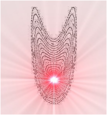
The NLSE and the SPE as approximations to a nonlinear wave equation

Tobias Schäfer

Department of Mathematics
College of Staten Island
&
Physics Department
Graduate Center
City University of New York

Fields Institute May 2011



A nonlinear wave equation

- From Maxwell's equations:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) E = \frac{\partial^2}{\partial t^2} \int \chi^{(1)}(t-\tau) E(x,\tau) d\tau + \chi^{(3)} \frac{\partial^2}{\partial t^2} E^3$$

- Neglects frequency dependence of radial modes
- Neglects retardation of nonlinear response
- Good model to study approximations of Maxwell's equations from theoretical point of view
- Numerics can be easily realized

Derivation of the NLSE

- Multiple scales $x_n = \epsilon^n x$ and $t_0 = t$, $t_1 = \epsilon t$.

$$E(x,t) = \epsilon A_0(x_1, x_2, ...; t_1)\mathrm{e}^{i(\tilde{\beta} x_0 - \tilde{\omega} t_0)} + \mathrm{c.c.}$$

- Linear equation easy:

$$\left(\frac{\partial^2}{\partial x_0^2} + \omega^2\right)\hat{E}_0 = -\omega^2 \hat{\chi}^{(1)}\hat{E}_0$$

- Solution is a wavepacket

$$\hat{E}_0 = \epsilon \hat{A}_0(x, \omega - \tilde{\omega})\mathrm{e}^{i\tilde{\beta} x_0}, \qquad \beta(\omega)^2 = (1 + \hat{\chi}^{(1)}(\omega))\omega^2$$

Solvability Conditions

- Order by order

$$\mathcal{O}(\epsilon^2): \qquad \frac{\partial A_0}{\partial x_1} + \tilde{\beta}' \frac{\partial A_0}{\partial t_1} = 0$$

$$\mathcal{O}(\epsilon^3): \qquad i\frac{\partial A_0}{\partial x_2} - \frac{\tilde{\beta}''}{2}\frac{\partial^2 A_0}{\partial t_1^2} + \frac{3\tilde{\omega}^2}{2\tilde{\beta}}|A_0|^2 A_0 = 0$$

- Next order often important, in particular for short pulses
- Generalized NLS with higher-order terms: Higher-order dispersion, Raman- and self-steepening terms

A particular susceptibility

- Particular approximation of the susceptiblity

$$\hat{\chi}^{(1)} = \hat{\chi}_0^{(1)} - \frac{\hat{\chi}_2^{(1)}}{\omega^2}$$

- Linear part in Fourier domain

$$\left(\frac{\partial^2}{\partial x^2} + \omega^2 \right) \hat{E} = -\omega^2 \left(\hat{\chi}_0^{(1)} - \frac{\hat{\chi}_2^{(1)}}{\omega^2} \right) \hat{E}$$

- Back in Time domain and after rescaling

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) E = \alpha E + \gamma \frac{\partial^2}{\partial t}(E^3)$$

Short-Pulse Equation

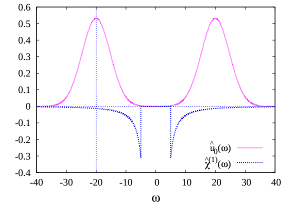
- Short pulses

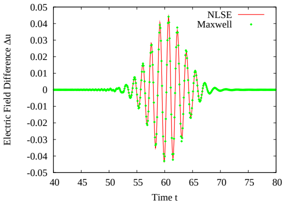
$$\begin{aligned} E &= \epsilon A_0 \left(\phi = \frac{x-t}{\epsilon}, x_1 = \epsilon x, \ldots \right) + \epsilon^2 A_1 + \ldots \\ E_{xx} &= \frac{1}{\epsilon} A_{0\phi\phi} + \epsilon A_{1\phi\phi} + 2\epsilon A_{0x_1\phi} + \ldots \\ E_{tt} &= \frac{1}{\epsilon} A_{0\phi\phi} + \epsilon A_{1\phi\phi} + \ldots \end{aligned}$$

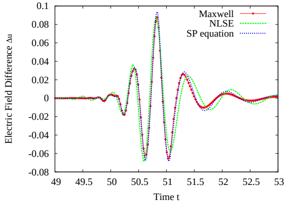
- Short pulse equation (S. and Wayne 2004)

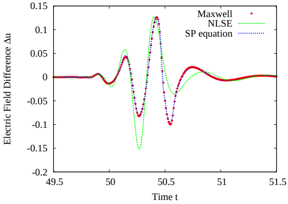
$$2\partial_{x_1}\partial_{\phi} A_0 = \alpha A_0 + \gamma \partial_{\phi\phi} A_0^3$$

Comparison to Maxwell's equations









(Chung, Jones, S., Wayne 2005)

Soliton solutions of the SPE

- Sakovich and Sakovich were able to prove integrability and to construct solitary solutions to the SPE

$$u = 4mn\frac{m\sin\psi\sinh\phi + n\cos\psi\cosh\phi}{m^2\sin^2\psi + n^2\cosh^2\phi}$$

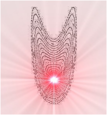
$$x = y + 2mn\frac{m\sin 2\psi - n\sinh 2\phi}{m^2\sin^2\psi + n^2\cosh^2\phi}$$

$$\phi = m(y+t), \psi = n(y-t), n = \sqrt{1-m^2}$$

- Conditions for nonsingular pulse: $m < m_{\mathrm{cr}} = \sin\pi/8$, for small m NLS-soliton-like pulse:

$$u(x,t) \approx 4m\cos(x-t)\mathrm{sech}(m(x+t))$$

The shortest solitary wave is about three cycles.

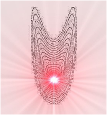


Higher-order SPE

- As for the NLSE, we can derive higher-order corrections
- Result: Higher-order SPE: (Chung, Kurt, S., in progress)

$$\begin{aligned} A_X &= \frac{\alpha}{2}\int A + \frac{\gamma}{2}(A^3)_\phi \\ &\quad -\frac{\epsilon^2}{2}\left(\frac{\alpha^2}{4}\int\int\int A + \frac{\alpha\gamma}{4}\int A^3 \right. \\ &\quad \left. +\frac{3\alpha\gamma}{4}A^2\int A + \frac{3\gamma^2}{4}A^2(A^3)_\phi\right) \end{aligned}$$

- Numerics: Dispersive term seems dominant for Sakovich soliton.



Nonlocal multiple scales

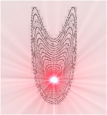
- Leading order:

$$\hat{E}_0(x_0, x_1, \omega) = \hat{A}_0(x_1, \omega)\mathrm{e}^{i\omega x} + \hat{B}_0(x_1, \omega)\mathrm{e}^{-i\omega x}$$

- First order:

$$\left(\frac{\partial^2}{\partial x_0^2} + \omega^2\right)\hat{E}_1 = -2\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_1}\hat{E}_0 - \omega^2\hat{\chi}^{(1)}(\omega)\hat{E}_0 - \frac{\omega^2}{(2\pi)^2}\mathcal{N}(E_0).$$

- Fundamental solutions of l.h.s: $\{\exp(i\omega x_0), \exp(-i\omega x_0)\}$
- Fredholm alternative: account for all possible frequency combinations in the nonlocal term that create resonances.
- Result: nonlocal equations for $\hat{A}_0$ and $\hat{B}_0$.



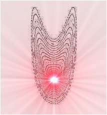
A pair of nonlocal equations

- Here they are: (Chung, S., 2007)

$$
\begin{aligned}
\frac{\partial}{\partial x_1} A_0 \quad &+ \quad \frac{1}{2}\frac{\partial}{\partial t}\Big(A_0^3 + 3B_{\text{zero}}A_0^2 + 3B_{\text{int}}A_0 \\
&+ \quad \int \chi^{(1)}(t-\tau)A_0(x_1,\tau)d\tau \Big) = 0, \\
\frac{\partial}{\partial x_1} B_0 \quad &- \quad \frac{1}{2}\frac{\partial}{\partial t}\Big(B_0^3 + 3A_{\text{zero}}B_0^2 + 3A_{\text{int}}B_0 \\
&+ \quad \int \chi^{(1)}(t-\tau)B_0(x_1,\tau)d\tau \Big) = 0.
\end{aligned}
$$

$$
A_{\text{zero}} = \frac{1}{2\pi}\hat{A}_0(0), \;\; A_{\text{int}} = \frac{1}{(2\pi)^2}\int \hat{A}_0(\omega_1)\hat{A}_0(-\omega_1)\, d\omega_1
$$

- A_0 and B_0 are coupled very *weakly*.



Pulse stabilization

- Nonlinearity and nonlocality are preserved:

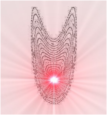
$$\frac{\partial}{\partial x_1} A_0 + \frac{1}{2} \frac{\partial}{\partial t} \left(A_0^3 + \int \chi^{(1)}(t - \tau) A_0(x_1, \tau) d\tau \right) = 0$$

- We can try to balance them to solve $\partial A_0 / \partial x_1 = 0$

$$A_0^3 + \int \chi^{(1)}(t - \tau) A_0(x_1, \tau) d\tau = 0.$$

- Example: Lorentz profile

$$\begin{aligned} A_0(t) &= \frac{\alpha}{1 + (\beta t)^2}, \\ \hat{\chi}^{(1)}(\omega) &= -\frac{1}{8} \frac{\alpha^2}{\beta^2} (\omega^2 + 3\beta\, |\omega| + 3\beta^2). \end{aligned}$$



A stochastic term in the susceptibility

- Stochastic wave equation:

$$\frac{\partial^2 E}{\partial x^2} - \frac{\partial^2 E}{\partial t^2} = (\alpha + \nu\xi(x))E(x,t) + \chi^{(3)}\frac{\partial^2}{\partial t^2}E(x,t)^3.$$

- Stochastic linear susceptibility modeled as white noise

$$\langle \xi(x)\xi(x')\rangle = \delta(x - x')$$

- How can we derive an averaged equation?

Techniques for coarse-graining noise

- Deterministic case: Use multi-scale expansion.
- Stochastic case: We have to find a way to combine multi-scale techniques and randomness.
- Three methods:
  1. Random solvability conditions
  2. Asymptotic expansion of the Fokker-Planck equation
  3. Path integrals
- (S., Moore, 2011)

A simple toy problem

- A linear SDE with a periodic coefficient

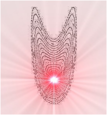
$$\dot{x} = d(t)x + \nu g(x(t)) + \sigma\xi(t), \qquad x(0) = a,$$

- Scales: $\sigma^2 \sim \nu \sim \epsilon$ and $T \sim \mathcal{O}(1)$.
- Both nonlinearity and randomness will come into play for times on the scale of $\mathcal{O}(1/\epsilon)$.
- First method: Random solvability conditions
- Multi-scale expansion of the SDE using $t_k = \epsilon^k t$.

$$x(t) = x_0(t_0, t_1) + \epsilon x_1(t_0, t_1) + \ldots$$

- Leading order $\mathcal{O}(1)$:

$$x_0(t_0, t_1) = \tilde{x}(t_1)\mathrm{e}^{R(t_0)}, \qquad R'(t_0) = d(t_0)$$



Random Solvability Conditions

- Next order can be written as

$$Lx_1 = x_{1t_0} - d(t_0)x_1 = -x_{0t_1} + \frac{\sigma}{\epsilon}\xi(t_0) + \frac{\nu}{\epsilon}g(x_0(t_0, t_1)).$$

- $\mathrm{Ker}(L^+)$ is generated by $\exp(-R(t_0))$
- Fredholm alternative yields

$$\tilde{x}' = \frac{\sigma}{\epsilon}\frac{1}{T}\int_0^T \xi(t_0)\mathrm{e}^{-R(t_0)}dt_0 + \frac{\nu}{\epsilon}\frac{1}{T}\int_0^T \mathrm{e}^{-R(t_0)}g\left(\tilde{x}(t_1)\mathrm{e}^{R(t_0)}\right)dt_0$$

- First term: *Slow noise* on the t_1-scale:

$$\tilde{x}' = \tilde{g}(\tilde{x}(t_1)) + \tilde{\sigma}\Xi(t_1).$$

Asymptotic expansion of the FPE

- Can we obtain the same result by borrowing tools from the PDE world? Yes!
- Fokker-Planck equation:

$$p_t = -\partial_x \left[(d(t)x + \nu g(x))p \right] + \frac{\sigma^2}{2} \partial_{xx} p, \qquad p(0,x) = \delta(x-a).$$

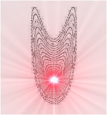
- Multi-scale expansion:

$$p(t,x) = p_0(t_0, t_1, x) + \epsilon p_1(t_0, t_1, x) + ...$$

- Leading order is solved by characteristics:

$$p_0(t_1, t_0, x) = \mathrm{e}^{-R(t_0)} \tilde{p} \left(t_1, \mathrm{e}^{-R(t_0)} x \right)$$

- $\mathrm{Ker}(L^+)$ is generated by $\phi(x, t_0) = \psi(x\mathrm{e}^{-R(t_0)})$



Fredholm Alternative

- Solvability condition:

$$
\begin{aligned}
0 \;=\; & \int_0^T \int_{-\infty}^{\infty} \psi(x\mathrm{e}^{-R(t_0)}) \left(-\mathrm{e}^{-R(t_0)} \tilde{p}_{t_1} \left(t_1, x\mathrm{e}^{-R(t_0)} \right) \right. \\
& - \frac{\nu}{\epsilon} \left(g'(x)\mathrm{e}^{-R(t_0)} \tilde{p} \left(t_1, x\mathrm{e}^{-R(t_0)} \right) + g(x)\mathrm{e}^{-2R(t_0)} \tilde{p}_X \left(t_1, x\mathrm{e}^{-R(t_0)} \right) \right) \\
& \left. + \frac{\sigma^2}{2\epsilon} \mathrm{e}^{-3R(t_0)} \tilde{p}_{XX} \left(t_1, x\mathrm{e}^{-R(t_0)} \right) \right) dx\, dt_0
\end{aligned}
$$

- Since ψ is an arbitrary function, this leads to an explicit equation for $\tilde{p}_1$ given by

$$
\begin{aligned}
\tilde{p}_{t_1} \;=\; & -\frac{\nu}{\epsilon T} \int_0^T g' \left(x\mathrm{e}^{R(t_0)} \right) \tilde{p} + g \left(x\mathrm{e}^{R(t_0)} \right) \mathrm{e}^{-R(t_0)} \tilde{p}_x \, dt_0 \\
& + \frac{1}{2} \left(\frac{\sigma^2}{\epsilon T} \int_0^T \mathrm{e}^{-2R(t_0)}\, dt_0 \right) \tilde{p}_{xx}
\end{aligned}
$$

Path integrals

- Stratonovich SDE:

$$\dot{x} = f(x, t; \epsilon) + \sqrt{\epsilon} g(x, t) \xi(t), \quad x(0) = a$$

- Path integral representation:

$$\begin{aligned} p(x, t) &= \int_{\mathcal{C}(x,t|a,0)} \mathcal{D}x(\tau) \, \mathrm{e}^{-\int_0^t L(x(\tau), \dot{x}(\tau), \tau) \, d\tau} \\ &= \int_{\mathcal{C}(x,t|a,0)} \mathcal{D}x(\tau) \mathcal{P}(x(\tau)) \end{aligned}$$

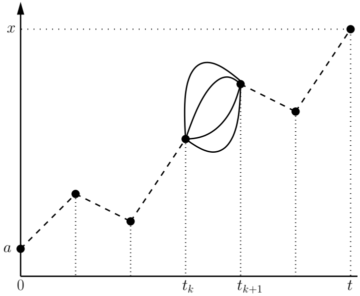
$$L = \frac{1}{2\epsilon} \sum_{i=1}^{n} h_{ik}^2 \left(\dot{x}_k - \left(f_k + \epsilon s \frac{\partial g_{kj}}{\partial x_l} g_{lj} \right) \right)^2 .$$

Decomposition of paths

- Introduce two time scales

$$p(x,t|a,0) \;=\; \int \left(\prod_{k=0}^{M} p(x_{k+1},t_{k+1}|x_k,t_k) \right) dx_1 ... dx_M$$

- Decomposition of paths: $\Gamma \in \mathcal{C}(x,t|a,0)$ can be written as $\Gamma = \sum_{k=0}^{M} \Gamma_k, \qquad \Gamma_k \in \mathcal{C}(x_{k+1},t_{k+1}|x_k,t_k)$.



Path integral hierarchy

- Evolution within on period: 1-period propagator

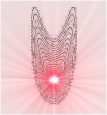
$$p(x_{k+1}, t_{k+1} | x_k, t_k) = \int_{\mathcal{C}(x_{k+1}, t_{k+1} | x_k, t_k)} \mathcal{P}(x(\tau)) \mathcal{D}x(\tau)$$

- Continuum limit:

$$\lim_{M \to \infty,\ \delta \to 0} \left(\prod_{k=1}^{M} \bar{p}(X_{k+1}, t_{1(k+1)} | X_k, t_{1k}) \right) = \bar{\mathcal{P}}(X(t_1)).$$

- Path integral hierarchy

$$p(x, t | a, 0) = \int \left(\lim \prod \int \mathcal{P}(X(\tau_0)) \mathcal{D}X(\tau_0) \right) \mathcal{D}X(\tau_1)$$



Application to toy problem

- 1-period propagator

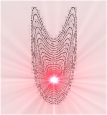
$$\mathcal{P}(x(s)) = \exp\left(-\frac{1}{2\sigma^2}\int_{t_k}^{t_{k+1}}(\dot{x}(s) - d(s)x(s) - \nu g(x(t)))^2 ds\right)$$

- multi-scale expansion: $x(t) = x_0(\tau_0, \tau_1) + \epsilon x_1(\tau_0, \tau_1) + \ldots$
- Scale separation in the Lagrangian:

$$L = \frac{1}{2\sigma^2}\left(\epsilon X'(\tau_1)\mathrm{e}^{R(s)} + \epsilon\left(x_{1s} - d(s)x_1\right) - \nu g\left(X(\tau_1)\mathrm{e}^{R(s)}\right)\right)^2$$

- Use semi-classical method to calculate 1-period propagator

$$\int_{t_k}^{t_{k+1}} L(x_{1c}, \dot{x}_{1c}, s)ds = \frac{1}{2\bar{\sigma}^2}\left(X' - \tilde{g}(X)\right)^2$$



Back to the stochastic wave equation:

- Use a coordinate transform for appropriate short-pulse scaling

$$E(x,t) = A\left(\phi \equiv \frac{t-x}{\epsilon}, x\right)$$

$$-\frac{2}{\epsilon}A_{\phi x} = (\alpha + \nu\xi(x))A - A_{xx} + \frac{1}{\epsilon^2}\chi^{(3)}(A^3)_{\phi\phi}$$

- Use now a multi-scale expansion of the form

$$A(\phi, x) = \epsilon M_0(\phi, x_0, x_1, ...) + \epsilon^2 M_1(\phi, x_0, x_1, ...), \qquad x_n = \epsilon^n x$$

- Leading order implies $M_0 = M_0(\phi, x_1, x_2, ...)$ as

$$-2M_{0\phi x_0} = 0$$

Stochastic Short Pulse Equation

- First nontrivial order:

$$-2M_{1\phi x_0} = 2M_{0\phi x_1} + (\alpha + \nu\xi(x_0))M_0 + \chi^{(3)}(M_0^3)_{\phi\phi}$$

- Solvability condition for M_1:

$$-2M_{0\phi x_1} = (\alpha + \nu\Xi(x_1))\,M_0 + \chi^{(3)}(M_0^3)_{\phi\phi}$$

- Slow noise: (remember $x_1 = \epsilon x$)

$$\Xi(x_1) = \sqrt{\epsilon}\int_0^1 \xi(x)dx$$

- Result: *Stochastic Short Pulse Equation* (Kurt, S., preprint 2011)
- For NLSE: Difficult (need higher order)

Stochastic Hamiltonian Systems

- Hamiltonian Dynamics and Noise:

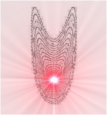
$$\dot{x} = \frac{\partial H}{\partial y} + \sigma \xi_1(t)$$
$$\dot{y} = -\frac{\partial H}{\partial x} + \sigma \xi_2(t)$$

- Conditional Probability Density

$$c(x, y, t) = \mathcal{P}\left((x, y, t) | (x_0, y_0, t_0)\right)$$

- Fokker-Planck Equation

$$c_t + u_1 c_x + u_2 c_y = \kappa \Delta c, \qquad \kappa = \frac{\sigma^2}{2}$$



Advection-Diffusion Equation

- Advection-Diffusion equation with time-dependent velocity field:

$$c_t + (u \cdot \nabla)\, c - \kappa\, \nabla^2\, c = 0.$$

- Specific form of the stream function:

$$\Psi(\xi, t) = \bar{\Psi}(\xi) f(t)$$

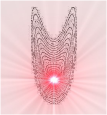
- equations of stream lines reduce to

$$\frac{d\xi}{dF} = \nabla^{\perp} \bar{\Psi}(\xi) \qquad F(t) = \int_0^t f(t')\, dt'$$

- integrability $\rightarrow$ action-angle variables

$$c_t - f(t)\omega(J)c_\theta - \kappa\, (\Gamma : \nabla\nabla + \delta \cdot \nabla)\, c = 0$$

- (S., Poje, Vukadinovic, 2010)



A modulated vortex

- Example: A modulated vortex:

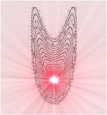
$$\Psi(t,x,y) = \ln\left(\sqrt{a^2+x^2+y^2}\right)\ f(t)$$

- After averaging:

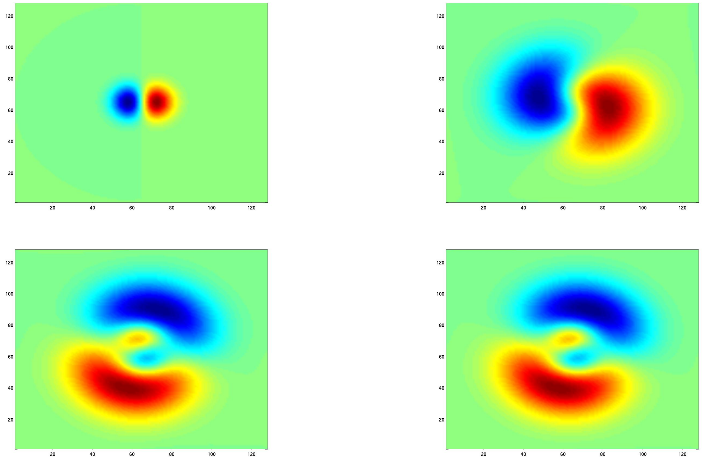
$$V_\tau + (\tilde{u}\cdot\nabla)V = \tilde{K} : \nabla\nabla V$$

- Advection:

$$\begin{aligned}\tilde{u}_1(x,y) &= \langle F\rangle\left(\left(\frac{\omega'}{r}+\omega''\right)+2\frac{\omega'}{r}\right)y + 2\langle F^2\rangle(\omega')^2 x\\ \tilde{u}_2(x,y) &= -\langle F\rangle\left(\left(\frac{\omega'}{r}+\omega''\right)+2\frac{\omega'}{r}\right)x + 2\langle F^2\rangle(\omega')^2 y\end{aligned}$$



Numerical Simulations



Understanding Advection and Diffusion

- Cole-Hopf transform for Burger's equation

$$c_t + c c_x = \kappa c_{xx}$$

- Gaussian initial condtion: $\kappa > 0$: no singularity
- Transformation to heat equation

$$c = -2\kappa \frac{v_x}{v} \quad \rightarrow \quad v_t = \kappa v_{xx}$$

- Transformation is singular

$$v = \mathrm{e}^{-\frac{1}{2\kappa}\int_{-\infty}^{x} c(x',t)dx'}$$

1-d parabolic equations

- Symmetrization of 1-d parabolic equations

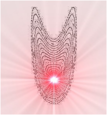
$$c_t + u(x,t)c_x = \kappa c_{xx}$$

- Can we 'remove' the advection term?
- Let's use a simple point transform:

$$\begin{aligned} c(x,t) &= \mathrm{e}^{\phi(x,t)} v(x,t) \\ c_t &= \phi_t \mathrm{e}^{\phi} v + \mathrm{e}^{\phi} v_t \\ c_x &= \phi_x \mathrm{e}^{\phi} v + \mathrm{e}^{\phi} v_x \end{aligned}$$

- Result:

$$\begin{aligned} v_t + (u - 2\kappa\phi_x)v_x &= \left(-\phi_t - u\phi_x + \kappa(\phi_{xx} + \phi_x^2)\right) v \\ &\quad + \kappa v_{xx} \end{aligned}$$



The point transform is singular

- Hence the 'correct' choice for ϕ is simply

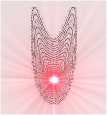
$$\phi_x = \frac{1}{2\kappa} u, \qquad \phi = \frac{1}{2\kappa} \int_{-\infty}^{x} u(x', t) dx'$$

- And the transformation is somewhat similar to Cole-Hopf:

$$v = \mathrm{e}^{-\frac{1}{2\kappa} \int_{-\infty}^{x} u(x',t) dx'} c$$

- Can we do something similar to the Fokker-Planck equation?
- Let's try it out:

$$c_t + 2u_1 c_x + 2u_2 c_y = \kappa \Delta c, \qquad c = \mathrm{e}^{\phi/\kappa} v$$



Point Transform yields...

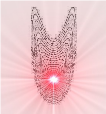
- As equation for v we find

$$
\begin{aligned}
v_t + Bv &= \frac{1}{\kappa} f v + \kappa \Delta v \\
f &= |\nabla \phi|^2 - 2u \cdot \nabla \phi \\
B &= 2(u - \nabla \phi) \cdot \nabla - \Delta \phi
\end{aligned}
$$

- Note: $B^+ = -B$, and $B = 0$ if u has a potential.
- For 'real' fluids we need to do more.
- Idea: Use a Lie transform

$$v = \mathrm{e}^{\kappa L} w, \qquad L^+ = L$$

- As $\kappa \to 0$, we can expand the result in powers of κ.



Baker-Campbell-Hausdorff...

- Transformed equation for w:

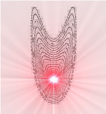
$$\begin{aligned} w_t \quad + \quad & Bw + \kappa[B, L]w = \\ & \left(\frac{1}{\kappa}f + [f, L] + \frac{\kappa}{2}[[f, L], L]\right) w \end{aligned}$$

- Killing skew-symmetric terms:

$$B = [f, L]$$

- What shall we choose for L? There are a lot of choices...
- Maybe a symmetric second-order differential operator?

$$\begin{aligned} L &= c_{11}\partial_x^2 + 2c_{12}\partial_x\partial_y + c_{22}\partial_y^2 + b_1\partial_x + b_2\partial_y \\ b_1 &= c_{11x} + c_{12y} \\ b_2 &= c_{12x} + c_{22y} \end{aligned}$$



Balance Equations

- Result: Conditions for L and ϕ:

$$
\begin{aligned}
c_{11} f_x + c_{12} f_y &= \phi_x - u_1 \\
c_{12} f_x + c_{22} f_y &= \phi_y - u_2
\end{aligned}
$$

- Evolution equation for w:

$$
\begin{aligned}
w_t &= \kappa A w \\
A &= -\frac{1}{2}[B, L] + \Delta + \frac{1}{\kappa^2} f
\end{aligned}
$$

- (S., Poje, Vukadinovic, preprint)

Summary & Acknowledgment

- NLSE and SPE approximate Maxwell's equations over a wide range. 'Overlap' when incorporating higher order terms?
- Non-local solitons as alternative approach for engineered susceptibilites.
- Variety of methods to coarse-grain noise in multi-scale systems.
- Stochastic SPE has been derived, stochastic NLSE requires higher-order expansion.
- New idea to use a combination of a point transform and a Lie transform to transform the Fokker-Planck to a new, symmetric equation.

- We gratefully acknowledge support from NSF under the grant DMS-0807396.