Tb Theorems on Upper Doubling Spaces

Henri Martikainen

Department of Mathematics and Statistics
University of Helsinki

July 27, 2011

Conference in Harmonic Analysis and Partial Differential Equations
in honour of Eric Sawyer
Fields Institute

Framework 1/3

Definition (Geometrically doubling metric spaces)

A metric space (X, d) is geometrically doubling if every ball $B(x, r) = \{y \in X : d(x, y) < r\}$ can be covered by at most N balls of radius $r/2$.

Definition (Upper doubling measures)

Consider a function $\lambda\colon X \times (0, \infty) \to (0, \infty)$ such that $r \mapsto \lambda(x, r)$ is non-decreasing and $\lambda(x, 2r) \leq C_\lambda \lambda(x, r)$. We say that a Borel measure μ on X is upper doubling with a dominating function λ, if $\mu(B(x, r)) \leq \lambda(x, r)$ for all $x \in X$ and $r > 0$.

Framework 2/3

Kernel estimates are tied to the given choice of λ.

Definition (Standard kernels)

Define $\Delta = \{(x,x) : x \in X\}$. A standard kernel is a mapping $K \colon X^2 \setminus \Delta \to \mathbb{C}$ for which we have for some $\alpha > 0$ and $C < \infty$ that

$$|K(x,y)| \le C \min\Big(\frac{1}{\lambda(x,d(x,y))}, \frac{1}{\lambda(y,d(x,y))}\Big), \qquad x \ne y,$$

$$|K(x,y) - K(x',y)| \le C \frac{d(x,x')^\alpha}{d(x,y)^\alpha \lambda(x,d(x,y))}, \qquad d(x,y) \ge 2d(x,x'),$$

and

$$|K(x,y) - K(x,y')| \le C \frac{d(y,y')^\alpha}{d(x,y)^\alpha \lambda(y,d(x,y))}, \qquad d(x,y) \ge 2d(y,y').$$

The smallest admissible C will be denoted by $\|K\|_{CZ_\alpha}$.

Framework 3/3

Definition (Accretivity)

A function b is called accretive if $\operatorname{Re} b \geq a > 0$ almost everywhere.

Definition (Weak boundedness property)

An operator T is said to satisfy the weak boundedness property if $|\langle T\chi_B, \chi_B\rangle| \leq A\mu(\Lambda B)$ for all balls B and for some fixed constants $A > 0$ and $\Lambda > 1$. The smallest admissible constant above is $\|T\|_{WBP_\Lambda}$.

Definition (BMO)

We say that $f \in L^1_{\mathrm{loc}}(\mu)$ belongs to $\mathrm{BMO}^p_\kappa(\mu)$, if for any ball $B \subset X$ there exists a constant f_B such that

$$\Big(\int_B |f - f_B|^p \, d\mu \Big)^{1/p} \leq L\mu(\kappa B)^{1/p}.$$

Global *Tb* Theorem on Upper Doubling Spaces

Theorem (Hytönen and Martikainen, J. Geom. Anal., to appear)

Let (X, d) be a geometrically doubling metric space which is equipped with an upper doubling measure μ. Let T be an $L^2(\mu)$-bounded Calderón–Zygmund operator with a standard kernel K, let b_1 and b_2 be two essentially bounded accretive functions, let $\alpha > 0$ and $\kappa, \Lambda > 1$ be constants. Then

$$\|T\| \lesssim \|Tb_1\|_{BMO^2_\kappa(\mu)} + \|T^* b_2\|_{BMO^2_\kappa(\mu)} + \|M_{b_2} T M_{b_1}\|_{WBP_\Lambda} + \|K\|_{CZ_\alpha}.$$

Here $M_b \colon f \mapsto bf$ is the multiplication operator with symbol b.

This is not the most general version (one can deal with quasi-metric spaces, use $BMO^1_\kappa(\mu)$ and use a bit more general notions of accretivity and weak boundedness property among other things). A vector-valued (UMD-valued) and some local versions also exist.

Examples 1/2

(1) The doubling theory is covered by taking $\lambda(x, r) = \mu(B(x, r))$.

(2) Usually in non-homogeneous analysis one assumes the control $\mu(B(x, r)) \leq Cr^m$. In that situation one can choose $\lambda(x, r) = Cr^m$.

(3) Recently Volberg and Wick obtained a characterization of measures μ in the unit ball $\mathbb{B}_{2n}$ of $\mathbb{C}^n$ for which the analytic Besov–Sobolev space $B_2^\sigma(\mathbb{B}_{2n})$ embeds continuously into $L^2(\mu)$. To obtain the characterization, they prove a $T1$ theorem for certain operators they say to be of Bergman type.

Their measures satisfy the upper power bound $\mu(B(x, r)) \leq r^m$, except possibly when $B(x, r) \subseteq H$, where $H = \mathbb{B}_{2n}$. One notes that this means that their measures are actually upper doubling with

$$\mu(B(x, r)) \leq \max(\delta(x)^m, r^m) =: \lambda(x, r),$$

where $\delta(x) = d(x, H^c)$.

Examples 2/2

(3) (continues from previous slide)
The kernel required by Volberg and Wick has the specific form

$$K(x,y) = (1 - \bar{x} \cdot y)^{-m}, \qquad x, y \in \bar{\mathbb{B}}_{2n} \subset \mathbb{C}^n.$$

One can check that the standard estimates of our theory are verified. Our metric $T1$ theorem then applies to give the same characterization as theirs.

(4) Adams and Eiderman recently successfully exploited a very nice trick of F. Nazarov, the philosophy of which is that some operators which are not initially Calderón–Zygmund with respect to the Euclidian metric can be made into such by cooking up a different metric. This means that metric *Tb* theorems can be useful even in $\mathbb{R}^n$ with the usual metric. In particular, Adams and Eiderman obtain by this strategy some results for capacities and Wolff potentials.

Randomization of Metric Dyadic Cubes of M. Christ 1/4

One key element of the proof of the Euclidian non-homogeneous *Tb* theorem by Nazarov, Treil and Volberg is to use random dyadic grids. In $\mathbb{R}^n$ this is done using the translation group. Therefore, the same strategy does not work in a general metric space.

The key is to look at the construction of M. Christ and not to use the cubes as a black box. Indeed, the construction depends on a certain process of selecting points (x_α^k) and building a spesific transitive relation $\leq$ between the pairs (k, α). The dyadic cubes of Christ are then defined by

$$Q_\alpha^k = \bigcup_{(\ell,\beta)\leq(k,\alpha)} B(x_\beta^\ell, \delta^\ell/100), \qquad \delta \leq 1/1000.$$

Randomization of Metric Dyadic Cubes of M. Christ 2/4

It turns out that this selection process may be appropriately randomized in a way that one gets the needed key property: for a fixed point it is unlikely to end up too close to the boundary of some cube.

Indeed, for a fixed $\epsilon > 0$ define $\delta_{Q_\alpha^k} = \{x : d(x, Q_\alpha^k) \leq \epsilon \ell(Q_\alpha^k) \text{ and } d(x, X \setminus Q_\alpha^k) \leq \epsilon \ell(Q_\alpha^k)\}$, where $\ell(Q_\alpha^k) = \delta^k$. We have

Lemma

For some fixed $x \in X$ and $k \in \mathbb{Z}$, there holds

$$\mathbb{P}(x \in \delta_{Q_\alpha^k} \textit{ for some } \alpha) \lesssim \epsilon^\eta$$

for some $\eta > 0$.

Randomization of Metric Dyadic Cubes of M. Christ 3/4

Besides being useful for our proof of the non-homogeneous metric *Tb* theorem, such metric randomization methods of cubes have found other applications. We briefly discuss the connection to the A_2 conjecture.

The A_2 conjecture has many proofs by now. However, all of them use Hytönen's representation of a general Calderón–Zydmund operator T in the weak sense as an average of certain Haar shifts:

$$\langle Tf, g\rangle = C\mathbb{E}_\beta \sum_{(m,n)\in\mathbb{Z}_+^2} 2^{-(m+n)\delta/2}\langle \mathbb{S}^\beta_{m,n} f, g\rangle.$$

Here one takes the average over all dyadic grids $\mathcal{D}^\beta$ and $S^\beta_{m,n}$ are certain Haar shifts related to these grids.

Randomization of Metric Dyadic Cubes of M. Christ 4/4

One detail here is that one usually uses two independent dyadic grids in the proofs of *Tb* theorems. In particular, certain important notions of goodness and badness of cubes are formulated relative to another grid. In the proof of the above representation formula, one uses such notions but always using only one grid.

Nazarov, Reznikov and Volberg have modified our method of randomization to be such that it works with only one grid and is enough to give the above representation theorem in geometrically doubling metric spaces. Using this they have also obtained a metric version of the A_2 theorem.