (thus getting a "more precise" integration by parts formula in the Wiener space)

Theorem. (A-Miranda-Pallara) *$|D_\gamma\chi_E|$ is concentrated on countably many graphs of entire Sobolev functions defined on hyperplanes of X.*

To make more precise the third question, we need a suitable notion of (cylindrical) codimension-one Hausdorff measure, introduced by Feyel-De la Pradelle.

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Codimension-one Hausdorff measures

Given a finite-dimensional subspace F of H, and the corresponding factorization $X = Y \oplus F$, Feyel-De la Pradelle defined

$$\mathscr{H}^{\infty-1}_F(A) := \frac{1}{\sqrt{2\pi}^m} \int_Y \int_{A_y} e^{-|x|^2/2}\, d\mathscr{H}^{m-1}_F(x)\, d\gamma^{\perp}(y)$$

and noticed the crucial monotonicity property $\mathscr{H}^{\infty-1}_F \leq \mathscr{H}^{\infty-1}_G$ whenever $F \subset G$.

Then, considering suitable families of subspaces that "invade" H, we can define several notions of codimension-one Hausdorff measure. In this lecture:

$$\mathscr{H}^{\infty-1} := \sup \left\{ \mathscr{H}^{\infty-1}_F : \ F \subset \tilde{H} \right\},$$

where $\tilde{H} = \{ \int \langle x^*, x \rangle x \, d\gamma : \ x^* \in X^* \}$.

Feyel-De la Pradelle prove that this measure coincides with the Airault-Malliavin one, on smooth level sets.

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SCUOLA NORMALE SUPERIORE

Codimension-one Hausdorff measures

Given a finite-dimensional subspace F of H, and the corresponding factorization $X = Y \oplus F$, Feyel-De la Pradelle defined

$$\mathscr{H}_F^{\infty-1}(A) := \frac{1}{\sqrt{2\pi}^m} \int_Y \int_{A_y} e^{-|x|^2/2} \, d\mathscr{H}_F^{m-1}(x) \, d\gamma^\perp(y)$$

and noticed the crucial monotonicity property $\mathscr{H}_F^{\infty-1} \leq \mathscr{H}_G^{\infty-1}$ whenever $F \subset G$.

Then, considering suitable families of subspaces that "invade" H, we can define several notions of codimension-one Hausdorff measure. In this lecture:

$$\mathscr{H}^{\infty-1} := \sup \left\{ \mathscr{H}_F^{\infty-1} : \; F \subset \tilde{H} \right\},$$

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SCUOLA NORMALE SUPERIORE

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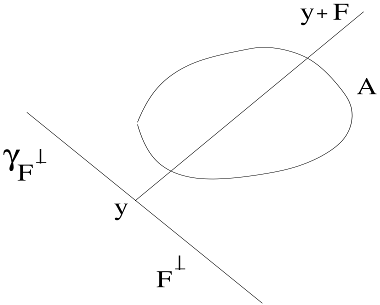
As illustrated in the picture, $\gamma_F^\perp \times \gamma_F$ is a factorization of γ induced by a m-dimensional subspace F of H (γ_F is the standard Gaussian in F, with the metric induced by H) and the sets A_y are the m-dimensional sections of A, keeping $y \in (I - \pi_F)(X)$ fixed.



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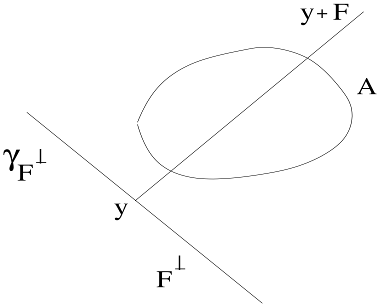

The OU semigroup is a substitute for the mean on balls

Theorem. (Stein, Rota) *Let P_t be a linear unitary semigroup in $L^2(X,\mu)$, and assume that for all $f \in L^2(X,\mu)$ the map $t \mapsto P_t f(x)$ is continuous in $(0,\infty)$ for μ-a.e. $x \in X$. Then*

$$\lim_{t \downarrow 0} P_t f(x) = f(x) \quad \text{for } \mu\text{-a.e. } x \in X.$$

Heuristically, since in finite dimensions P_t is a mean value of mean values on balls (mostly of radius $\sim \sqrt{t}$), we may think to use the Ornstein-Uhlenbeck semigroup T_t also to define measure-theoretic boundaries, and this makes sense also in infinite dimensions.

This is in agreement with the pioneering definition of perimeter De Giorgi, based on the heat semigroup, and on the semigroup based proofs of the isoperimetric inequality (Ledoux):

$$\sqrt{\frac{\pi}{t}} \int T_{t/2}\chi_E \, T_{t/2}\chi_{X \setminus E} \, dx \le P(E).$$

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SCUOLA NORMALE SUPERIORE

Points of density 1/2

Density theorem. (A-Figalli) *If E is a Borel set with finite perimeter in (X,γ), it holds:*

$$\lim_{t\downarrow 0}\int_X |T_t\chi_E-\frac{1}{2}|^2\,d|D_\gamma\chi_E|=0.$$

Warning. Here one has to work with $T_t\chi_E$ as pointwise defined by Mehler's formula and the choice of a Borel representative is important.

Definition. (Points of density 1/2) Let $t_i\downarrow 0$ be such that $\sum_i\sqrt{t_i}+\|T_{t_i}\chi_E-\frac{1}{2}\|_{L^1(|D_\gamma\chi_E|)}<\infty$. We define

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Representation of $|D_\gamma \chi_E|$

A drawback of this definition is its dependence upon (t_i). Nevertheless, the density theorem ensures that $|D_\gamma \chi_E|$ is concentrated on $E^{1/2}$ and the next result shows that the dependence on (t_i) is mild:

Representation theorem. (A-Figalli) *$E^{1/2}$ has finite $\mathscr{H}^{\infty-1}$-measure and*

$$|D_\gamma \chi_E| = \mathscr{H}^{\infty-1} \llcorner E^{1/2}.$$

This improves an earlier result by Hino. Given a nondecreasing family $\mathcal{F} = \{F_m\}_{m \geq 1}$ of subspaces of $\tilde{H}$ whose union is dense in H, he defined *cylindrical essential boundary* the set

$$\partial^*_{\mathcal{F}} E := \liminf_m \partial^*_{F_m} E, \quad \text{where} \quad \partial^*_{F_m} E := \{(y, z) : \ z \in \partial^* E_y\}.$$

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Then, Hino proved that

$$(*) \qquad |D_\gamma \chi_E| = \mathscr{H}_{\mathcal{F}}^{\infty-1} \llcorner \partial_{\mathcal{F}}^* E \qquad \text{with} \qquad \mathscr{H}_{\mathcal{F}}^{\infty-1} := \sup_m \mathscr{H}_{F_m}^{\infty-1}.$$

The drawback in (*) is that both objects in the r.h.s. a priori depend on $\mathcal{F}$, while the l.h.s. does not. Indeed, it seems quite hard in general to compare

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and even to compare their null sets.

A consequence of our result is that we can replace in (*) coordinate-free objects, namely the Feyel-De La Pradelle codimension-one Hausdorff measure and $E^{1/2}$.

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Some ideas from the proofs

We focus on the density theorem. Recall that in finite dimensions

$$\lim_{t\downarrow 0}\int_X |T_t\chi_E-\frac{1}{2}|^2\,d|D_\gamma\chi_E|=0$$

holds simply because $T_t\chi_E\to 1/2$ *pointwise* on the reduced boundary, on which $|D_\gamma\chi_E|$ is concentrated. In turn, the convergence to $1/2$ of $T_t\chi_E$ relies on a blow-up analysis, a tool we cannot use in infinite dimensions.

I will present first a soft and quite general argument that provides w^*-convergence of $T_t\chi_E$ to $1/2$ in $L^\infty(X,|D_\gamma\chi_E|)$. Then we will see how one can show that

$$\limsup_{t\downarrow 0}\int_X |T_t\chi_E|^2\,d|D_\gamma\chi_E|\le\frac{1}{4}|D_\gamma\chi_E|(X).$$

This allows to improve the convergence from weak to strong.

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Soft proof of w^*-convergence

Suffices to show that any weak* limit point g of $T_t\chi_E$ as $t \downarrow 0$ satisfies $g \geq 1/2\,|D_\gamma\chi_E|$-a.e. in X.

Fix $A \subset X$ open and set $f_t = T_t\chi_E$, then

$$|D_\gamma(f_t\chi_E)|(A) \leq \int_A f_t\,d|D_\gamma\chi_E| + \int_{E\cap A} |\nabla f_t|\,d\gamma.$$

Since $\nabla f_t\gamma = D_\gamma(T_t\chi_E) = e^{-t}T_t^* D_\gamma\chi_E$, we can estimate

$$|D_\gamma(f_t\chi_E)|(A) \leq \int_A f_t\,d|D_\gamma\chi_E| + e^{-t}\int_X T_t(\chi_{E\cap A})\,d|D_\gamma\chi_E|.$$

Since $T_t(\chi_{E\cap A}) \leq f_t$ and tends to 0 out of $\overline{A}$, as $t \downarrow 0$ we get

$$|D_\gamma\chi_E|(A) \leq 2\int_{\overline{A}} g\,d|D_\gamma\chi_E|$$

for any limit point g.

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Estimate of $\int |T_t\chi_E|^2\, d|D_\gamma\chi_E|$

Let $X = Y \oplus F$ be a factorization of X, with the corresponding factorization $\gamma = \gamma_F^\perp \otimes \gamma_F$, and $F \subset \tilde H$ finite dimensional.
Set $x = (y,z) \in Y \oplus F$, $E_y := \{z :\ (y,z) \in E\} \subset F$ and notice that, obviously

$$\lim_{t\downarrow 0} \int_Y \int_F |T_t^F \chi_{E_y}|^2\, d|D_{\gamma_F}\chi_{E_y}|\, d\gamma^\perp(y) = \frac{1}{4}\int_Y |D_{\gamma_F}\chi_{E_y}|(F)\, d\gamma^\perp(y).$$

We have to carefully estimate the error we make when we replace the OU semigroup in F by the "global" OU semigroup T_t. Another error, easier to handle, arises from the replacement of

$$\int_Y |D_{\gamma_F}\chi_{E_y}| d\gamma^\perp(y) \qquad \text{with} \qquad |D_\gamma \chi_E|.$$

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The error estimate relies on three ingredients.

$$T_t f(y,z) = T_t^Y\big(y' \mapsto T_t^F f(y',\cdot)(z)\big)(y) \qquad \text{(factorization of } T_t\text{)}$$

$$\int_Y |g - T_t g|\, d\gamma \le c\sqrt{t} \int_Y |\nabla g|\, d\gamma \qquad \text{(Poincaré inequality)}$$

Since we are integrating against *singular* measures $\sigma = |D_{\gamma_F}\chi_{E_y}|$ in F, the Poincaré inequality is not sufficient to conclude. We need also the dimension-free estimate

$$\limsup_{t\downarrow 0} \sqrt{t}\, T_t^* \sigma \le \gamma.$$

For general measures σ the blow-up rate of $T_t^*\sigma$ as $t \downarrow 0$ is $\sqrt{t}^{-m}$, m being the dimension of F, but rectifiability of σ leads to a blow-up rate independent of m.

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Open problems

1. What about log-concave measures γ? As far as I know, there is no general integration by parts formula or analogue of Cameron-Martin space in this context.

2. Can we show that $\mathscr{H}^{\infty-1}(X \setminus (E^0 \cup E^{1/2} \cup E^1)) = 0$?

3. What about higher (finite) codimension theory? Still the integral-geometric approach of Feyel-De La Pradelle and Hino works, but some "global" and coordinate-free concepts seem to be missing.

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Some references

[1] H. Airault and P. Malliavin, *Intégration géométrique sur l'espace de Wiener.* Bull. des Sciences Math., **112** (1988), 25–74.

[2] L. Ambrosio, M. Miranda and D. Pallara, *Sets with finite perimeter in Wiener spaces, perimeter measure and boundary rectifiability.* Discrete Contin. Dyn. Syst. Series A, **28** (2010), 591–606.

[3] L. Ambrosio and A. Figalli, *Surface measures and convergence of the Ornstein-Uhlenbeck semigroup in Wiener spaces.* Preprint, 2010.

[4] E. De Giorgi, *Su una teoria generale della misura* $(r-1)$*-dimensionale in uno spazio ad* r *dimensioni.* Ann. Mat. Pura Appl., **4** (1954), 191–213.

[5] D. Feyel and A. De la Pradelle, *Hausdorff measures on the Wiener space.* Potential Anal., **1** (1992), 177–189.

[6] M. Fukushima, *BV functions and distorted Ornstein-Uhlenbeck processes over the abstract Wiener space.* J. Funct. Anal., **174** (2000), 227–249.

[7] M. Hino, *Sets of finite perimeter and the Hausdorff–Gauss measure on the Wiener space.* J. Funct. Anal., **258** (2010), 1656–1681.

[8] M. Ledoux, *Semigroup proof of the isoperimetric inequaliy in Euclidean and Gaussian spaces.* Bull. Sci. Math., **118** (1994), 485–510.

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