A Generalization of Caffarelli's Contraction Theorem via (reverse) Heat-Flow

Emanuel Milman

University of Toronto
and Fields Institute

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November 5, 2010

joint with Young-Heon Kim, UBC.

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Outline

- Optimal Transport and Caffarelli's Contraction Theorem.
- Recall some applications of CCT.
- State Generalization of CCT.
- New Applications.
- Ideas in Proof.
- Challenge.

Optimal Transport

Monge (Monge–Kantorovich) Transport Problem ($\sim$ 1781)

Let μ, ν be two Borel probability measures on $\mathbb{R}^n$, $\mu << Leb$, $\int |x|^2 d\mu, \int |x|^2 d\nu < \infty$.
Among all maps $T : \mathbb{R}^n \to \mathbb{R}^n$ pushing forward μ onto ν, ($\nu = \mu \circ T^{-1}$, "$T : \mu \mapsto \nu$"), minimize $\int |T(x) - x|^2 d\mu(x)$.

Thm (Brenier '91, McCann '95)

Minimizing $T = T_{opt}$ exists, unique (μ-a.e.), and characterized by $T_{opt} = \nabla \varphi$, $\varphi : \mathbb{R}^n \to \mathbb{R}$ convex.

Thms (Caffarelli 90's)

Regularity Theory for the Monge–Ampére equation:

$$detD^2\varphi(x) = \frac{f(x)}{g(\nabla\varphi(x))} \; , \; \mu = f(x)dx \; , \; \nu = g(x)dx \; .$$

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$$Q(x) = \langle Ax, x\rangle \;\; (A \geq 0) \;\; , \;\; V \text{ is convex} \;\; .$$

Then $T_{opt} : \mu \mapsto \nu$ is a contraction:

$$|T_{opt}(x) - T_{opt}(y)| \leq |x - y| \quad \forall x, y \in \mathbb{R}^n \; .$$

Applications:

- Transferring isoperimetric and Sobolev inequalities.
- Transferring Poincaré inequalities (Brascamp–Lieb, B-conjecture).
- Correlation Inequalities.
- More...

Applications only require that there is *some* map $T : \mu \mapsto \nu$.

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Example: Gaussian Correlation Conjecture

- γ_n - standard Gaussian measure on $(\mathbb{R}^n, |\cdot|)$.
- K, L - two convex subsets of $\mathbb{R}^n$, $K = -K$, $L = -L$.

Gaussian Correlation Conjecture (Das Gupta, Eaton, Olkin, Perlman, Savage and Sobel 1972)

$$\gamma_n(K \cap L) \geq \gamma_n(K)\gamma_n(L) \quad ?$$

Pitt (1977) confirmed for $n = 2$, $n \geq 3$ still open.

Thm (Hargé 1999, Cordero-Erausquin 2002)

True if K (or L) is a centered ellipsoid.

Choosing a degenerate ellipsoid $K = \{x; |\langle x, \theta \rangle| \leq c\}$, this recovers Khatri, Šidák 1967 (holds for other radial measures).

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Cordero–Erausqin's proof

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If $\mathcal{E}$ is a centered ellipsoid, then:

$$\gamma_n(\mathcal{E} \cap L) \geq \gamma_n(\mathcal{E})\gamma_n(L) \quad \forall L = -L \text{ convex} \quad .$$

If $\mathcal{E} = A(B_2^n)$, set $\mu = \gamma_n \circ A$. Want to show:

$$\mu(B_2^n \cap L) \geq \mu(B_2^n)\mu(L) \quad \forall L = -L \text{ convex} \quad .$$

Set $\nu = (\mu \mathbf{1}_L)/\mu(L)$. Want to show:

$$\nu(B_2^n) = \frac{\mu(B_2^n \cap L)}{\mu(L)} \geq \mu(B_2^n) \; .$$

By Caffarelli's Thm, $T_{opt} : \mu \mapsto \nu$ is a contraction & $T_{opt}(0) = 0$, so $T_{opt}(B_2^n) \subset B_2^n$, and hence $B_2^n \subset T_{opt}^{-1}(B_2^n)$. Therefore:

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Very useful in applications.

However, leaves room for improvement:

- $\mu =$ Gaussian.
- T_{opt} optimal-transport map:
  - Non-constructive.
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Main Result (Generalization of Caffarelli's Theorem)

Thm (Kim and M. 2010)

Fix $\mathbb{R}^n = E_0 \oplus E_1 \oplus \ldots \oplus E_k$ orthogonal decomposition.
Assume $\mu = \exp(-U(x))dx$ and $\nu = \mu \exp(-V)$, where:

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$$(\rho_i^{(2)} \geq 0) \text{ and } \rho_i^{(3)} \leq 0 \text{ on } \mathbb{R}_+ \;\; (\text{e.g. } \rho_i(t) = t^{p_i}, \; p_i \in (0,2]) \; ;$$

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- $E_0 = \mathbb{R}^n$ - recovers Caffarelli's Thm (with different T).
- $E_1 = \mathbb{R}^n$ - $U = \rho(|x|), V = v(|x|)$ are radial.
- $E_0 = 0$ and $dim(E_i) = 1$ - $U = \sum \rho_i(|x_i|) \Rightarrow \mu$ is product ; V is convex and unconditional.

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Generalizes all the applications mentioned.

Corollary (Generalized Correlation Inequality)

Fix $\mathbb{R}^n = E_0 \oplus E_1 \oplus \ldots \oplus E_k$ orthogonal decomposition. Assume $\mu = \exp(-U(x))dx$, with:

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Let $L = -L$ convex such that:

$$\exists C_L \subset \mathbb{R}^{dim E_0 + k} \quad \mathbf{1}_L(x) = \mathbf{1}_{C_L}(P_{E_0} x, |P_{E_1} x|, \ldots, |P_{E_k} x|) \; .$$

Let $K = -K$ such that if $(x_0, x_1, \ldots, x_k) \in K$ then:

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Then:

$$\mu(K \cap L) \geq \mu(K)\mu(L) \; .$$

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Other Applications: Transferring isoperimetric inequalities

Fact: Lipschitz maps transfer isoperimetric inequalities

Assume $T : (\Omega_1, d_1, \mu_1) \rightarrow (\Omega_2, d_2, \mu_2)$, $T : \mu_1 \mapsto \mu_2$ and:

$$d_2(T(x), T(y)) \leq d_1(x, y) \quad \forall x, y \in \Omega_1 \ .$$

If $\mu_1(\partial A) \geq \mathcal{I}(\mu_1(A)) \ \ \forall A$, then $\mu_2(\partial B) \geq \mathcal{I}(\mu_2(B)) \ \ \forall B$.

Corollary of Generalized CCT (particular case)

Let $\mu = c \exp(-\sum_{i=1}^n |x_i|^p)$, $p \in (0, 2]$. It is known that:

$$\mu(\partial A) \geq \mathcal{I}_{p,n}(\mu(A)) \ .$$

Set $\nu = (\mu \mathbf{1}_L)/\mu(L)$, $L \subset \mathbb{R}^n$ unconditional and convex. Then:

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The Construction of T

We construct $T^{-1} : \nu \mapsto \mu$, and show it is expanding.

T^{-1} is constructed by following heat-flow with drift, transforming $\nu = \mu \exp(-V)$ into μ [$= \exp(-U(x))dx$].

Denote by $P_t(f_0)$ the solution to:

$$\frac{d}{dt} f = Lf := \Delta f - \langle \nabla f, \nabla U \rangle \ , \ f|_{t=0} = f_0 \ .$$

$P_t = \exp(tL)$ is the associated semi-group.

L and P_t are self-adjoint on $L_2(\mu)$, $\mu = \exp(-U(x))dx$.
P_t preserves total μ-mass, and (under mild assumptions) converges to the constant stationary distribution:

$$P_t(\exp(-V)) \to \int \exp(-V) d\mu = |\nu| = 1 \ .$$

Interpolation:

$$\nu_0 = \nu = \mu \exp(-V) \ \ \to \ \ \nu_t := \mu P_t(\exp(-V)) \ \ \to \ \ \nu_\infty = \mu \ .$$

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$$\frac{d}{dt} P_t(\exp(-V)) = (\Delta - \langle \nabla, \nabla U \rangle) \, P_t(\exp(-V)) \quad \text{(Eulerian)} \, .$$

Construct S_t as flow along time-dependent vector-field W_t:

$$\frac{d}{dt} S_t(x) = W_t(S_t(x)) \, , \ S_0 = Id \quad \text{(Lagrangian)} \, .$$

To determine W_t, use Continuity Equation:

$$\frac{d}{dt}\nu_t + \nabla \cdot (\nu_t W_t) = 0 \ \Rightarrow \ W_t := -\nabla \log P_t(\exp(-V)) \text{ (advection)} \, .$$

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The Reduction

Reduction

When does heat-flow w.r.t. $\mu = \exp(-U(x))dx$:

$$\frac{d}{dt} f = (\Delta - \langle \nabla f, \nabla U \rangle) f \;,\; f|_{t=0} = f_0 \;,$$

preserve log-concavity: $-\log P_t(f_0)$ remains convex $\forall t \geq 0$?

Example: $\mu =$ Gaussian ($U = Q$) "Ornstein–Uhlenbeck"

By the Mehler formula, it is known that:

$$P_t(f_0) = \text{rescaled}_t\text{-}f_0 * \text{Gaussian}_t \;.$$

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The Reduction (continued)

Thm (Kolesnikov '01)

For general initial log-concave data $f_0 = \exp(-V)$,
only Ornstein–Uhlenbeck ($U = Q$) preserves log-concavity.

Observation ("inherent tradeoff")

For less general convex V, log-concavity may still be preserved by more general U.

Thm (Kim and M. 10)

Let $P_t(f_0)$ denote solution to:

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Idea of Proof

Classical PDE problem (Korevaar, Caffarelli–Spruck, Kawohl ...). One of key methods is maximum principle.

New - Geometric ideas + technical points previously not treated:

- Reduce to Dirichlet boundary valued problem on $B(R) \times [0, T]$:

$$\frac{d}{dt} f = (\Delta - \langle \nabla f, \nabla U \rangle) f \; , \; f|_{t=0} = f_0 \; , \; f|_{\partial B(R) \times [0,T]} \equiv 0 \; .$$

- Show that f remains log-concave near the parabolic boundary $\partial B(R) \times [0, T]$: $D^2(-\log f) = -\frac{D^2 f}{f} + \frac{\nabla f \otimes \nabla f}{f^2} \geq 0$?
  - Problem: $f \notin C^{2;1}(\overline{B(R)} \times [0, T])$!
  - $\partial_{n,n}$ - assume $\langle \nabla f_0, n \rangle > 0$ and use Hopf strong principle.
  - $\partial_{t,*}$ - use strong convexity of $\partial B(R)$ + extra regularity.
- Set $V := -\log f$, and assume V_0 is strictly convex. V satisfies:

$$\frac{d}{dt} V = \Delta V - \langle \nabla V, \nabla U \rangle - \langle \nabla V, \nabla V \rangle \; , \; V|_{t=0} = V_0 \; .$$

Apply maximum principle to $D_{e,e} V$. Assume $D_{e,e} V(x_0, t_0) = 0$:

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Epilogue: T vs. T_{opt}

Intuitively, $T \neq T_{opt}$ generically.

$T = T_{opt}$ in dimension 1, or in 1-dimensional situations.

Example (where everything is computable):

$$\mu = \exp(-\langle Ax, x\rangle) \; , \; \nu = \mu \exp(-\langle Bx, x\rangle) \; , \; A, B \geq 0 \; .$$

$$T_{opt} = A^{1/2}(A^{1/2}(A+B)A^{1/2})^{-1/2}A^{1/2} \; , \; \frac{d}{dt} S_t = B_t S_t \; .$$

Fact

When A, B commute, $T = T_{opt}$.

Conjecture

When A and B do not commute, $T \neq T_{opt}$ generically.

Can show that generically for $n \geq 2$, along the flow, DS_t is not symmetric except at discrete set of times, so S_t is not the gradient of a potential, i.e. not interpolating optimal map.

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