A MASS TRANSPORT APPROACH FOR THE RELATIVISTIC HEAT EQUATION

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The relativistic heat equation

We study the Cauchy problem for "a relativistic version" of the heat equation

$$\partial_t \rho = \text{div}\,(\rho \frac{\nabla \rho}{\sqrt{\rho^2 + |\nabla \rho|^2}}) = \text{div}\,(\rho \frac{\nabla \log \rho}{\sqrt{1 + |\nabla \log \rho|^2}}). \qquad (1)$$

introduced to impose an upperbound for the propagation velocity.
(Ref Brenier (01), Rosenau (92), Mihalas-Mihalas (84))

Assumptions $\rho(t, x)$ with $t \in [0, T]$ and $x \in \Omega$, bounded domain of $\mathbf{R}^d$ and

$$0 < m \leq \rho_0 \leq M \text{ and } \int_0^T \int_\Omega \rho dx dt = 1.$$

Andreu, Caselles, Mazòn (ref Non Linear Analysis and JEMS 2005)
Andreu, Caselles, Mazòn, Moll (ref Arch Ration Mech Anal 2006)

Mass Transport Strategy

The aim of our work is to implement a different point of view following the ideas of Jordan, Kinderleherer, Otto
SIAM J.Math. Anal.(98)

$\longrightarrow$ construction of solutions of general equation of the type

$$\partial_t \rho = \mathrm{div}\ (\rho \nabla c^*(\nabla(F'(\rho))))$$

F is a convex Entropy and c is a convex cost function + cond
c^* is a mobility function, defined as the Legendre transform of c i.e.

$$c^*(x) = \sup_{y \in \mathbf{R}^d} x \cdot y - c(y).$$

$\longrightarrow$ time discrete scheme involving a double minimization process.

Time discrete scheme

Let $P(\Omega)$ be the set of probability measures on Ω,

$\rho_0 \in P(\Omega)$ given, find $\rho^h(t,x) \in P(\Omega)$ defined by

$$\begin{cases} \rho^h(0,x) = \rho_0(x) \\ \rho^h(t,x) = \rho_i^h(x) \quad \text{for } t \in]ih;(i+1)h] \quad h \text{ being the time step} \end{cases}$$

where

$$\rho_i^h(x) = \text{argmin}_\rho\Big(\int_\Omega F(\rho(y))dy + h \inf_{\gamma \in \Gamma_i^h(\rho_{i-1}^h,\rho)} \int_{\Omega\times\Omega} c\Big(\frac{x-y}{h}\Big)d\gamma(x,y)\Big),$$

$\Gamma_i^h(\rho_{i-1}^h,\rho)$ is the set of transport plans between ρ_{i-1}^h and ρ.

$\longrightarrow$ Generalisation of discrete gradient flow to general convex cost function

Ref Ambrosio Gigli Savaré Lectures in math ETH (2005),

Villani Graduate studies in Math AMS (2003)

Previous results

- Jordan-Kinderlehrer-Otto SIAM Journ Math Anal (98)

$c(z) = \frac{|z|^2}{2}$ and $F(\rho) = \beta^{-1}\rho \log \rho - V\rho$, V given.
$\longrightarrow$ Linear Fokker Plank equation

$$\partial_t \rho = \text{div } (\nabla V \rho) + \beta^{-1} \Delta \rho.$$

- Otto Preprint (96) $c(z) = \frac{|z|^q}{q}$ and $F(\rho) = \frac{n\rho^m}{m(m-1)}$
where $m = n + \frac{p-2}{p-1}$, $\frac{1}{p} + \frac{1}{q} = 1$, $p \geq 2$

$\longrightarrow$ Doubly degenerate equation

$$\partial_t \rho = \text{div } (|\nabla \rho^n|^{p-2} \nabla \rho^n)$$

- Agueh Adv Diff Equ (05) $\beta|z|^q \leq c(z) \leq \alpha(|z|^q + 1)$
and F convex + displacement convex

$\longrightarrow$ A large set of equation

$$\partial_t \rho = \text{div } (\rho \nabla c^*(\nabla(F'(\rho))))$$

Cost Function and Entropy for the relativistic heat equation

The relativistic heat equation does not belong to those sets of equations since it corresponds to the cost function

$$c(z) = \begin{cases} 1 - \sqrt{1 - |z|^2} & \text{if } |z| \leq 1 \\ +\infty & \text{if } |z| > 1 \end{cases}$$

and the Entropy $F(\rho) = \rho \log \rho - \rho$.

- This cost function is strictly convex and discontinuous ($c(z) = \infty$ if $|z| > 1$)
- ∇c^* – and then the velocity $\nabla c^*(\nabla F'(\rho))$ – is bounded $\Longrightarrow$ characteristic property of a relativistic phenomenum.

General assumptions on the Cost and on the Entropy

This present work will in fact apply for any

- Cost function

$$c(z) = \begin{cases} \tilde{c}(|z|) \geq 0 & \text{if } |z| \leq 1 \\ +\infty & \text{if } |z| > 1 \end{cases}$$

where $\tilde{c}$ is strictly convex, $C^0[0,1] \cup C^2([0,1[)$,
with $|\nabla c(z)| \to \infty$ when $|z| \to 1$.

- Entropy function $F \in C^2(\mathbf{R})$ satisfying $\frac{F(\lambda)}{\lambda} \to \infty$ when $\lambda \to \infty$
and $\lambda^d F(\lambda^{-d})$ is convex (displacement convexity)
Ref McCann Adv Math (97).

Formal argument I

Step 1 Find for every time intervall $[ih; (i+1)h]$

- the optimal transport plan γ_i^h,
- its second marginal ρ_i^h
- the associated optimal map (i.e. $\gamma_i^h = \delta(x - S_i^h(y))$).

The existence of ρ_i^h is ensured by the double minimization process.

Problem 1 Definition of the map

Step 2 Derive the Euler-Lagrange equations (derivation / ρ and then ∇)

$$\nabla(F'(\rho(y))) = \nabla c(\frac{S_i^h(y) - y}{h}) \implies \frac{S_i^h(y) - y}{h} = \nabla c^*(\nabla(F'(\rho(y))))$$

since $\nabla c^*(\nabla c(z)) = z$.

Problem 2 supp $\gamma_i^h \subset \{|x - y| \leq h\}$ but if $|x - y| = h$,

then $\nabla c(\frac{x-y}{h})$ is not defined.

Formal argument II

Step 3 Obtain an approximate time discrete equation for ρ^h by $\times \rho \nabla \phi$

$$\frac{\rho_i^h - \rho_{i-1}^h}{h} = \text{div}(\rho_i^h \nabla c^*(\nabla(F'(\rho_i^h))) + \text{Correction terms in O(h)}.$$

Step 4 It remains to pass to the limit when the time step h goes to zero

$$\Longrightarrow \quad \partial_t \rho = \text{div}(\rho \nabla c^*(\nabla(F'(\rho))).$$

Problem 3 We want to use a monotonicity argument (Minty-Browder)

to identify the limit but lack of regularity of ρ.

Construction of the optimal map (drop the h)

The classical result of Gangbo McCann CRAS (95) can not be applied here. Indeed it is based on the Kantorovich duality:

$$\int_{\Omega\times\Omega} c(x-y)d\gamma_{opt}(x,y) = \int_{\Omega}\phi(x)d\rho_0(x) + \int_{\Omega}\psi(y)d\rho_1(y)$$

where ϕ is the c-transform of ψ, i.e.

$$\phi(x) = \inf_y (c(x-y) - \psi(y))$$

When $c \in C^{1,\alpha}$, ψ is Lipschitz and then differentiable which means that when $(x,y) \in \text{supp}\gamma_{opt}$, we have

$$\nabla c(x-y) = -\nabla\psi(y) \text{ and then } x = y + \nabla c^*(-\nabla\psi(y))$$

In our case, ψ is no more Lipschitz + problem when $|x-y| = 1$.

The strategy consists in introducing a mollified problem.

The mollified problem

We introduce the Yoshida mollification of the convex function c (Ref Brezis Operateur maximaux monotones et semi-groupes de contraction dans les espaces de Hilbert (73)) by a
Convexification of c^*

$$c^{\epsilon *}(x) = c^*(x) + \frac{\epsilon}{2}|x|^2$$

$\Longrightarrow$ Mollification of c

$$c^{\epsilon}(x) = \inf_{z \in \mathbf{R}^d} (c(x-z) + \frac{|z|^2}{2\varepsilon})$$

and we define $\gamma_i^{\epsilon h}$, the optimal transport plan between ρ_{i-1}^{h} and $\rho_i^{\epsilon h}$ obtained when the minimization involves the mollified cost function c^{ϵ}.

Kantorovich duality and optimal map for the mollified problem

$$\int_{\Omega\times\Omega} c^{\epsilon}(x-y)d\gamma^{\epsilon}(x,y) = \int_{\Omega}\phi^{\epsilon}(x)d\rho_0(x) + \int_{\Omega}\psi^{\epsilon}(y)d\rho^{\epsilon}(y)$$

where the potential function satisfies

$$\phi^{\epsilon}(x) = \inf_{y\in\mathbf{R}^d}(c^{\epsilon}(x-y) - \psi^{\epsilon}(y)).$$

Gangbo-McCann $\Rightarrow$ existence of a map

$$S^{\epsilon}(y) = y + \nabla c^{\epsilon *}(-\nabla\psi^{\epsilon}(y))$$

Agueh $\Rightarrow$ Euler-Lagrange eq.

$$S^{\epsilon}(y) = y + \nabla c^{\epsilon *}(\nabla(F'(\rho^{\epsilon}(y))))$$

that leads to $\psi^{\epsilon}(y) = -F'(\rho^{\epsilon}(y))$

Limiting process to obtain the optimal map I

1- the limit $\lim_{\epsilon \to 0} \gamma_i^\epsilon = \gamma_i$ and
supp $\gamma_i \subset \{(x, y)$ such that $|x - y| < 1\} \cup Z_i$ where $\gamma_i(Z_i) = 0$.

2- $\rho^\epsilon(y)$ is bounded in $W^{1,1}(\Omega)$ since the displacement convexity of the Entropy leads to a Fisher information–entropy inequality

$$\int \rho^\epsilon \nabla c^{\epsilon *}(\nabla(F'(\rho^\epsilon))) \cdot \nabla(F'(\rho^\epsilon))dy \leq \int [F(\rho_0(y)) - F(\rho^\epsilon(y))]dy.$$

Then $\rho_i^\epsilon \to_{\epsilon \to 0} \rho_i \in BV(\Omega)$ weak in $BV(\Omega)$, strong in $L^1(\Omega)$

$\implies \rho_i$ is approximatively differentiable
$\implies$ avoid a.e. the undetermination of the map.

Limiting process to obtain the optimal map II

3- Kantorovich duality

$$\int_{\Omega\times\Omega} c(x-y)d\gamma_i(x,y) = \int_\Omega \phi_i(x)d\rho_{i-1}(x) + \int_\Omega \psi_i(y)d\rho_i(y).$$

with $\phi_i(x) = \inf_{y\in\overline{\Omega}}(c(x-y) - \psi_i(y))$ and $\psi_i(y)) = -F'(\rho_i)$

$\Longrightarrow$ Optimal map $+$ Euler-Lagrange equation

$$S_i(y) = y + \nabla c^*(-\nabla\psi_i(y)) = y + \nabla c^*(\nabla(F'(\rho_i(y)))).$$

Monge-Kantotovich problem for the relativistic cost

New collaboration with J. Bertrand. Let μ_0 and μ_1 be two given compact supported probabilities. We look for

$$\inf_{\pi\in\Gamma(\mu_0,\mu_1)}\int_{\Omega_0\times\Omega_1} c(x-y)d\pi(x,y).$$

If μ_0 and μ_1 are too far from each other, the cost will be infinite. If they are too close (e.g. $\mathrm{dist}(\Omega_0,\Omega_1)$ too small), we don't see the difficulties of the cost.

$+$ extension of the problem to the restriction of a strictly convex function on the bowl.

Parametrized problem

We introduce a parametrized problem

$$C(t) = \inf_{\pi \in \Gamma(\mu_0,\mu_1)} \int_{\Omega_0 \times \Omega_1} c(\frac{x-y}{t}) d\pi(x,y).$$

Properties of C

1. C is decreasing.
2. There exists a critical T such that $t < T \rightarrow C(t) = \infty$ and
   $t > T \rightarrow C(t) < \infty$
3. $c(T) < \infty$

Ideas for the construction of a map for $t > T$

We construct a Kantorovich potential

- finite, Ω covered by a finite family of tubular neighborhood
- a.e. differentiable on $p_x(Supp(\pi_{opt}) \cap \{|x-y| < t\})$ as extension of a sequence of Lipschitz functions

$+$ use argument of Champion, De Pascale, Juutinen, SIAM J. Math. Anal. 2008 to prove

- $p_x(\{|x-y| < t\} \cap Supp(\pi_{opt})) \cap p_x(\{|x-y| = t\} \cap Supp(\pi_{opt})) = \emptyset$.
- ${\pi_{opt}}_{|(p_x(\{|x-y|=t\} \cap Supp(\pi_{opt})) \times \Omega)}$ is supported by the graph of an application

Particularity of the relativistic cost

The function $C(t)$ is a.e. differentiable $+$

$$\forall \delta > 0 \quad C'(t) > K\nabla c(1-\delta)\pi^t_{opt}(|x-y| \geq (1-\delta)t)$$

$\rightarrow$ for almost every $t > T$, $\pi^t_{opt}(|x-y| = t) = 0$.

From the time discrete equation to the continuous equation

By multiplying the Euler-Lagrange equation by $\rho_i^h \nabla \phi$, we obtain the approximate time discrete equation

$$\frac{\rho_i^h - \rho_{i-1}^h}{h} = \operatorname{div}(\rho_i^h \nabla c^*(\nabla(F'(\rho_i^h))) + \text{Correction terms in O(h)}.$$

and when h goes to zero, we obtain the continuous equation

$$\partial_t \rho = \operatorname{div}\,(\rho A)$$

where ρ is the $L^1([0, T] \times \Omega)$ limit of ρ^h
and A is the w*$L^\infty(\Omega)$ limit of $\nabla c^*(\nabla(F'(\rho^h))$.

It remains to identify the limit A

Identification of the limit A

We use a monotonicity argument (Minty-Browder) by proving for any test functions $\zeta \geq 0$ and ξ (Ref Evans Weak convergence method for nonlinear partial differential equations)

$$\int \xi(\rho A - \rho \nabla c^*(\zeta))(\nabla(F'(\rho)) - \zeta) \geq 0 \quad (*)$$

which implies that $A = \nabla c^*(\nabla(F'(\rho)))$
Indeed, it implies

$$(\rho A - \rho \nabla c^*(\zeta))(\nabla(F'(\rho)) - \zeta) \geq 0$$

and then by taking $\zeta = \nabla(F'(\rho)) + \gamma\chi$, $\gamma \nearrow 0$ and $\gamma \searrow 0$, we deduce the result.
To obtain equation $(*)$, we pass to the limit when $h \to 0$ in

$$\int \xi(\rho^h \nabla c^*(\nabla(F'(\rho^h))) - \rho^h \nabla c^*(\zeta))(\nabla(F'(\rho^h)) - \zeta) \geq 0$$

obtained thanks to the monotonicity of ∇c^*.

Limiting process in the Minty Browder argument

Instead of passing to the limit in the non linear term
we write the Fisher information- Entropy inequality

$$\int \rho^h \nabla c^*(\nabla(F'(\rho^h))) \cdot \nabla(F'(\rho^h)) \leq \int [F(\rho_0(y)) - F(\rho^h(T,y))]dy,$$

the strong convergence of ρ^h leads to

$$\int [F(\rho_0(y)) - F(\rho^h(T,y))]dy \longrightarrow \int [F(\rho_0(y)) - F(\rho(T,y))]dy = -$$

and finally, we obtain by using the equation that,

$$\int \rho^h \nabla c^*(\nabla(F'(\rho^h))) \cdot \nabla(F'(\rho^h)) \leq \int \rho A \nabla(F'(\rho))$$

Problem define all the terms when $\rho \in L^1_w([0,T], BV(\Omega))$.

Need regularity for $\partial_t \rho$ to multiply the equation by $\nabla(F'(\rho))$
$+$ Def of functions of BV functions and L^1 lower-semicontinuity
Ref De Cicco, Fusco, Verde J. Convex Anal. (05).
Ref Andreu, Caselles, Mazon Arch Rat Mech (05).

Main Result

Theorem

(i) **Support of the optimal measure: Finite speed of propagation**

$$\text{supp } \gamma_i^h \subset \{(x,y) \quad | \quad \frac{|x-y|}{h} < 1\} \cup Z_i^h \text{ with } \gamma_i^h(Z_i^h) = 0.$$

(ii) **Euler-Lagrange equation: a discrete scheme**

$$\gamma_i^h(x,y) = \delta(x - S_i^h(y)) \text{ with } S_i^h(y) = y + h\nabla c^*(\nabla(F'(\rho_i^h(y)))).$$

(iii) **Convergence of the measure ρ^h.** Up to a subsequence,

$\rho^h \longrightarrow \overline{\rho}$ in $L^1([0,T] \times \Omega)$

and $\overline{\rho} \in W = L^\infty([0,T] \times \Omega) \cap L^1_w([0,T], BV(\Omega))$.

(iv) **Limiting equation** $\overline{\rho}$ is a solution to

$$\partial_t \overline{\rho} = \text{div}(\overline{\rho} \nabla c^*(\nabla(F'(\overline{\rho})))).$$