Optimization, Knots and Differential Equations

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Joint work with

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In Memory of Jerry Marsden

1942 – 2010

Three Related Optimization Problems

- minimize $\|x\|$
  subject to constraints

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- x is a chain or cochain
  constraint involves a boundary matrix

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Given finite simplicial complex X (dim $n+1$)

$$\begin{aligned} &\min \|x\|_1 \\ &\text{subject to } x = c + \partial_{n+1} y \end{aligned}$$

where x, c are n-chains.

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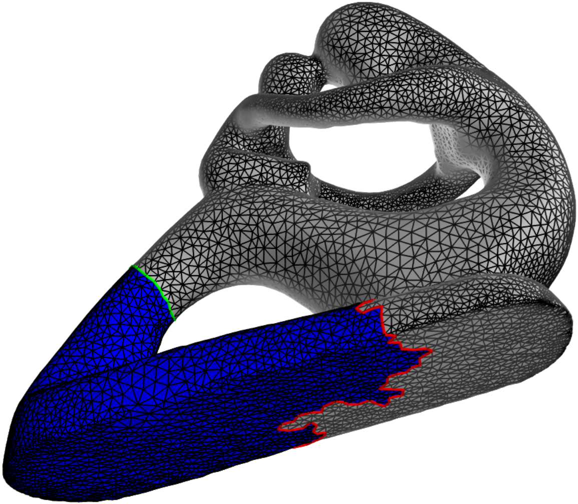
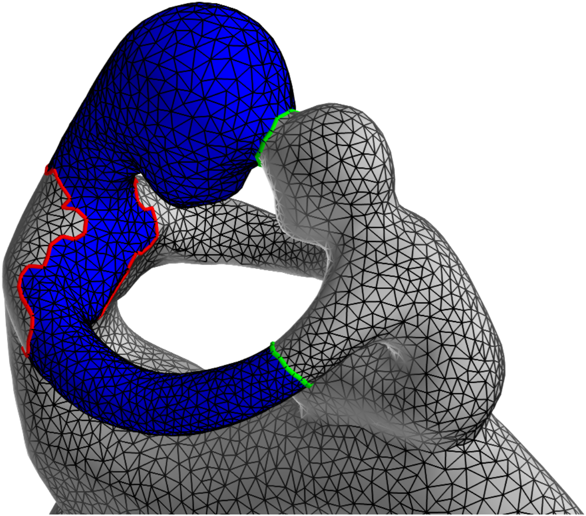
- **NP**-hard if x, y, c are $\mathbb{F}_2$ valued
- If x, y, c are $\mathbb{Z}$ valued ...

  Polynomial time if X is relatively torsion free





Main Result

Theorem (Dey-H.-Krishnamoorthy)

For a finite simplicial complex K of dimension $> n$, the boundary matrix ∂_{n+1} is totally unimodular if and only if $H_n(L, L_0)$ is torsion-free, for all pure subcomplexes L, L_0 in K of dimension $n+1$ and n respectively, where $L_0 \subset L$.

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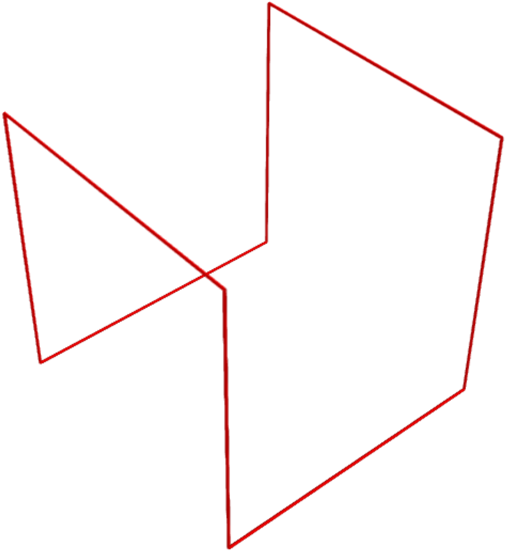
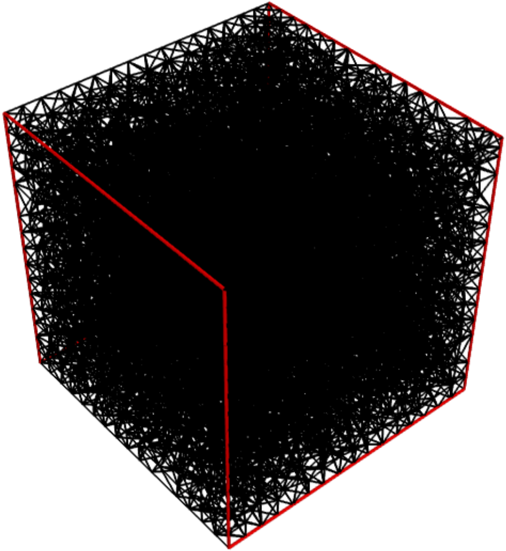
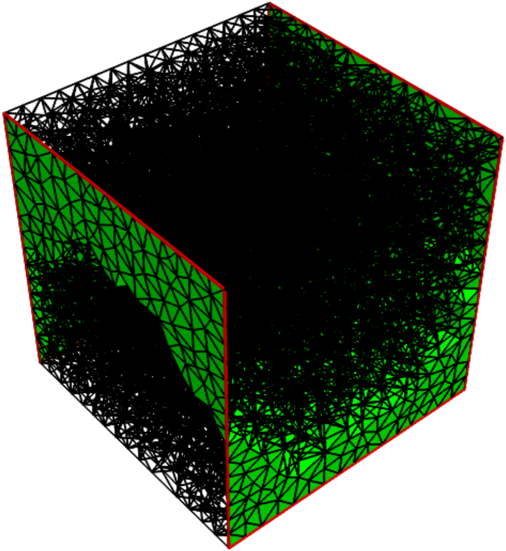
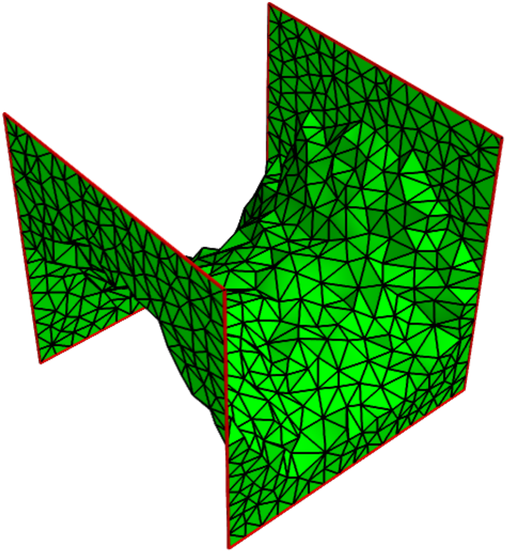
Optimal Bounding Chain Problem (OBCP)

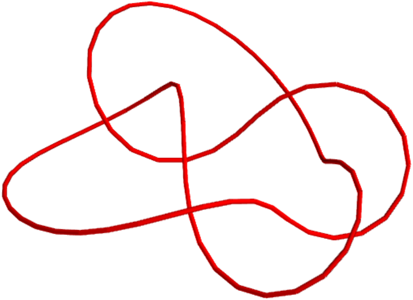
$$\begin{aligned} &\min \ \|x\|_1 \\ &\text{subject to} \ \ \partial\, x = b \end{aligned}$$

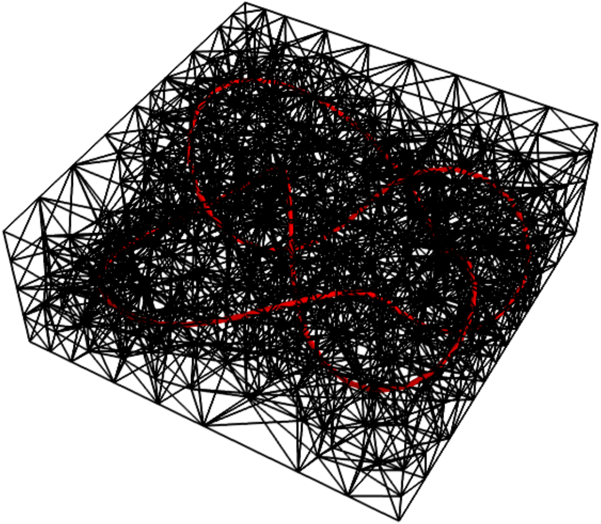


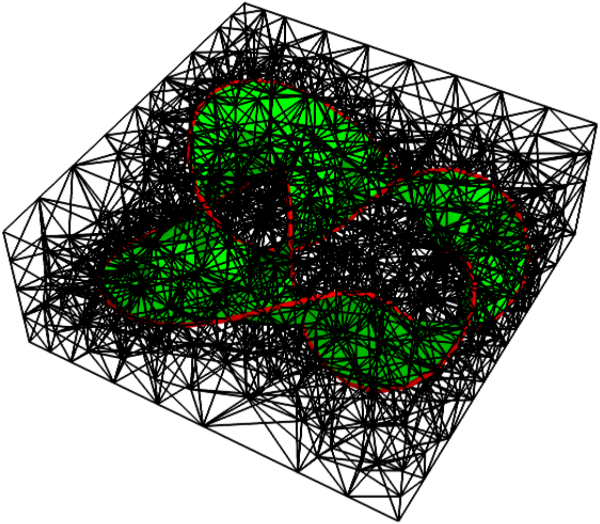


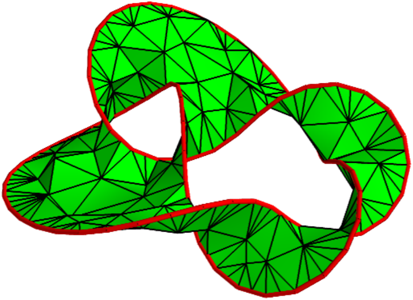








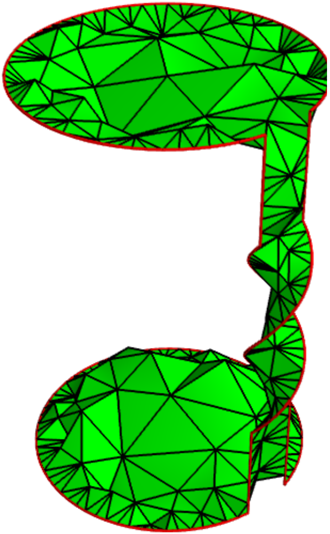


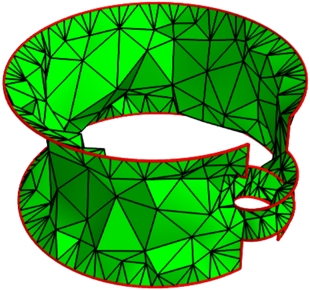


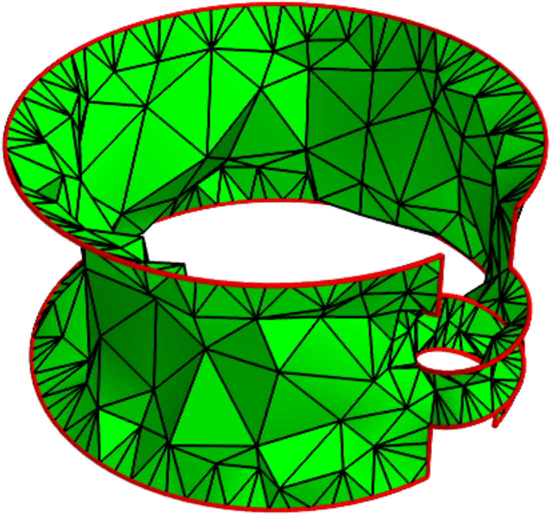
Using above and tools from low-dimensional topology:

Theorem (Dunfield-H.)

Let Y be an orientable manifold and K be a knot embedded in Y. If $H_2(Y; \mathbb{Z}) = 0$, the Least Spanning Area problem for K can be solved in polynomial time.







Optimal Cohomologous Cocycle Problem

Given closed form c:

$$\begin{aligned} &\min \|x\|_2 \\ &\text{subject to } x = c + \mathrm{d}\, y \end{aligned}$$

- Minimizer is a harmonic form.
- There is a unique harmonic form in each cohomology class.

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Given cocycle c:

$$\begin{aligned}&\min \|x\|_2\\&\text{subject to } x = c + \partial^T\, y\end{aligned}$$

- Minimizer is a harmonic cochain.

Hilbert Complexes

[Brüning and Lesch, 1992, Arnold, Falk and Winther, 2010]

- Given a sequence of Hilbert spaces W^k with inner products $\langle \cdot, \cdot \rangle$ and closed densely defined d^k with $\mathrm{d}^{k+1} \circ \mathrm{d}^k = 0$
- (W, d) is called a Hilbert complex
- $V^k \subset W^k$ domain of d^k with inner product

$$\langle u, v \rangle_{V^k} = \langle u, v \rangle_{W^k} + \langle \mathrm{d}^k u, \mathrm{d}^k v \rangle_{W^{k+1}}$$

- (V, d) is a bounded Hilbert complex called domain complex
- Let d^* be the adjoint of d
- $\Delta := \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d}$ is called the Laplace-deRham operator

de Rham Complex

$W^k = L^2\Lambda^k =$ Square integrable forms

$V^k = H\Lambda^k =$ forms in $L^2\Lambda^k$ with weak d in $L^2\Lambda^{k+1}$

de Rham Complex

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$$H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl};\Omega) \xrightarrow{\text{curl}} H(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega)$$

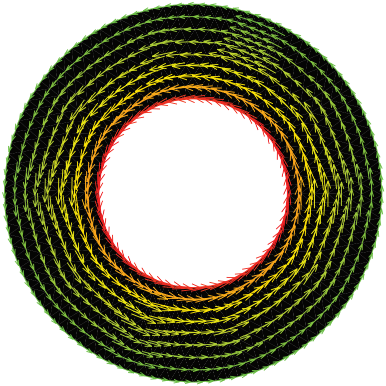
Hodge Decomposition in Finite Dimensions

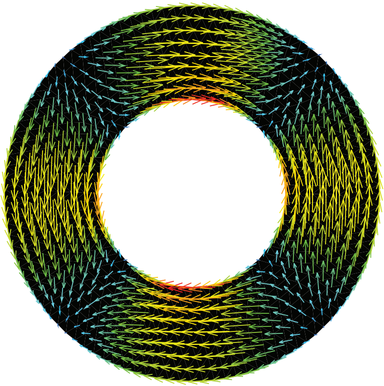
X, Y, Z finite-dimensional inner product spaces

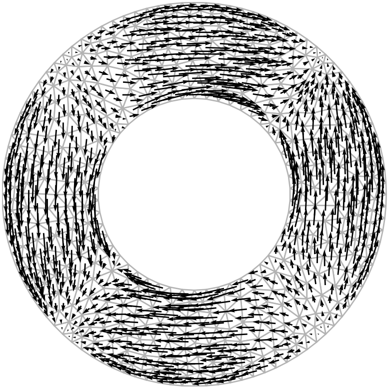
$$X \xrightarrow{A} Y \xrightarrow{B} Z \qquad B \circ A = 0$$
$$X \xleftarrow{A^T} Y \xleftarrow{B^T} Z$$

$$Y = \operatorname{im} A \oplus \operatorname{im} A^\perp = \operatorname{im} A \oplus \ker A^T$$
$$Y = \operatorname{im} A \oplus \operatorname{im} B^T \oplus (\ker B \cap \ker A^T)$$
$$Y = \operatorname{im} A \oplus \operatorname{im} B^T \oplus \ker \Delta$$
$$\Delta = A A^T + B^T B$$

[Arnold, Falk and Winther, 2010]







Poisson's Equation

$$\Delta\, u = f$$

$$\mathrm{d}\,\mathrm{d}^* u + \mathrm{d}^* \mathrm{d}\, u = f$$

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$$f = \mathrm{d}(\mathrm{d}^* u) + \mathrm{d}^*(\mathrm{d}\, u)$$

Not well-posed (missing harmonic part)

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$$\langle \mathrm{d}^* u, \mathrm{d}^* v \rangle + \langle \mathrm{d}\, u, \mathrm{d}\, v \rangle + \langle p, v \rangle = \langle f, v \rangle\,, \quad v \in V^k \cap V_k^*$$

Mixed Formulation

$$\begin{aligned}\Delta\, u &= f - p\\ \mathrm{d}\,\mathrm{d}^* u + \mathrm{d}^*\mathrm{d}\, u &= f - p\end{aligned}$$

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$$\langle u, q \rangle = 0 \quad q \in \mathfrak{H}^k$$

Finite Element Approximation

$$\begin{aligned}
\langle \sigma, \tau \rangle - \langle \mathrm{d}\, \tau, u \rangle &= 0\,, & \tau &\in V^{k-1}\,,\\
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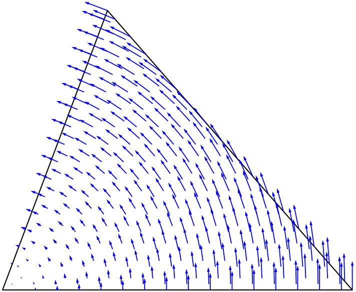
Example of V_h

Whitney Forms

Whitney map: $C^k \to L^2\Lambda^k$
For a triangle:

0-forms Barycentric coordinates

1-forms $\mu_0 \, \mathrm{d} \, \mu_1 - \mu_1 \, \mathrm{d} \, \mu_0$ etc.



Harmonic Cochains

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\end{aligned}
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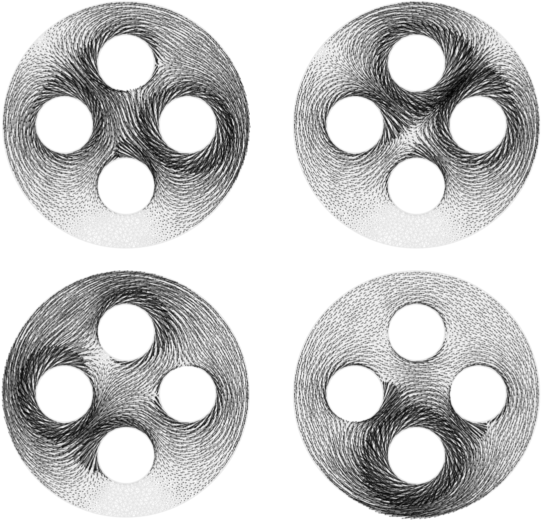
$$
\begin{bmatrix} * & -\mathrm{d}^T * \\ * \mathrm{d} & \mathrm{d}^T * \mathrm{d} \end{bmatrix} \begin{bmatrix} \sigma_h \\ u_h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Harmonic Cochains

Eigenvector Method

Compute eigenvectors of zero eigenvalue



Harmonic Cochains

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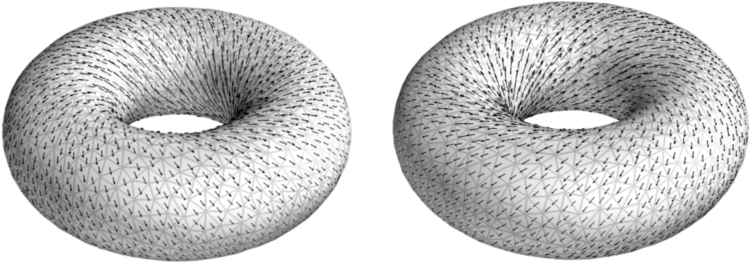


Harmonic Cochains

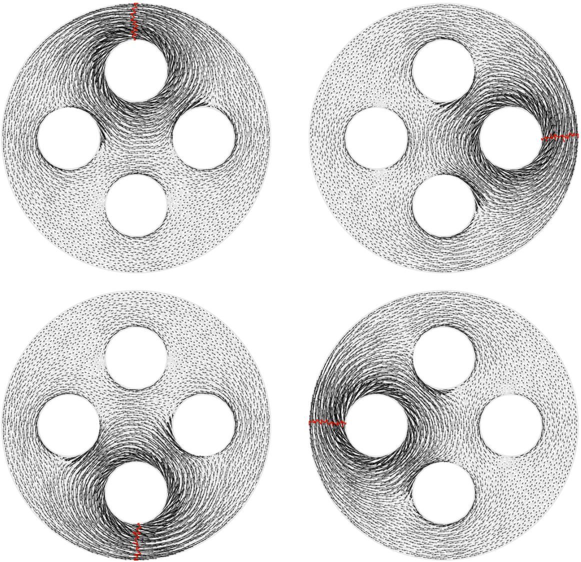
Least Square Methods

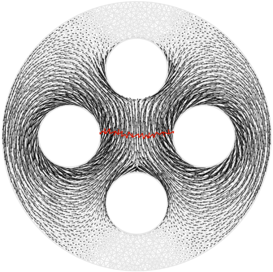
Let ω be a cocycle. Solve for α

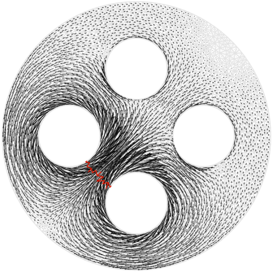
$$\partial * \partial^{\mathsf{T}} \alpha = -\partial * \omega$$

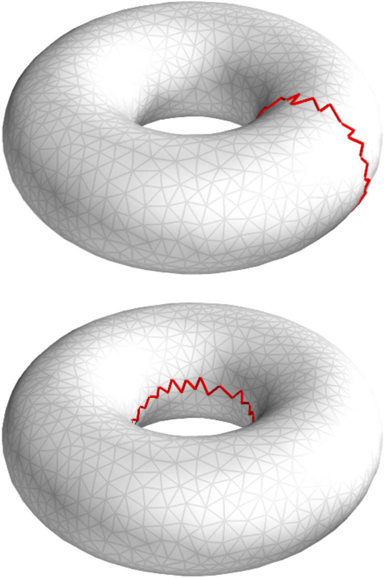
Theorem (Demlow-H.)

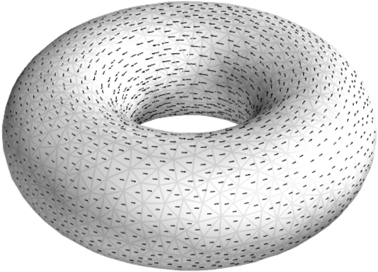
$\omega + \partial^{\mathsf{T}} \alpha$ *solves the finite element equations for Whitney forms.*

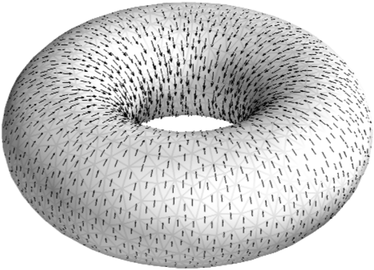












Hodge-deRham Isomorphism

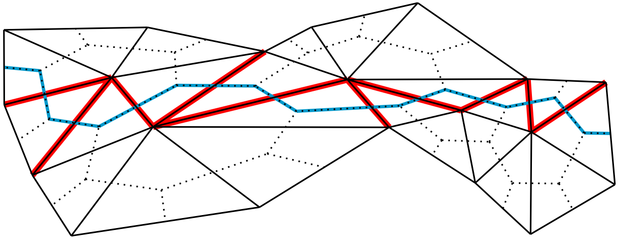
For a boundaryless manifold M: $H^k(M;\mathbb{R}) \cong \mathcal{H}^k(M) = \ker \Delta_k$.
For manifolds M with boundary: $H^k(M;\mathbb{R}) \cong \mathcal{H}^k_N(M)$ and
$H^k(M, \partial M;\mathbb{R}) \cong \mathcal{H}^k_D(M)$.

Harmonic forms : $\ker \Delta$
Harmonic fields : $\mathcal{H}(M) = \ker \mathrm{d} \cap \ker \mathrm{d}^*$
Harmonic Neumann fields : $\mathcal{H}_N(M)$ normal component 0
Harmonic Dirichlet fields : $\mathcal{H}_D(M)$ tangential component 0

$$\mathcal{H}^k_N(M) \cong H^k(M;\mathbb{R}) \cong H_{n-k}(M, \partial M;\mathbb{R})$$
$$\mathcal{H}^k_D(M) \cong H^k(M, \partial M;\mathbb{R}) \cong H_{n-k}(M;\mathbb{R})$$

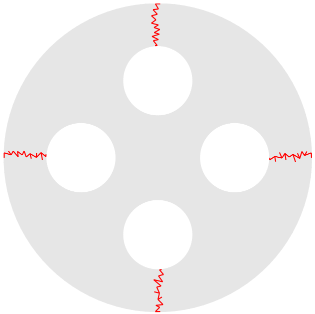
What is Needed From Computational Topology ?

Cocycles

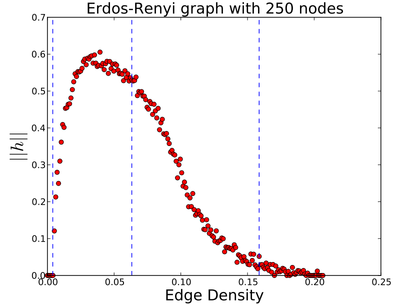


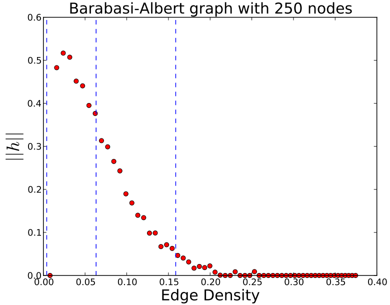
What is Needed From Computational Topology ?

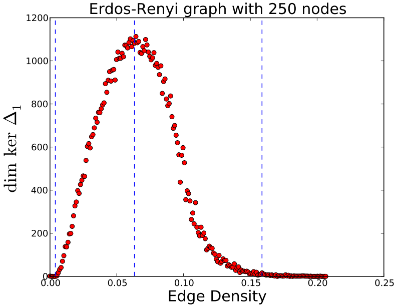
Cocycles

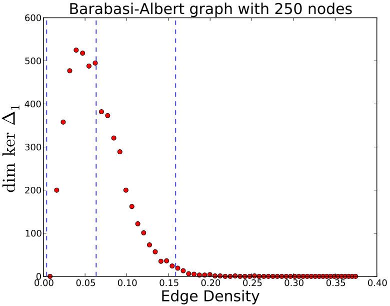


What can be Given to Computational Topology ?









What can be Given to Computational Topology ?

$$\Delta\, u = \lambda u$$

Conclusions

Unifying theme : minimize norm subject to constraints involving boundary or coboundary

min $\|x\|_1$	min $\|x\|_1$	min $\|x\|_2$
subject to	subject to	subject to
$x = c + \partial_{n+1}\, y$	$\partial_n\, x = b$	$x = c + \partial_k^T\, y$

- 1-norm and $\mathbb{Z}$ yields results in computational topology
- 2-norm and $\mathbb{R}$ yields results in numerical PDEs
- On graphs these themes yield ranking algorithms
- Simple application of computational topology to PDEs

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