Billiard scattering by rough obstacles and optimal mass transportation

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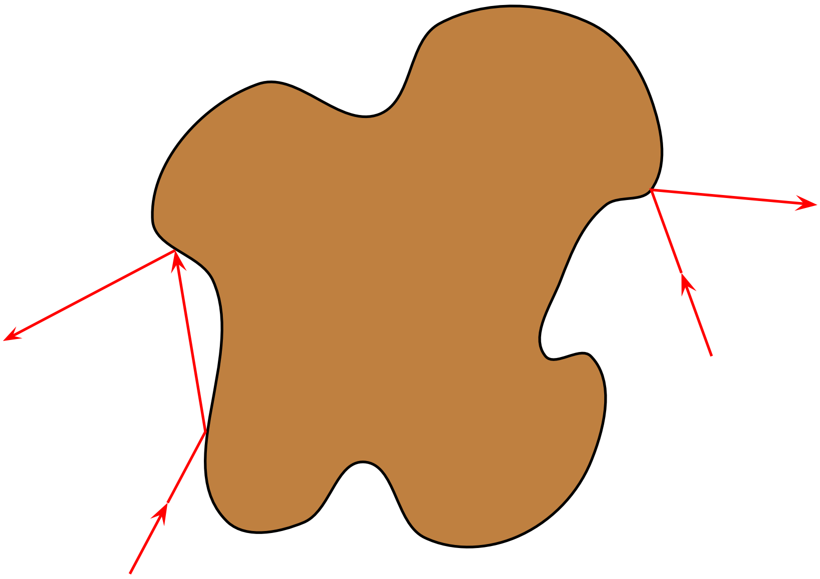
Billiard scattering by a (generally) nonconvex obstacle.



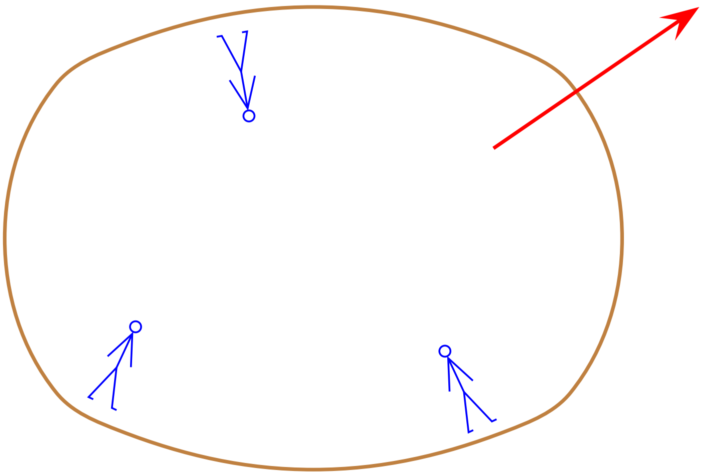
MathSciNet displays nearly 200 articles with the phrase "rough surface" in the title.

In the literature, roughness is modeled by periodic functions, fractal functions, random Gaussian or non-Gaussian functions, etc.

Consider a motivating example:

A spaceship making a long galactic voyage.

In the voyage it will traverse huge interstellar clouds, and its velocity will slow down as a result of collisions with cloud particles.



There is no control of spaceship rotations; it is supposed that the spaceship is slowly and uniformly rotating.

Initially the spaceship is a convex body.

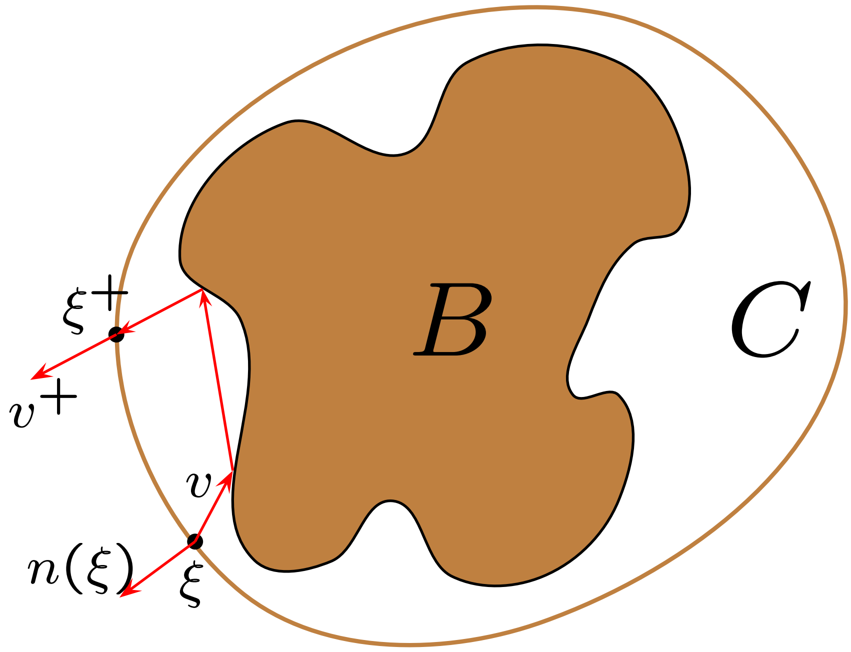
The Mission Control Center wants to apply a riffling (roughening) on its surface so as to minimize the resistance of the cloud.

At the same time, a malicious member of the staff wants to maximize the satellite resistance.

Thus, the problem is twofold: *minimize* or *maximize* the resistance by riffling.

First, define the resistance of a body (a bounded set with piecewise smooth boundary) $B \subset \mathbb{R}^d$.

Generally, B is nonconvex.



Def. 1. The measure $\nu_{B,C}$ (the law of scattering by B) describes the joint distribution of the triple (v, v^+, n) for an incident particle taken at random, where $v \in S^{d-1}$ is the velocity of incidence, $v^+ \in S^{d-1}$ is the velocity of the reflected particle, and $n \in S^{d-1}$ is the outer normal to ∂C at ξ.

More precisely, let $B \subset C$, with C convex. Define the measure μ_C on $\partial C \times S^{d-1}$ by

$$d\mu_C(\xi, v) = (n(\xi) \cdot v)_- \, d\xi \, dv.$$

Let $v^+ = v^+_{B,C}(\xi, v)$ be the final velocity of the particle. Define

$$T : (\xi, v) \mapsto (v, v^+_{B,C}(\xi, v), n(\xi)).$$

By definition, $\nu_{B,C}$ is the push-forward measure $\nu_{B,C} := T^{\#}\mu_C$.

In the convex case $B = C$ one has

$$d\nu_{C,C}(v, v^+, n) =$$

$$= (n \cdot v)_- \, \delta(v^+ - (v - 2(v \cdot n)n)) \, dv \, d\tau_C(n).$$

<u>Def. 2.</u> The resistance of B is

$$R(B) = \int_{(S^{d-1})^3} (v - v^+) \cdot v \; d\nu_{B,C}(v, v^+, n).$$

It does not depend on the ambient convex body C.

Using the law of elastic reflection, $v^+ = v - 2(v \cdot n)n$ and denoting by τ_C the surface measure of C, one easily calculates the resistance of the convex body C:

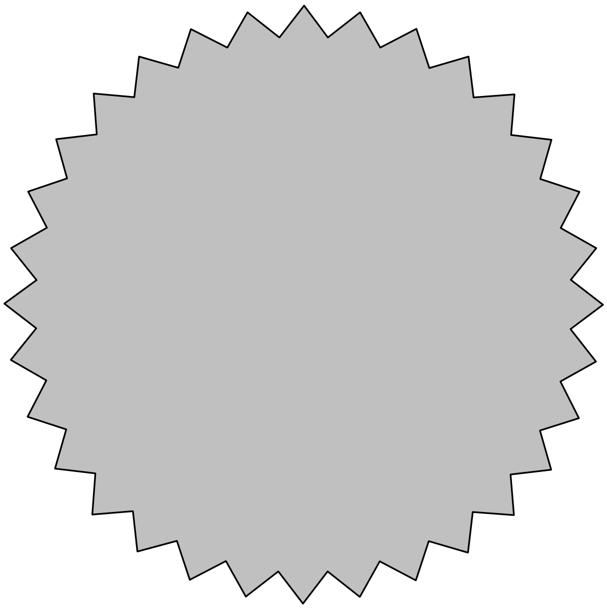
$$R(C) = \int_{(S^{d-1})^2} 2(v \cdot n)^2 \, (v \cdot n)_- \, dv \, d\tau_C(n) =$$

$$= |\partial C| \cdot \int_{S^{d-1}} 2(v \cdot n)_-^3 \, dv.$$

That is, $R(C)$ is proportional to the area of the surface $|\partial C|$.

The resistance can be easily increased by roughening (to the joy of the malicious person). Consider several examples.

Example 1.



The boundary ∂C is substituted with a broken line with small segments making nearly 45^0 with ∂C. For the resulting body C_∇ one has

$$\frac{R(C_\nabla)}{R(C)} = \sqrt{2} \approx 1.4142.$$

In the limit, where the segment size goes to 0, one gets a "rough body" with infinitely small dimples on its surface.

Example 2. In the more general case, where adjacent segments of the broken line make nearly the angle α with each other, $\pi/2 < \alpha < \pi$, the resistance of the body C_α bounded by this line satisfies the relation

$$r(\alpha) := \frac{R(C_\alpha)}{R(C)} =$$

$$= \frac{3}{2}\left(1 - \sin\frac{\alpha}{2}\right) + \frac{3}{4}\sin\frac{3\alpha}{2} + \frac{1}{4}\sin\frac{5\alpha}{2} - \frac{1}{2}\cos\alpha +$$

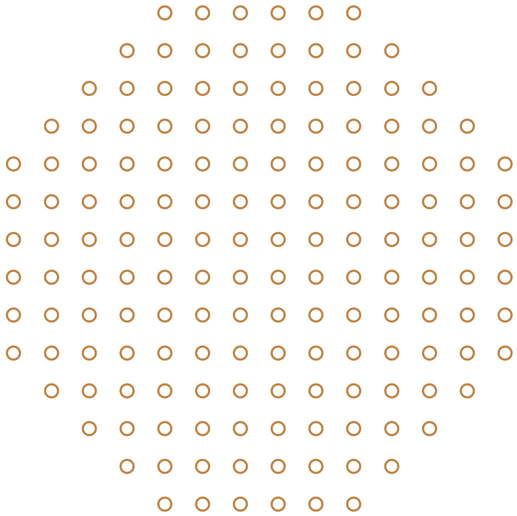
$$+ \frac{3(1-\cos 2\alpha)\left(1 - \sin\frac{\alpha}{2}\right)}{4\sin\frac{\alpha}{2}} + \frac{\cos 3\alpha - 9\cos\alpha}{8\sin\frac{\alpha}{2}}.$$

In particular, one has

$$r(\pi) = 1, \quad r(2\pi/3) = \frac{5}{8} + \frac{1}{\sqrt{3}} \approx 1.2024$$

$$\text{and} \quad r(\pi/2) = \sqrt{2} \approx 1.4142.$$

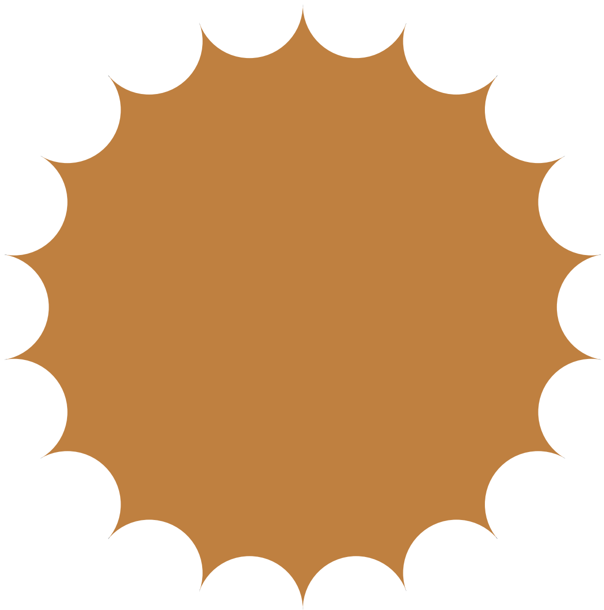
Example 3.



In the limit where the radius of balls goes to zero, the angle of incidence and the angle of the scattered particle become independent. A a result, for the body $C_\circ$ composed of these balls one has

$$\frac{R(C_\circ)}{R(C)} = \frac{3}{4}\left(1 + \frac{\pi^2}{16}\right) \approx 1.2126.$$

Example 4.



The set $C_{\smile}$ is obtained from C by taking away small semicircles. Then one has

$$\frac{R_{\smile}}{R(C)} = \frac{3\pi}{8} \approx 1.1781.$$

We want to minimize or maximize the value

$$\frac{R(\tilde{C})}{R(C)}$$

over all C obtained by roughening C.

But first, we have to define the notion of a rough body.

Def. 3. We say that a sequence of bodies B_m represents a rough body $\mathcal{B}$ obtained by roughening C, if
(a) $B_m \subset C$ and $\lim_{m\to\infty} |C \setminus B_m| = 0$;
(b) the sequence $\nu_{B_m,C}$ weakly converges, $\nu_{B_m,C} \to \nu_{\mathcal{B}}$ as $m \to \infty$.

The limiting measure $\nu_{\mathcal{B}}$ is called the law of scattering by the rough body $\mathcal{B}$.

Two sequences B_m and B'_m are called *equivalent*, if the limiting measure is the same, $\lim_{m\to\infty} \nu_{B_m,C} = \lim_{m\to\infty} \nu_{B'_m,C}$.

By definition, a rough body $\mathcal{B}$ is a class of equivalence of sequences $\{B_m\}$ and $\{B'_m\}$ satisfying (a) and (b).

Let $\pi_{v,n}$ and $\pi_{v^+,n}$ be projections on the subspaces $\{v, n\}$ and $\{v^+, n\}$, respectively, and let τ_C be the surface measure of C.

Def. 4. The set of measures ν on $S^{d-1}_{\{v\}} \times S^{d-1}_{\{v^+\}} \times S^{d-1}_{\{n\}}$ satisfying

(a) $d\pi^{\#}_{v,n}\nu(v, n) = (v \cdot n)_- dv\, d\tau_C(n)$

(b) $d\pi^{\#}_{v^+,n}\nu(v^+, n) = (v^+ \cdot n)_+ dv^+\, d\tau_C(n)$

(c) ν is invariant w.r.t. the exchange $(v, v^+, n) \mapsto (-v^+, -v, n)$

is denoted by Γ_C.

Main Theorem.

$\{\nu_{\mathcal{B}} : \mathcal{B}$ is obtained by roughening $C\} = \Gamma_C$.

The resistance of the rough body is the limit

$$R(\mathcal{B}) := \lim_{n\to\infty} R(B_n) =$$

$$= \iiint_{(S^{d-1})^3} (v - v^+) \cdot v \; d\nu_{\mathcal{B}}(v, v^+, n).$$

Note that $(v - v^+) \cdot v = \frac{1}{2}|v - v^+|^2$.

The problems of maximal and minimal resistance for rough bodies then read as follows.

Problems.

(a) Find
$\sup\{\frac{R(\mathcal{B})}{R(C)} : B$ is obtained by roughening $C\}$.

(b) Find
$\inf\{\frac{R(\mathcal{B})}{R(C)} : B$ is obtained by roughening $C\}$.

They are reduced to maximizing and minimizing specific resistance of an infinitely small flat surface. One writes down: $R(\mathcal{B}) =$

$$= \int_{S^{d-1}} d\tau_C(n) \iint_{(S^{d-1})^2} \frac{|v - v^+|^2}{2} \, d\nu_{\mathcal{B}}(v, v^+|n),$$

where $\nu_{\mathcal{B}}(\cdot|n)$ is the conditional measure (conditional law of resistance) with n fixed.

Problems of maximum and minimum specific resistance read as follows:

$$\text{find} \quad \frac{\inf_{\nu\in\Gamma_{\lambda_-,\lambda_+}}\mathcal{F}(\nu)}{\mathcal{F}(\nu_0)}, \quad \frac{\sup_{\nu\in\Gamma_{\lambda_-,\lambda_+}}\mathcal{F}(\nu)}{\mathcal{F}(\nu_0)}, \quad (1)$$

where

$$\mathcal{F}(\nu) = \iint_{(S^{d-1})^2} \frac{1}{2}|v - v^+|^2 \, d\nu(v, v^+), \quad (2)$$

$\Gamma_{\lambda_-,\lambda_+}$ is the set of measures on $(S^{d-1})^2$ with fixed marginals λ_-, λ_+, where

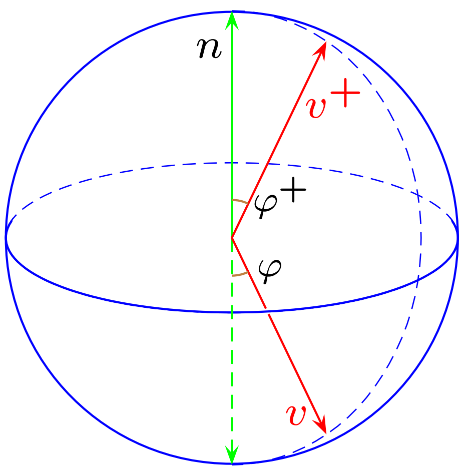
$$d\lambda_-(v) = (v\cdot n)_- dv, \;\; d\lambda_+(v) = (v\cdot n)_+ dv$$

and $\nu_0 \in \Gamma_{\lambda_-,\lambda_+}$ corresponds to elastic reflections:

$$d\nu_0(v, v^+) = (v\cdot n)_- \, \delta(v^+ - v + 2(v\cdot n)n)\, dv.$$

By (1) we compare, at each point of the surface, the resistance of an elementary surface of the rough body with the resistance of a flat elementary surface (of the same area).

The incident flow is associated with the lower hemisphere, and the reflected flow, with the upper hemisphere. The roughness defines a transformation from the lower to the upper hemisphere. The mass distribution of the incident (reflected) flow is given by λ_- (λ_+). The local cost of the transportation is $\frac{1}{2}|v - v^+|^2$. The total cost (specific resistance) is the integral $\mathcal{F}(\nu)$ (2).



Sup-problem. The optimal measure is supported on the subspace $v = v^+$.

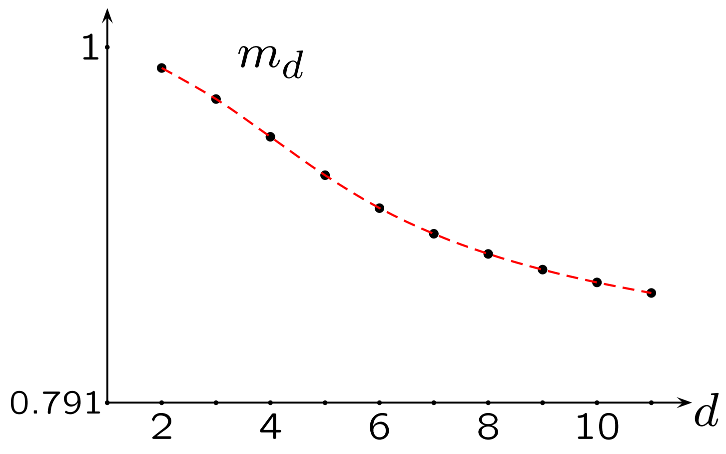
$$\frac{\sup \mathcal{F}(\nu)}{\mathcal{F}(\nu_0)} = \frac{d+1}{2}.$$

Inf-problem. Due to radial symmetry of the problem, it can be reduced to the following 1D one:

$$\text{find} \quad \inf_{\nu \in \Gamma_{\lambda_d,\lambda_d}} \iint_{[0,\pi/2]^2} (1+\cos(\varphi+\varphi^+))\, d\nu(\varphi,\varphi^+),$$

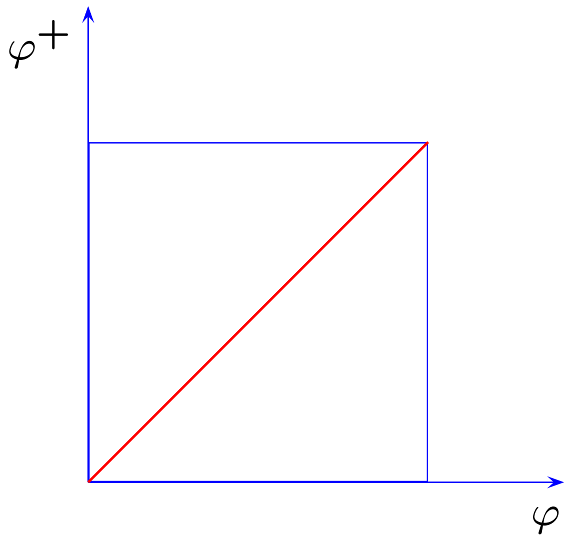
where λ_d is given by
$d\lambda_d(\varphi) = \sin^{d-2}\varphi \cos\varphi \, d\varphi$.

$$\frac{\inf \mathcal{F}(\nu)}{\mathcal{F}(\nu_0)} = m_d.$$

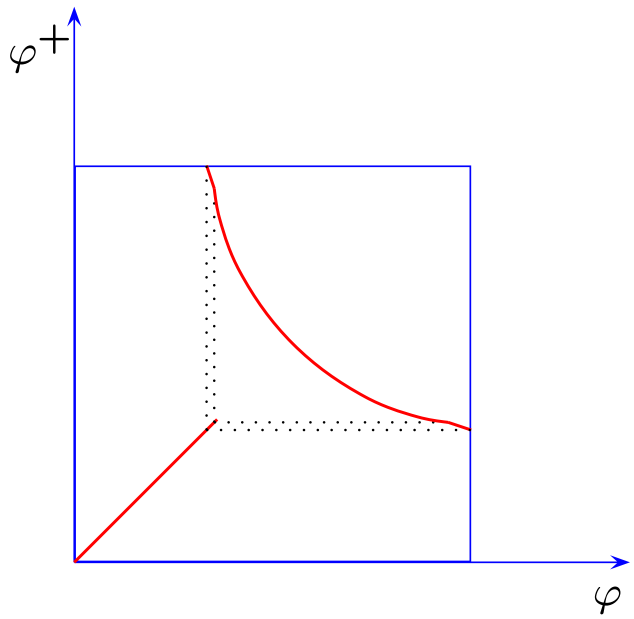


In particular, in the 2D case
$\inf(\text{Resist}) \approx 0.98782$, $\sup(\text{Resist}) = 1.5$.

in the 3D case
$\inf(\text{Resist}) \approx 0.96945$, $\sup(\text{Resist}) = 2$.



Elastic reflections: $\varphi = \varphi^+$.



Optimal scattering law ($d = 2$).