PROPERTIES OF k-SCHUR FUNCTIONS, k-BRANCHING AND THE k-POSET

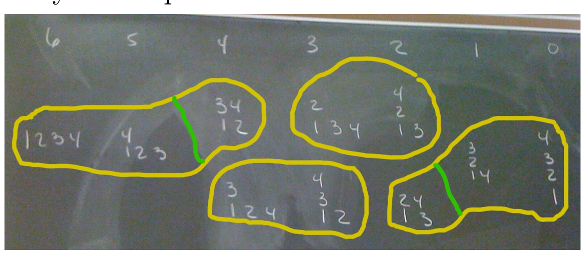
NOTES FROM THE TALK OF LUC LAPOINTE

Abstract. First Lecture: Definitions We will present the various conjecturally equivalent characterizations of k-Schur functions $s_\lambda^{(k)}[X;t]$: the tableau atom definition, the algebraic definition using Jings operators and the definition as sums over strong tableaux with spin. We will give an overview of the properties of k-Schur functions and specify whether they are known to hold in each characterization.

Second lecture: Atomic properties. We will focus on the tableau definition of k-Schur functions. We will discuss the notions of katabolism, cyclage, Lascoux-Schutzenberger action of the symmetric group on words, etc. We will introduce the concept of copies of atoms and explain the meaning of the Pieri rule and branching coefficients (decomposition of k-Schurs into k+1-Schurs) in this context.

Third lecture: k-poset We will introduce a poset (the k-poset) on a certain type of partitions called k-shapes that allows to give an explicit expression for the branching coefficients. We will explain how the k-poset is compatible with the concept of charge and with the Lascoux-Schtzenberger action of the symmetric group on words. Finally, we will present a conjecture that relates the k-poset and tableau atoms, and give some open problems that arise from the k-poset.

The first talk I showed you this picture.



Here is an example of how we build this with Atoms.

$$\mathbb{A}_{22}^{(2)} = \young(1122) + \young(2,112) + \young(22,11)$$

$$\mathcal{F}(\mathbb{A}_{22}^{(2)}) = s_{22} + ts_{31} + t^2 s_4$$

Notice how this appears on the standard tableaux. How do we standardize in a way that we see these copies of the Atom?

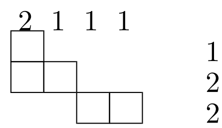
We looked very hard to match up the column strict tableaux with the standard tableaux. It seems as though it was difficult to match these up. I don't want to say that our original defintion was the wrong direction to go, but that maybe there is another way of constructing the Atoms that reflects better the standard tableaux we found.

From the computer we had a way of to associate a k-tableau for every standard tableau so we had lots of experimental values but not in general.

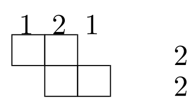
Instead we used the $s_\lambda^{(k)}$

Pushout bijection

k-shape starting with a core λ $k = 3$



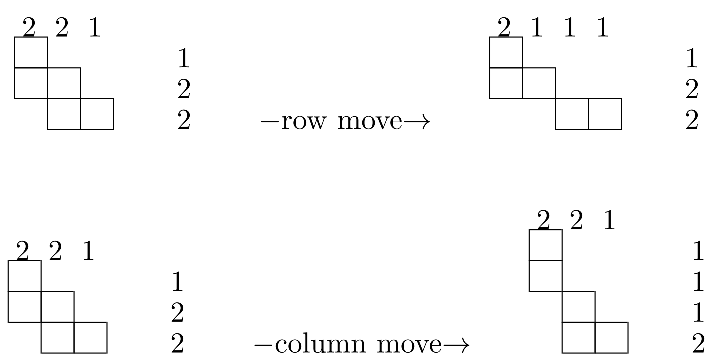
Definition: a k-shape is a partition such that $rs(\lambda)$ and $cs(\lambda)$ re partitions. In particular, k-cores and $k+1$-cores are k shapes. Here is an example of something which is not a k-shape:



the reason is that $cs(\lambda) = (1, 2, 1)$ which is not a partition.

Poset on k-shapes consisting of relations that are row moves

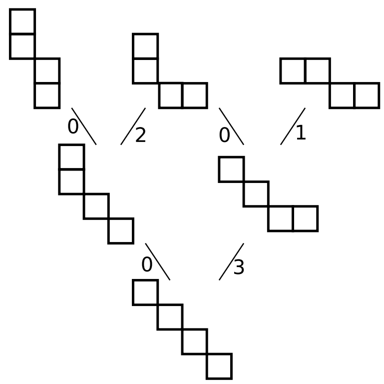
Example $k = 3$,



There is a row move (resp. column move) from μ to λ (μ and λ are the same shape) if $rs(\lambda) = rs(\mu)$ (resp $cs(\lambda) = cs(\mu)$) and λ/μ is a horizontal strip.

The charge of a move from μ to λ is $\begin{cases} 0 & if\ row\ move \\ |\lambda/\mu| & if\ column\ move \end{cases}$

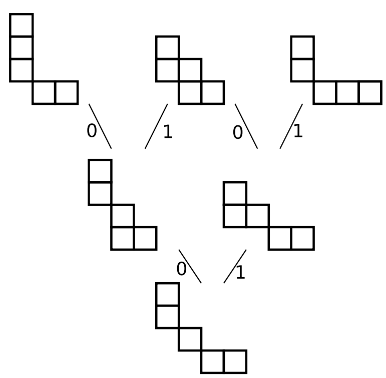
$k = 2$



$$s^{(1)}_{(1111)} = s^{(2)}_{(1111)} + (t^2 + t^3)s^{(2)}_{(211)} + t^4 s^{(2)}_{(22)}$$

One caveat, two paths are equivalent if they differ by a sequence of diamonds that preserve the charge so that in the example below there is only one equivalent path to the final shape (221).

$k = 3$



$$s^{(2)}_{(2111)} = s^{(3)}_{(2111)} + ts^{(3)}_{(221)} + t^2 s^{(3)}_{(311)}$$

Note in the examples above, there is one example where the diamond does not preserve the charge, and in the example below there is an equivalence in the diamond (and hence only one corresponding term).

N.B. in the paper the definiton is a bit more complicated, but conjectureally these are these same.

The theorem (holds only for $t = 1$ but is conjecturally true for t arbitrary).

$$s^{(k)}_{\lambda} = \sum d^{(k+1)}_{\lambda\mu} s^{(k+1)}_{\lambda}$$

and

$$\mathfrak{S}_\lambda^{(k+1)} = \sum d_{\lambda\mu}^{(k+1)} \mathfrak{S}_\lambda^{(k)}$$

where $d_{\lambda\mu}^{(k+1)}$ is the sum over all classes of paths $\sum_{[P]} t^{ch(P)}$.

How do we prove this?

We show that there is a bijection start with $T^{(k+1)} \to$ by a path to another $T^{(k)}$ then there is a bijection from $T^{(k+1)}$ to pairs $(T^{(k)}, P)$ where P is a path.

Then there is a picture square with a weak cover and a path. -¿P-¿ c — — -¿P-¿

compatibilty with the usual concept of charge.

***** insert other picture here ****

Maria Helena is looking for the meaning of charge?

connection with atoms.

Somehow what seems to be coming out of this is the idea of copies of atoms. What we don't know for sure is that

Conjecture:

$$\mathbb{A}_T^{(k)} = \{V(\text{standard tableau})|k - \text{tableau associated to } V \text{ is } U^{(k)}\}$$

We say that $\mathbb{A}_T^{(k)}$ is a copy of $\mathbb{A}_\lambda^{(k)}$ and we observe that $\mathcal{F}(\mathbb{A}_T^{(k)}) = t^* \mathcal{F}(\mathbb{A}_\lambda^{(k)})$.

What about strong tableaux?

start with a minimal $U^{(k)}$

We are looking for a bijection between words and a pair consisitng of a k-tableau and a dual-k-tableau that is compatible with the charge where $P^{(k)}$ is a weak k-tableau and $Q^{(k)}$ is a strong k-tableau.

$$\begin{array}{c} (P,Q) \\ \downarrow\uparrow \\ (P^{(n-1)}, Q^{(n-1)}) \\ \downarrow\uparrow \\ w \leftrightarrow (P^{(n-2)}, Q^{(n-2)}) \\ \downarrow\uparrow \\ \vdots \\ \downarrow\uparrow \\ (P^{(k)}, Q^{(k)}) \end{array}$$

$ch(P^{(k)}) + spin(Q^{(k)}) = ch(P)$

This bijection would imply the k-branching for t-generic. Luc is offering \$2000 for a good result.