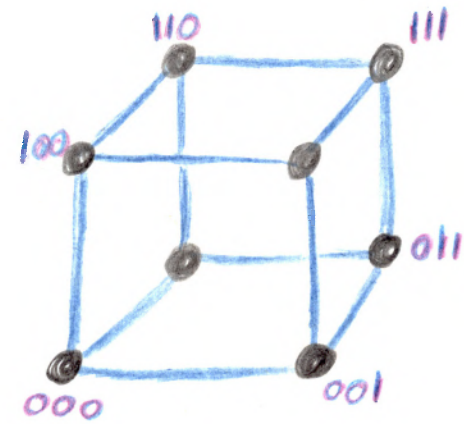
Duality and Exponential Graphs

Claude Tardif
Royal Military College

Prologue: the cube



$$V(Q_n) = \{ f : \mathbb{Z}_n \to \{0,1\} \}$$

$$E(Q_n) = \{ f - g : \exists!\, i : f(i) \neq g(i) \}$$

Thm

$$\boxed{Q_n \text{ is bipartite}}$$

Proof's algorithm:

input: n, $f \in V(Q_n)$

$$S = \sum_{i=1}^{n} f(i)$$

if S is even, output red

else output blue

Other algorithm:

input: $n, f \in V(Q_n)$
output: red

Part I : R_n

by El-Zahar and Sauer

$$V(R_n) = \left\{ \begin{array}{l} f : \mathbb{Z}_{2n+1} \to \{0,1,2\} \text{ such that} \\ \text{① } f(0) \neq f(1) \\ \text{② } \left|\{ i : f(i) \neq f(i+2) \}\right| \text{ is even} \end{array} \right\}$$

$$E(R_n) = \{ f - g : \forall i,\ f(i) \neq g(i+1),\ g(i) \neq f(i+1) \}$$

Thm $\boxed{R_n \text{ is bipartite}}$

Proof: Suppose

$$f_0 - f_1 - f_2 - \cdots f_{2m-1} - f_{2m}$$

Lemma 1: $|\{i : f_i(0) \neq f_{i+2}(0)\}|$ is odd

Lemma 2: $|\{i : \{f_i(0), f_{i+1}(0), f_{i+2}(0)\} = \{0, 1, 2\}\}|$

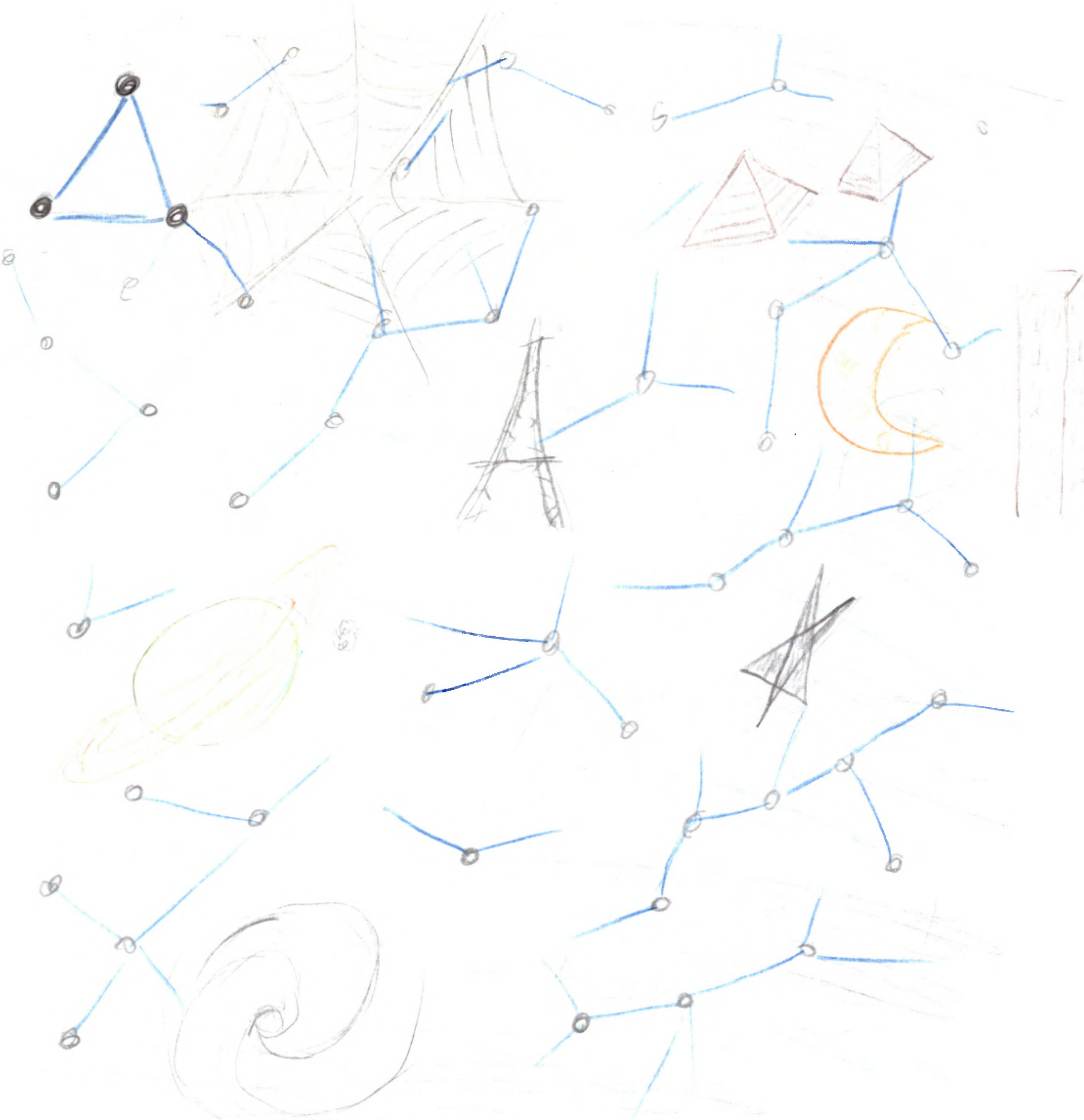
is odd (≥ 1 !)

$$\begin{tikzcd}
f_i(0) \arrow[dr, phantom, "\neq" description] & f_{i+1}(0) \arrow[d, phantom, "\neq" description] & f_{i+2}(0) \arrow[dl, phantom, "\neq" description] & \rightarrow 0, 1, 2 \\
& f_{i+1}(1) & & \rightarrow ?
\end{tikzcd}$$

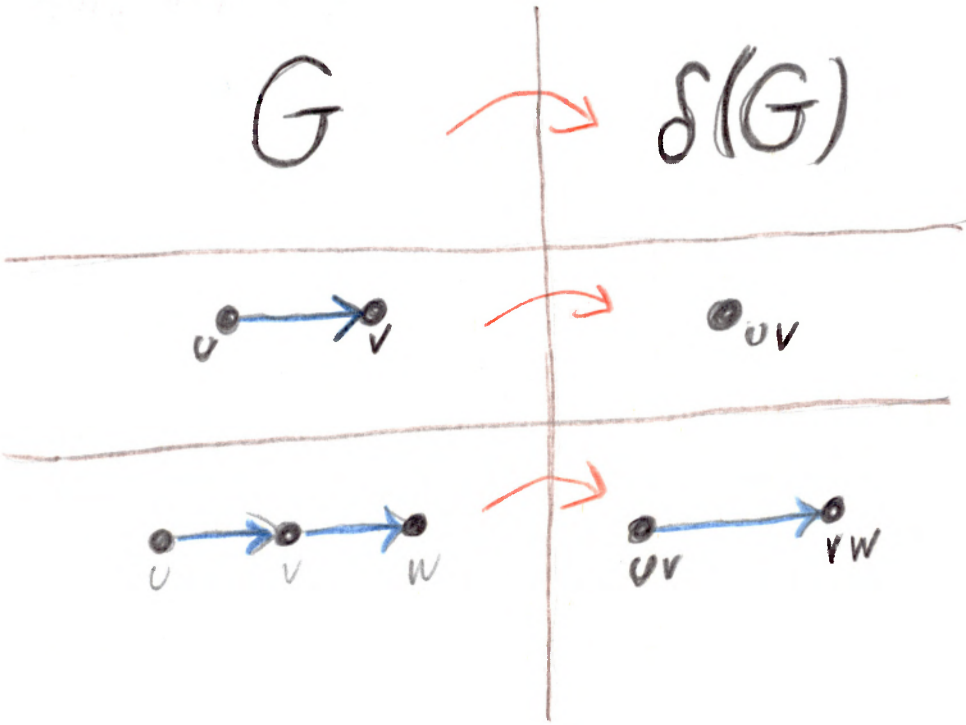
Input: $n, f \in R_n$ → Algorithm → Output: Colour of f

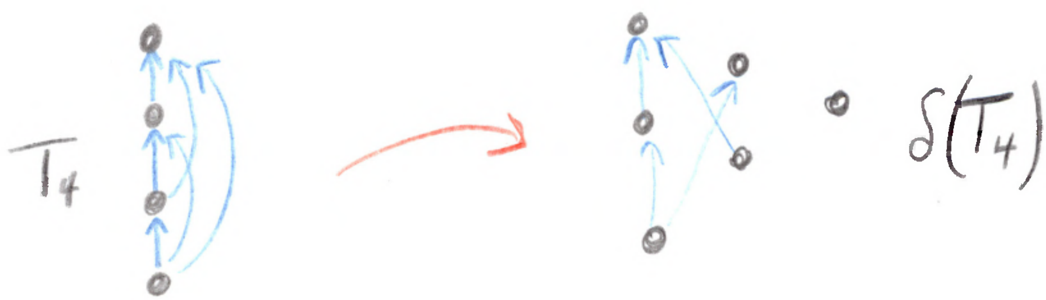
```
g = (0, 1, 0, 0, ..., 0)
while (f not reachable from g)
    g++
if (f reachable from g in R_n^2)
    output red
else
    output blue
```

Part II : $K_3^{K_3^{\delta^k(T_n)}}$



The arc graph $\delta(G)$





$$\chi(\delta(G)) \approx \lg(\chi(G))$$

$$\left(\chi(\delta^k(G)) \approx \lg^{(k)}(\chi(G)) \right)$$

Exponential graphs

K^G : vertices: $f: V(G) \to V(K)$

arcs: $f \to g \iff G: u \to v \Rightarrow K: f(u) \to g(v)$

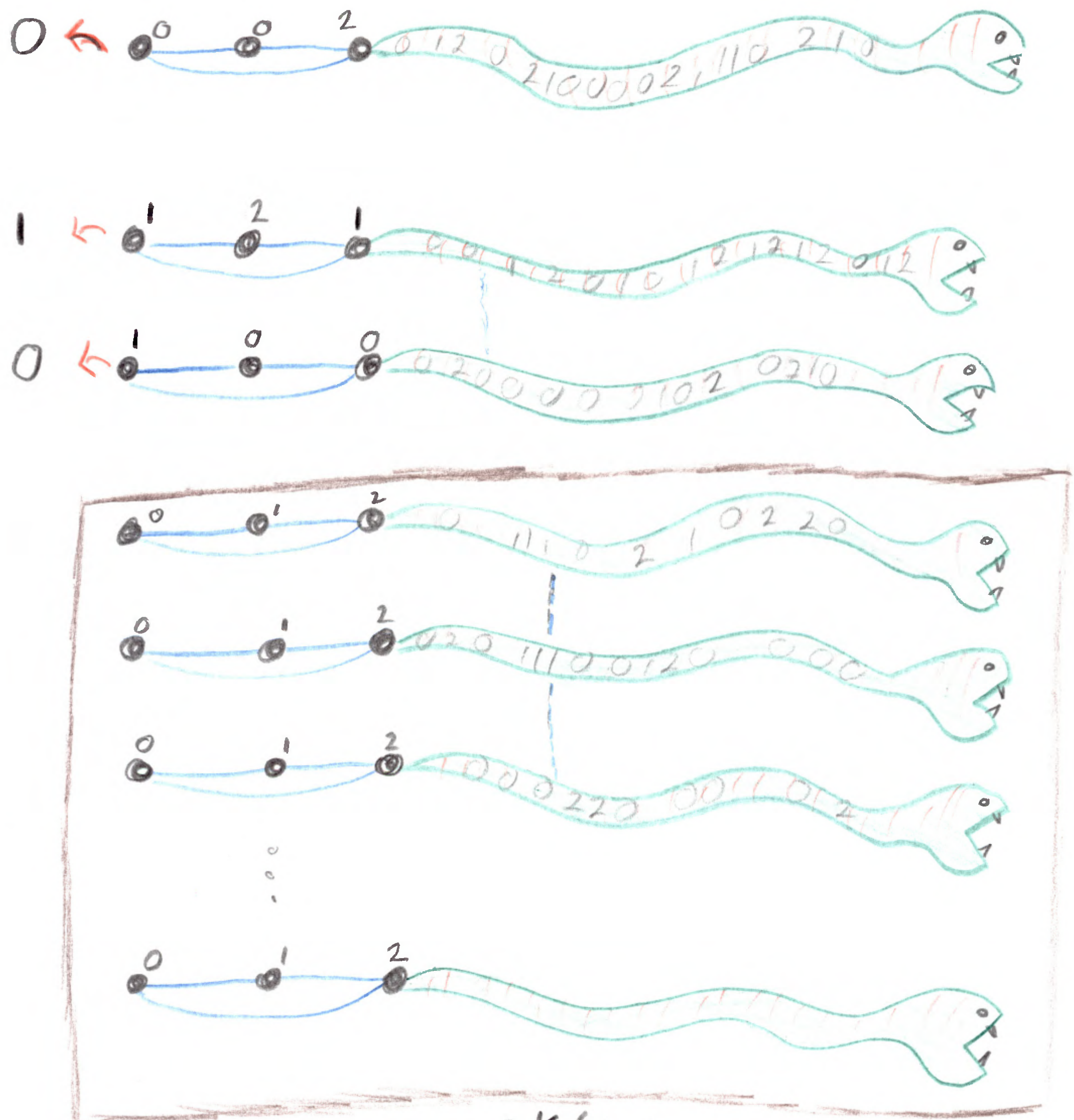
$$\boxed{G \times H \to K \iff H \to K^G}$$

- $G \to K^{K^G}$
- $K^G \leftrightarrow K^{K^{K^G}}$

Thm

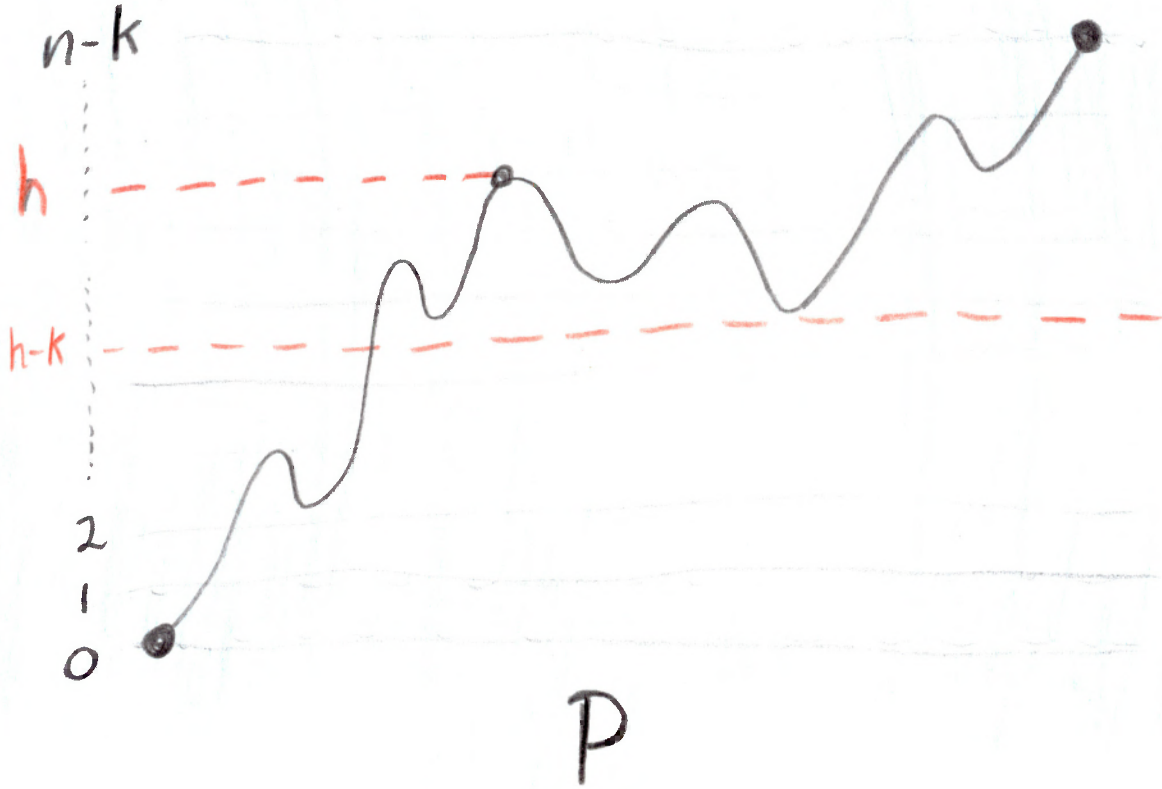
$$\boxed{K_3^{K_3^{\delta^K(T_n)}} \leftrightarrow K_3 \cup \delta^K(T_n)}$$

Components of $K_3^{K_3^{\delta^K(T_n)}}$



$\hookrightarrow \delta^K(T_n)$

Obstructions of $\delta^k(T_n)$

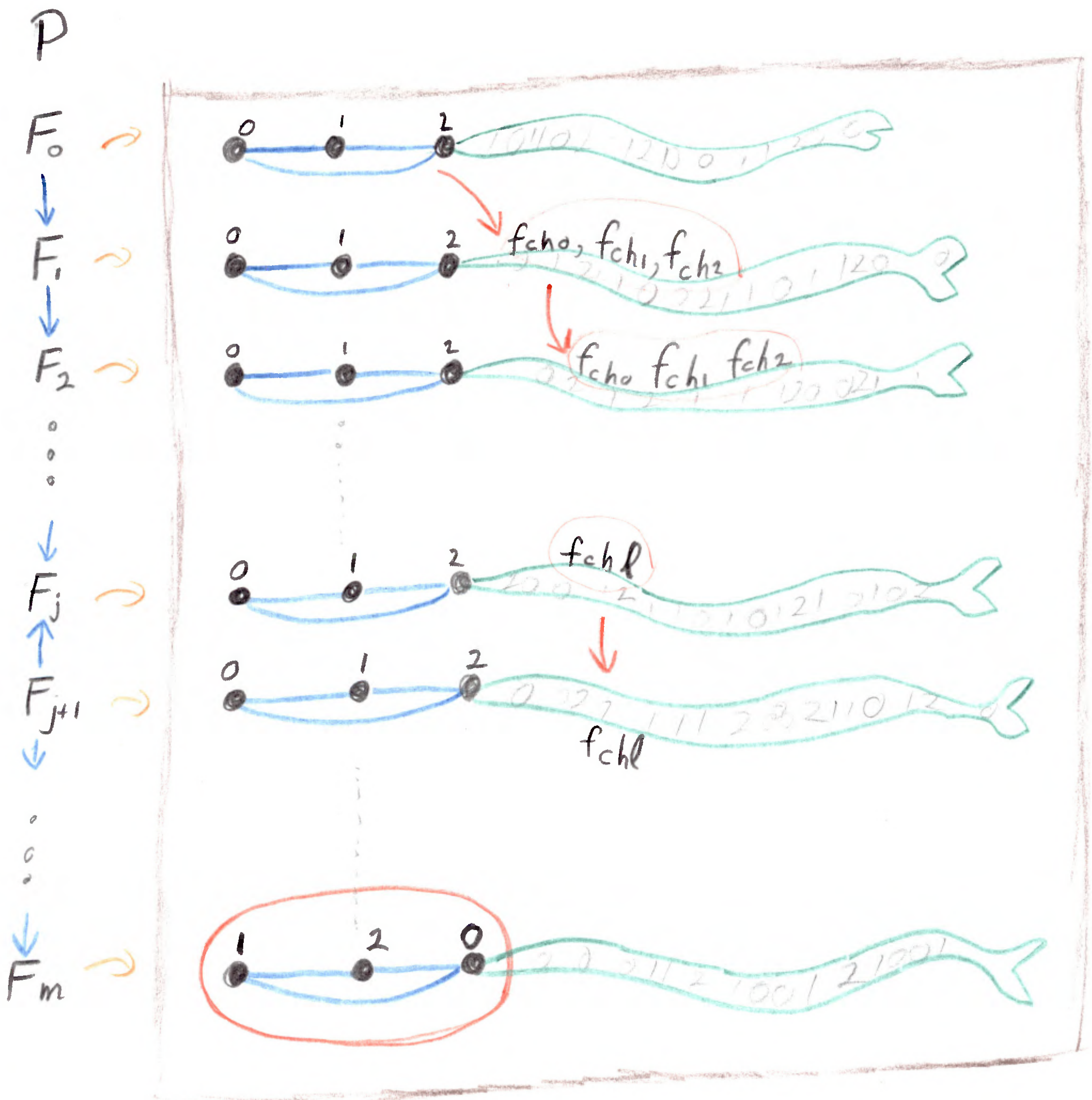


Special vertices of $K_3^{\delta^k(T_n)}$

$$f_{c,h,\ell} \in V\left(K_3^{\delta^k(T_n)}\right)$$

$$f_{ch\ell}(i_0, i_1, \ldots, i_k) = \begin{cases} \ell \text{ if } i_c \leq h \\ \ell+1 \text{ otherwise} \end{cases}$$

Suppose...



Contradiction!