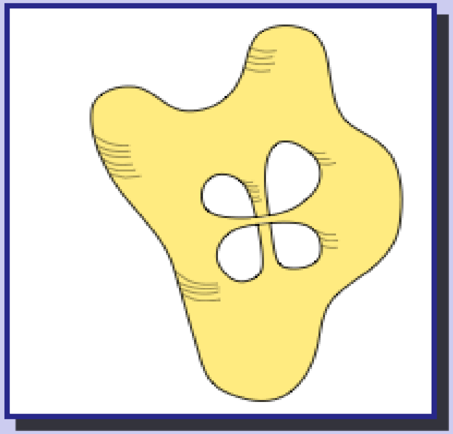
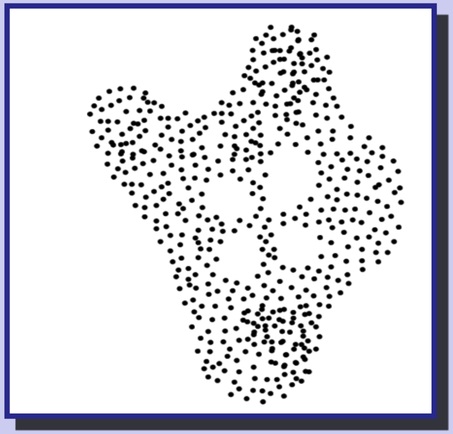
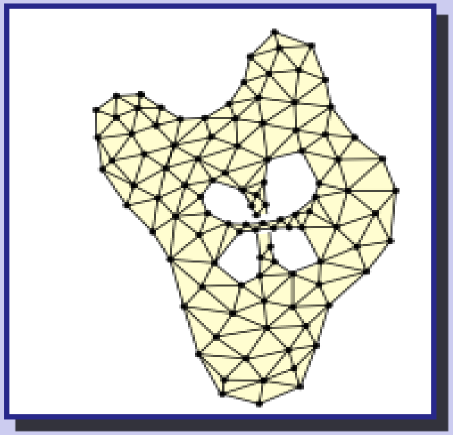
Vietoris-Rips complexes also provide topologically correct reconstructions of sampled shapes

Dominique Attali, Andre Lieutier and David Salinas

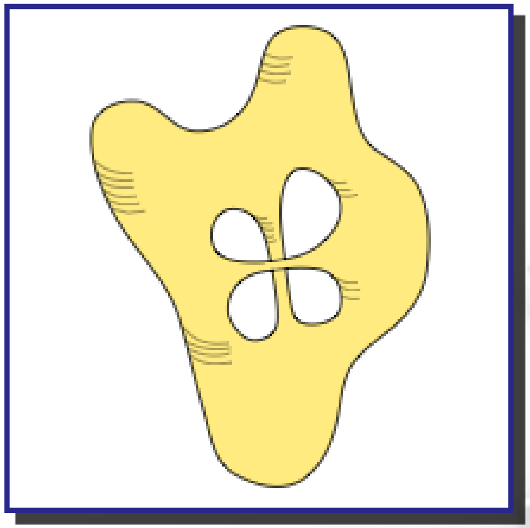






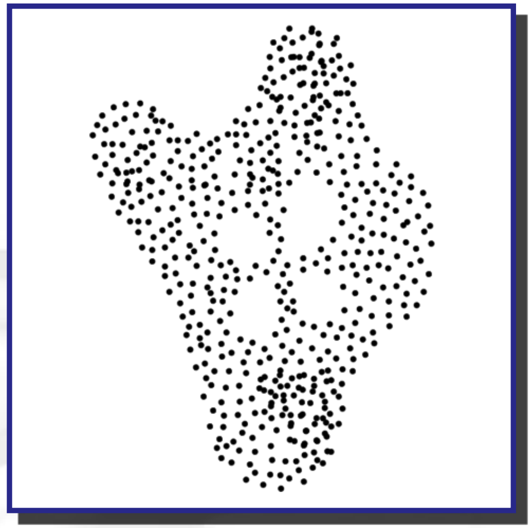
Shape Reconstruction

UNKNOWN



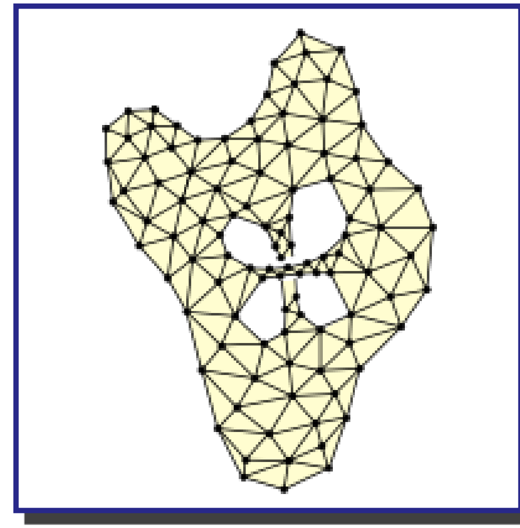
Shape

INPUT



Sample

OUTPUT



Approximation
of the shape

- geometrically accurate
- topologically correct

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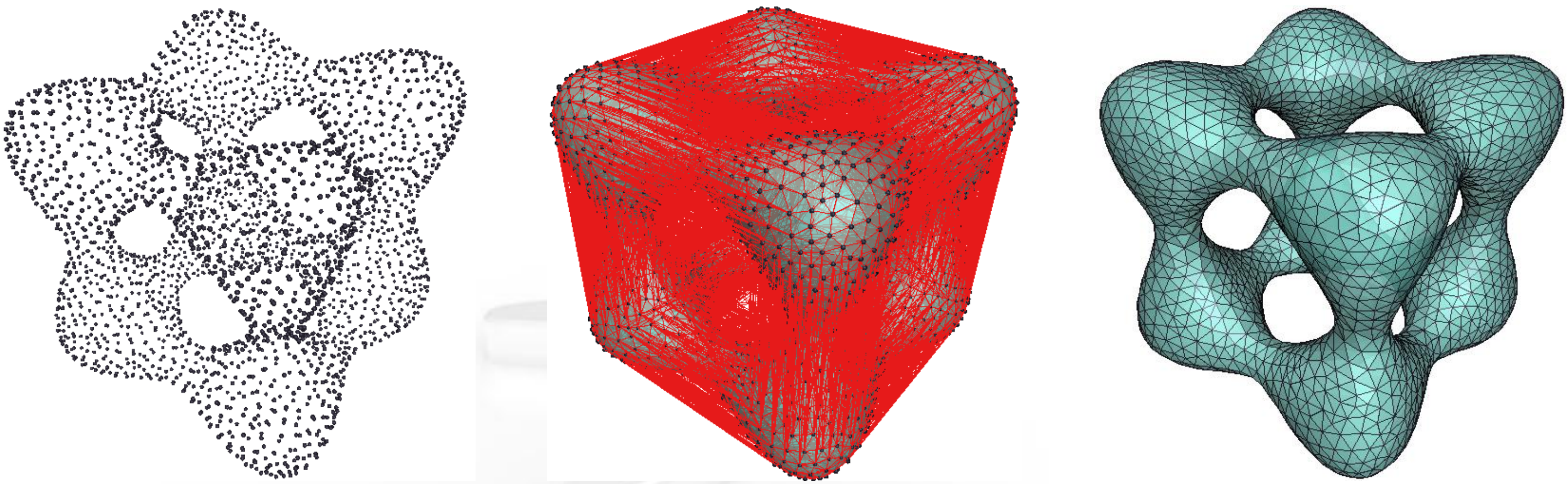
Algorithms in 3D



Delaunay triangulation

select a subset of the Delaunay triangulation

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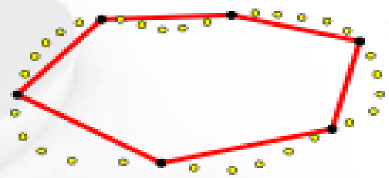
Algorithms in High Dimemsions

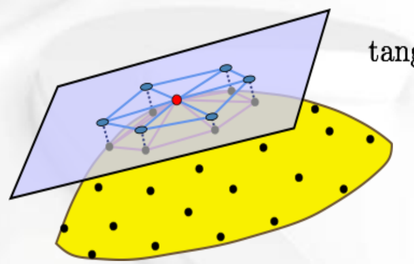
How can we reconstruct without having to build the whole Delaunay triangulation ?

- weak Delaunay triangulation
  [de Silva 2008]



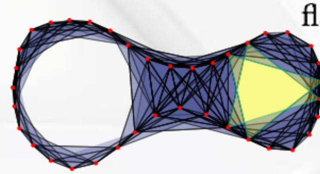
landmarks

- tangential Delaunay complexes
  [Boissonnat & Ghosh 2010]



tangent plane

- Rips complexes
  [Attali, Lieutier & Salinas 2011]



flag complexes

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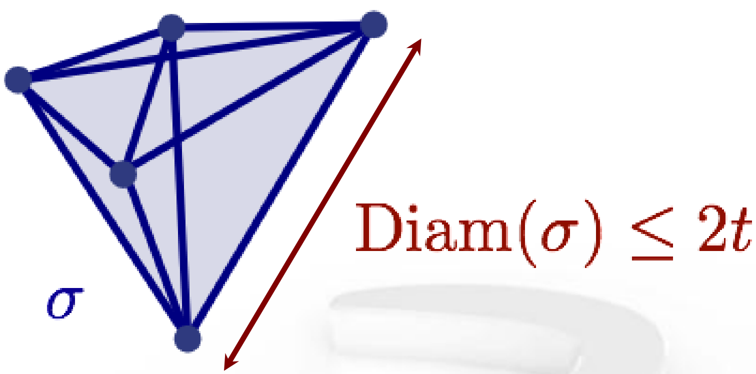
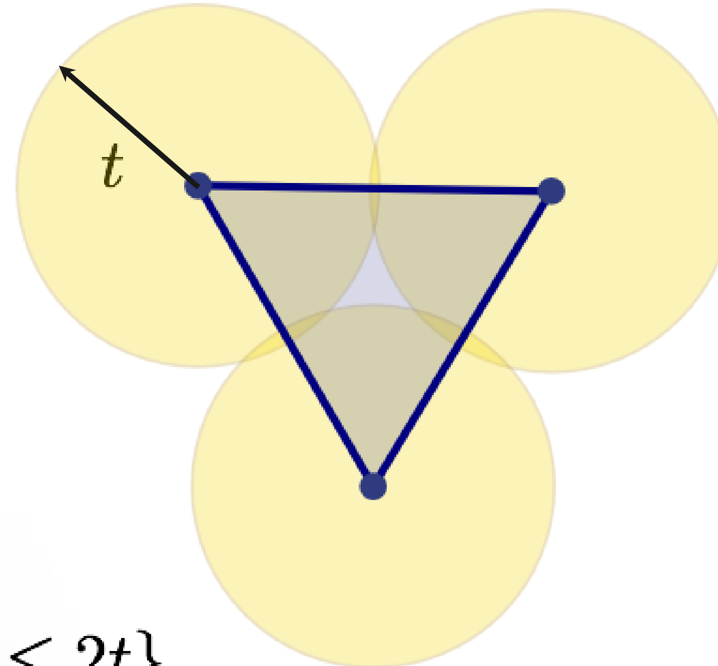
Rips Complex (or Vietoris-Rips complex)

P finite set of points





$$\mathcal{R}(P,t) = \{\sigma \mid \emptyset \neq \sigma \subset P, \mathrm{Diam}(\sigma) \leq 2t\}$$

Rips complexes are **flag** (or **clique**) **complexes**, they can therefore be represented by their 1-skeleton, which is a **graph**:

This allows a parcimonious representation even in high ambient dimension

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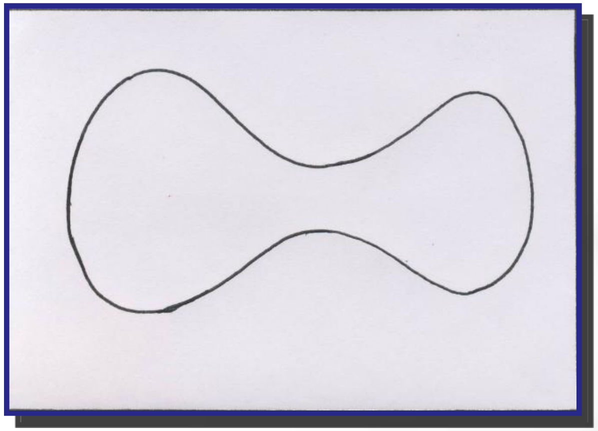
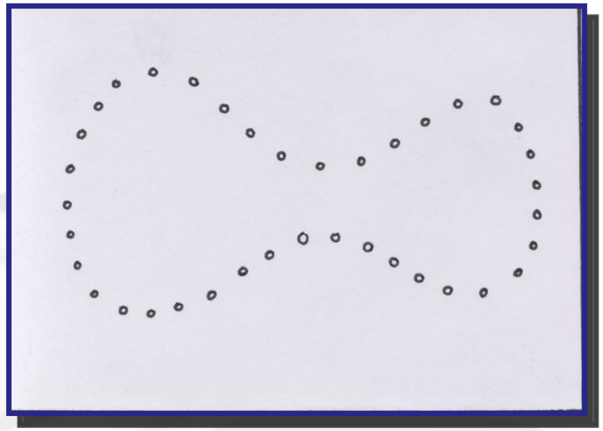
A Simple Algorithm

UNKNOWN



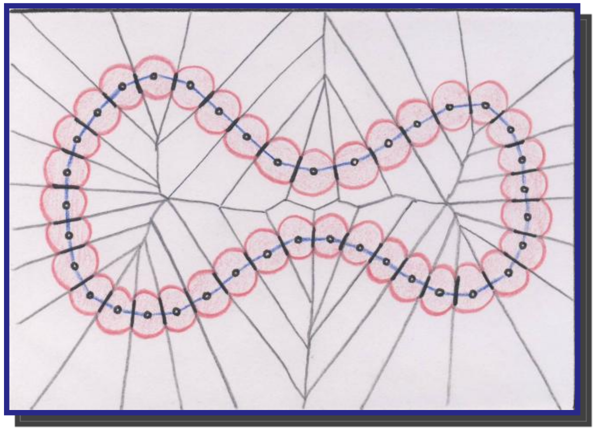
Shape

INPUT



Sample

OUTPUT



α-offset = union of balls with radius α centered on the sample

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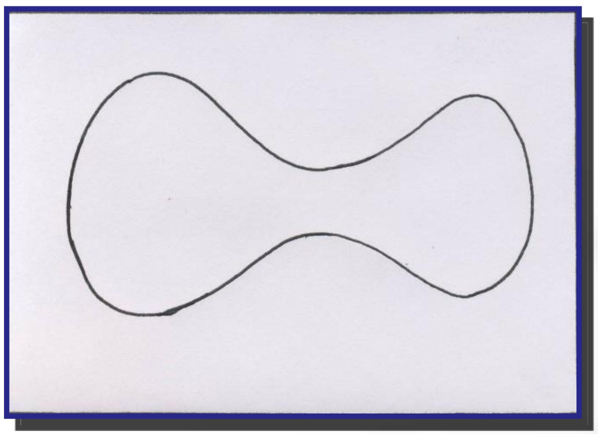
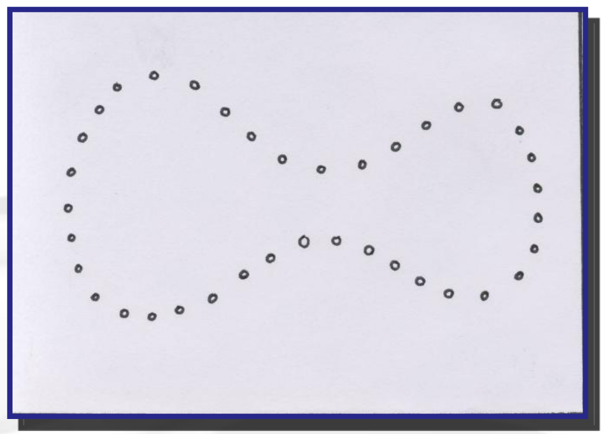
A Simple Algorithm

UNKNOWN



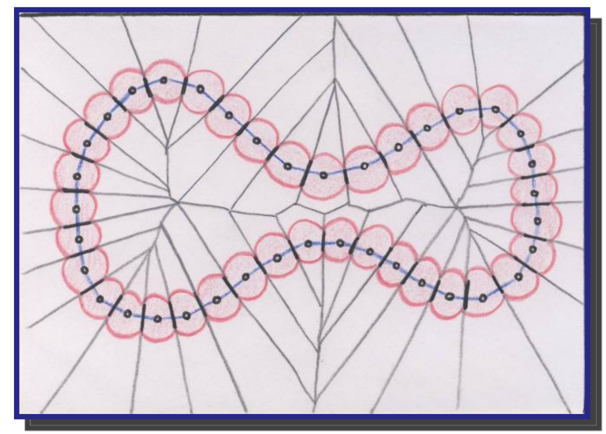
Shape

INPUT



Sample

OUTPUT



From Nerve Theorem:

α-offset $\simeq$ α-complex

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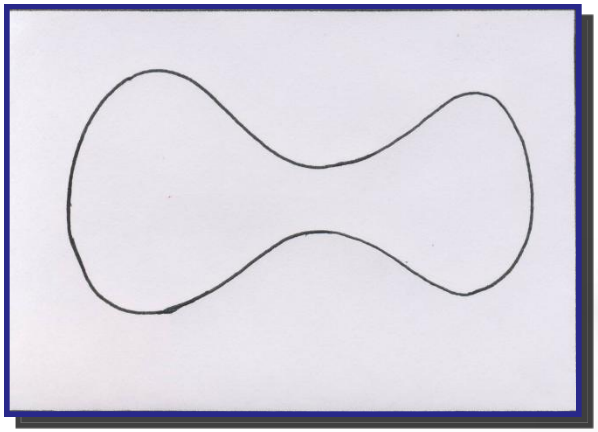
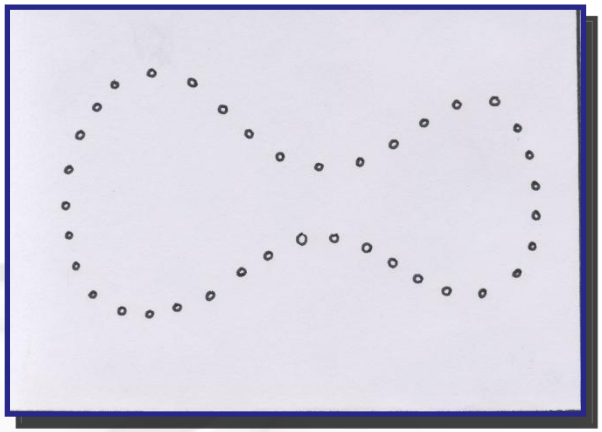
A Simple Algorithm

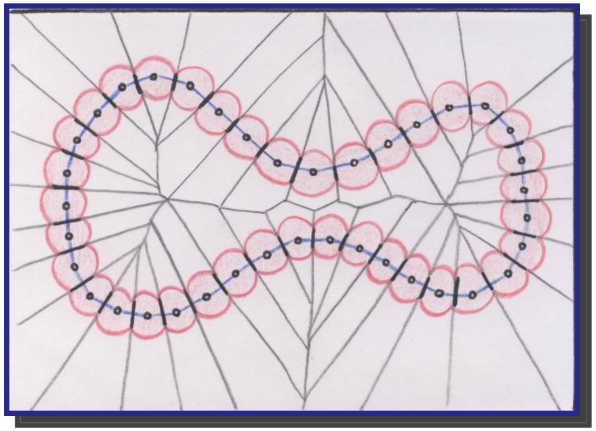
Shape

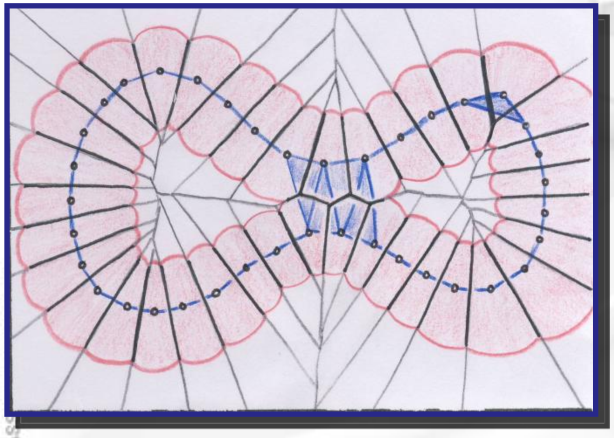


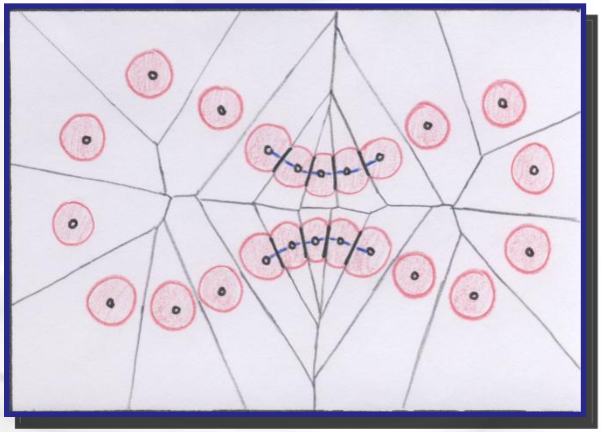
Sample

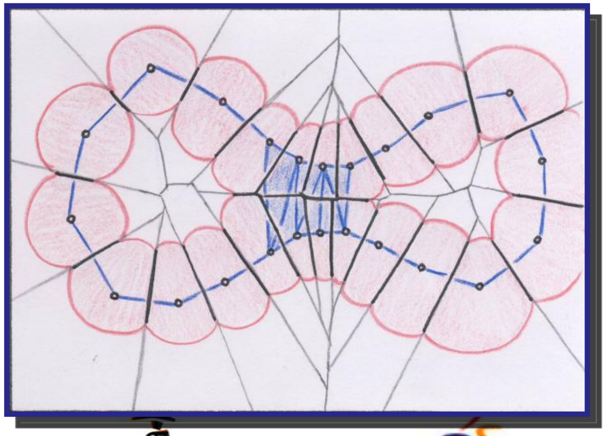


OUTPUT

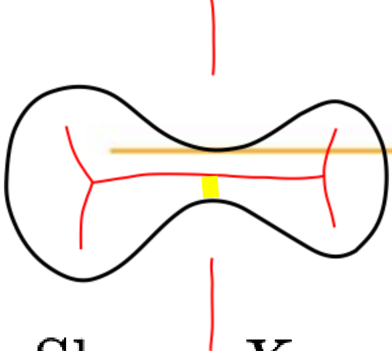


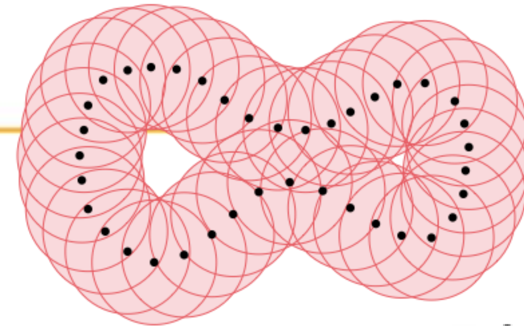


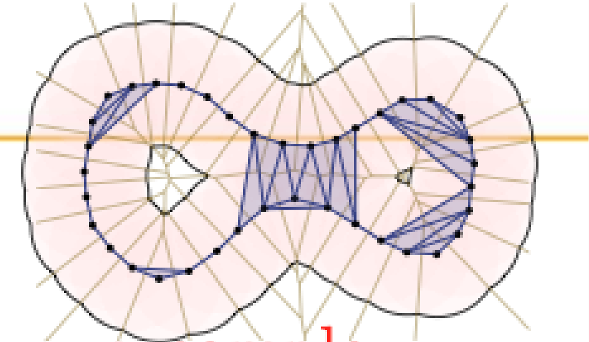




Shape X







$P^\alpha \simeq$ α-complex

Reconstruction theorem

Sampling conditions

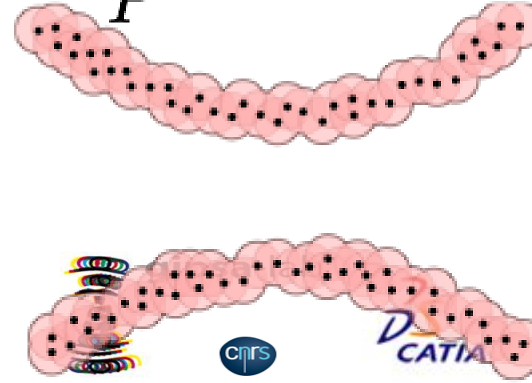
[Niyogi Smale Weinberger 2004]

$d_H(X,P) \leq (3-\sqrt{8})\,\mathrm{Reach}(X)$

$\alpha = (2+\sqrt{2})d_H(X,P)$

Similar result:
Chazal Lieutier 2006

X $\simeq$ P P^α



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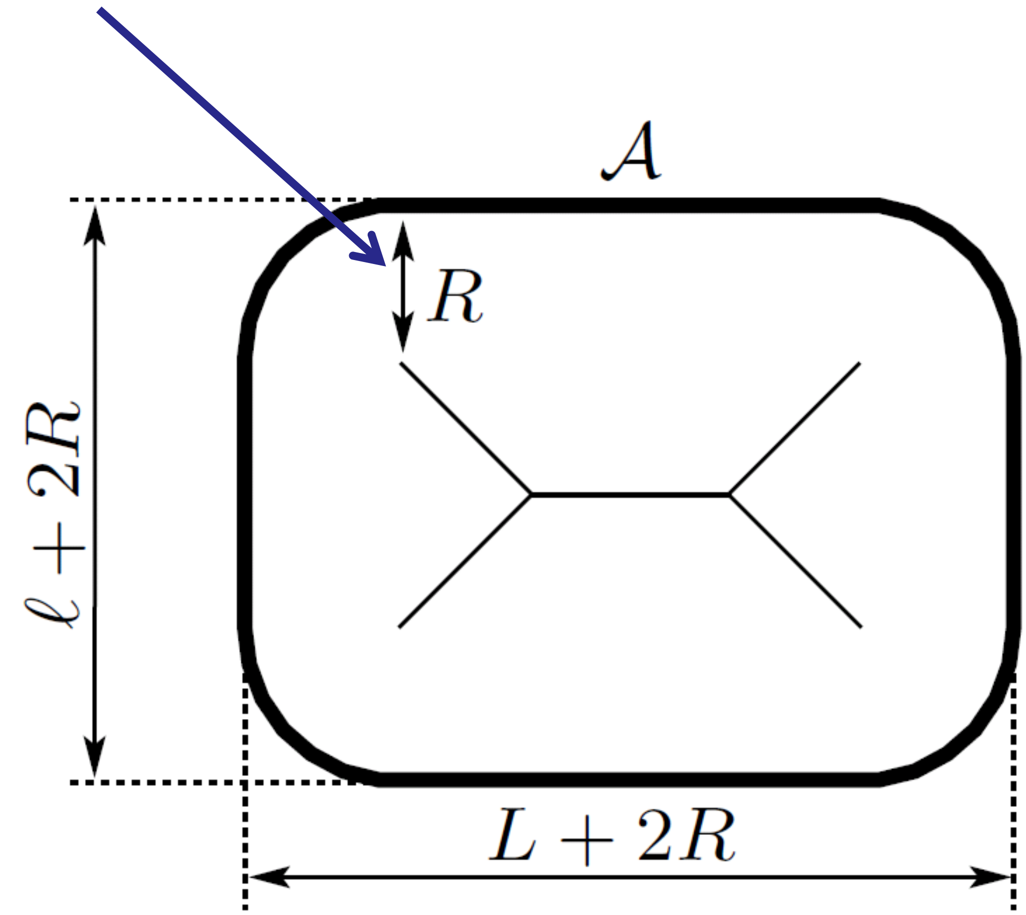
$\mathrm{Reach}(\mathcal{A})$



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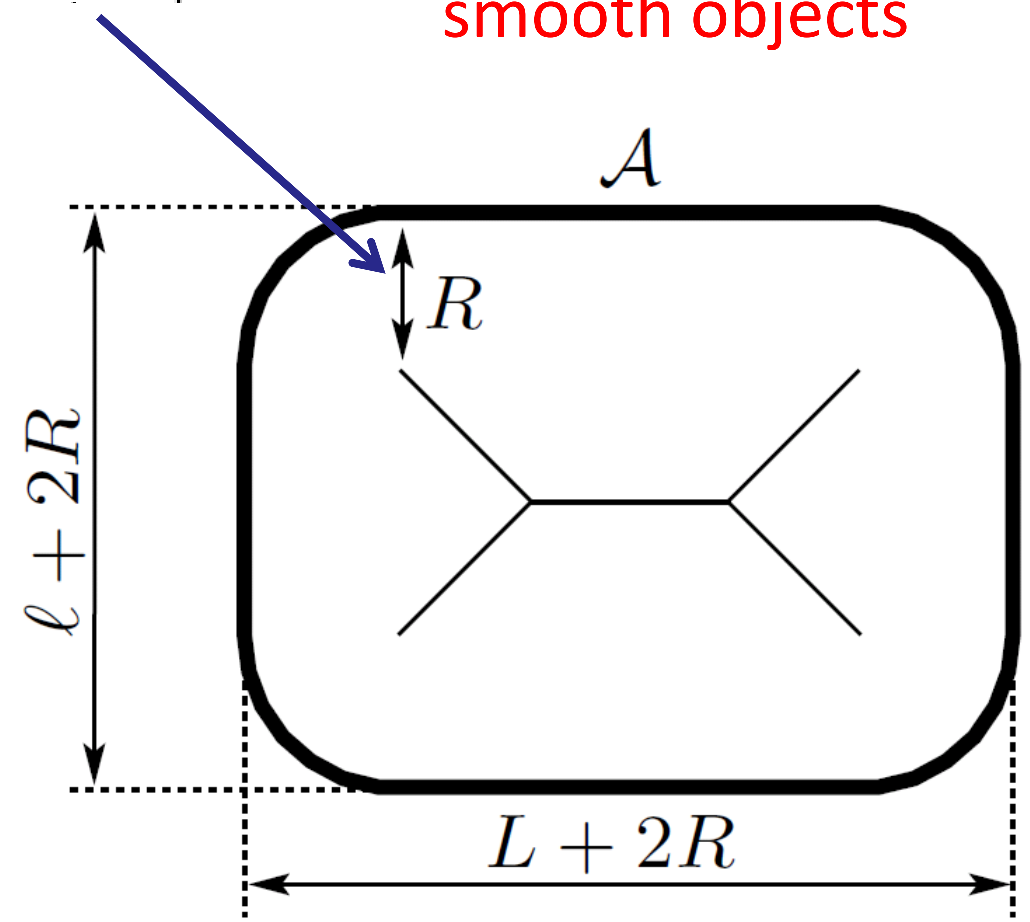
$\text{Reach}(\mathcal{A})$

But reach vanishes on non smooth objects



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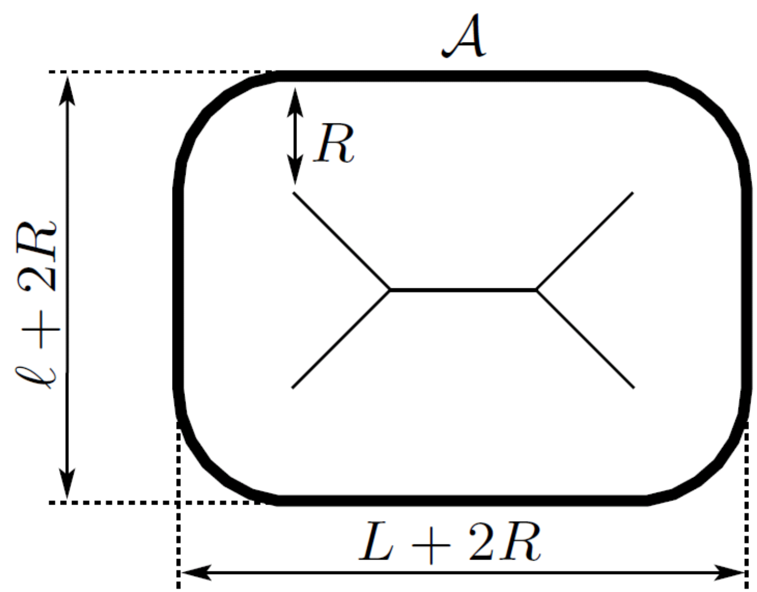
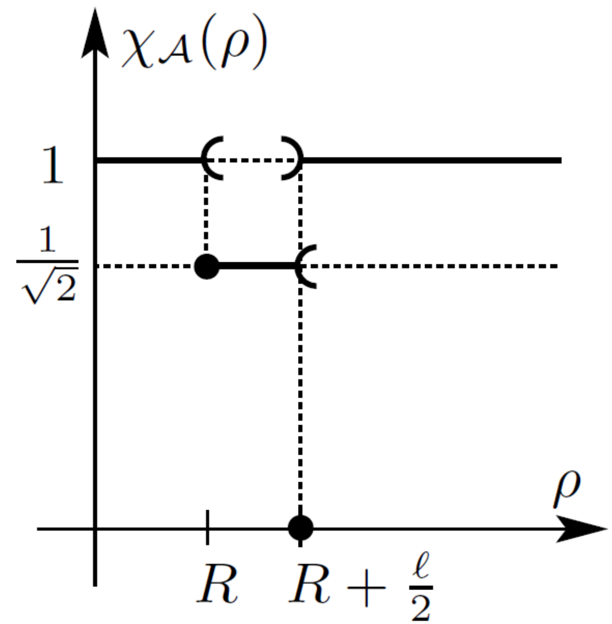
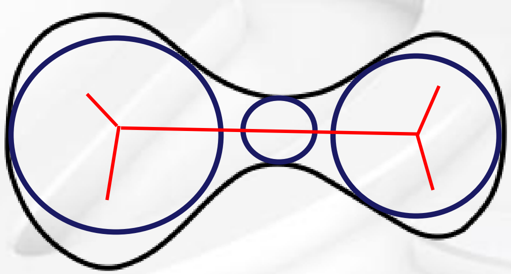
Beyond the reach : WFS and μ-reach





Critical function



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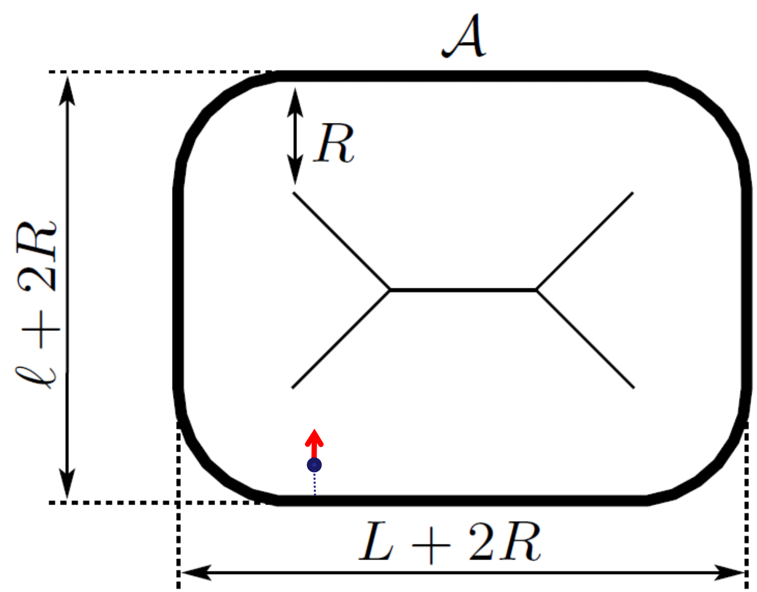
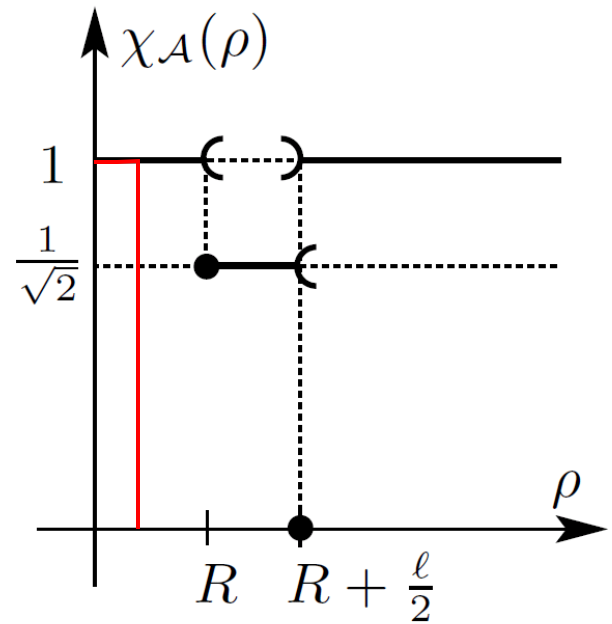
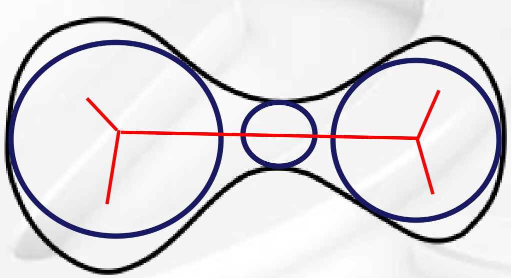
Beyond the reach : WFS and μ-reach





Critical function



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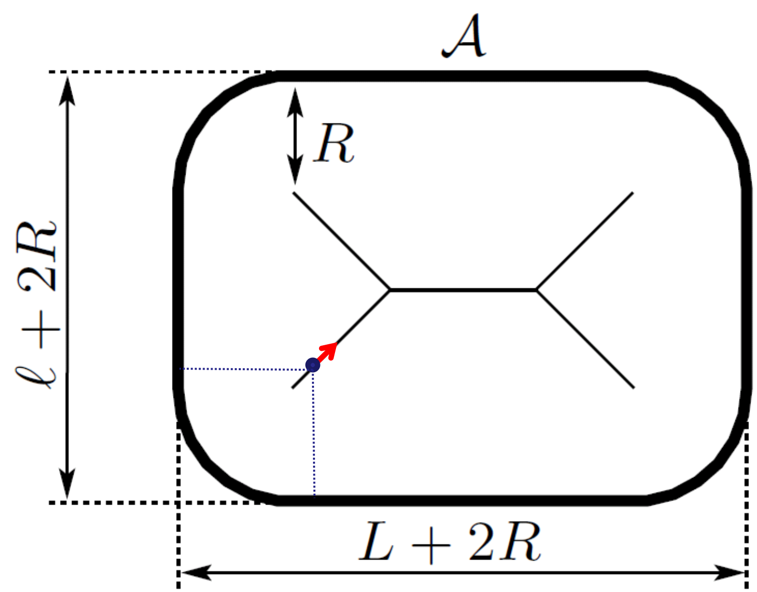
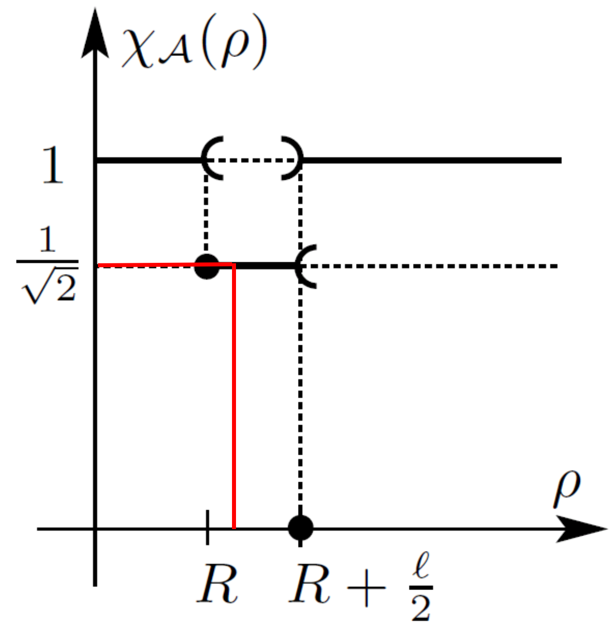
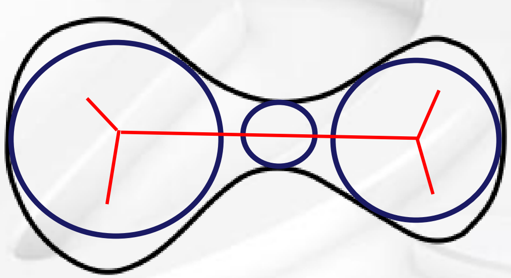
Beyond the reach : WFS and μ-reach





Critical function



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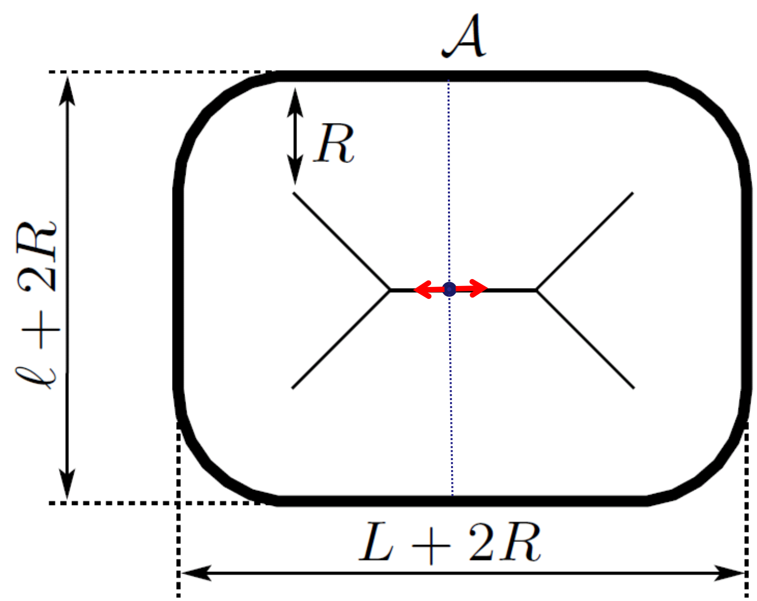
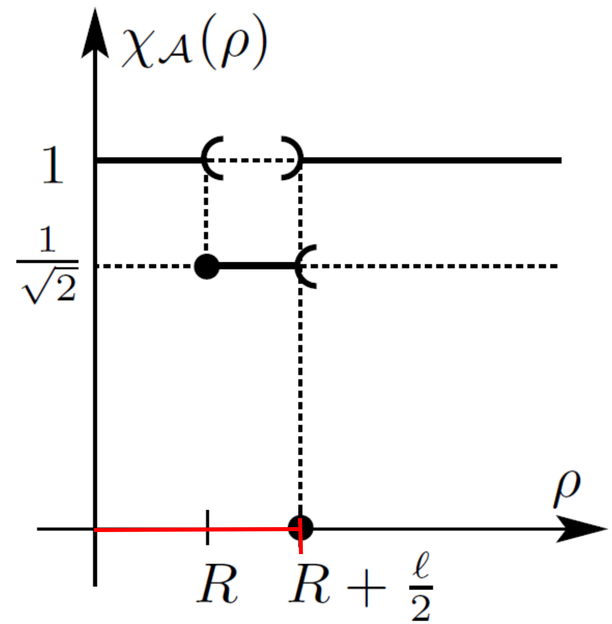
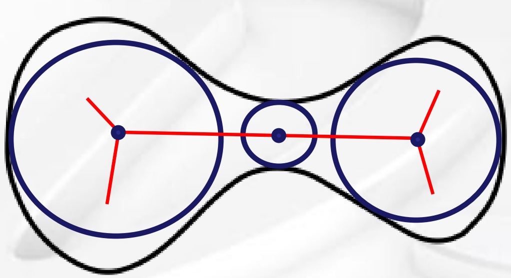
Beyond the reach : WFS and μ-reach





Critical function



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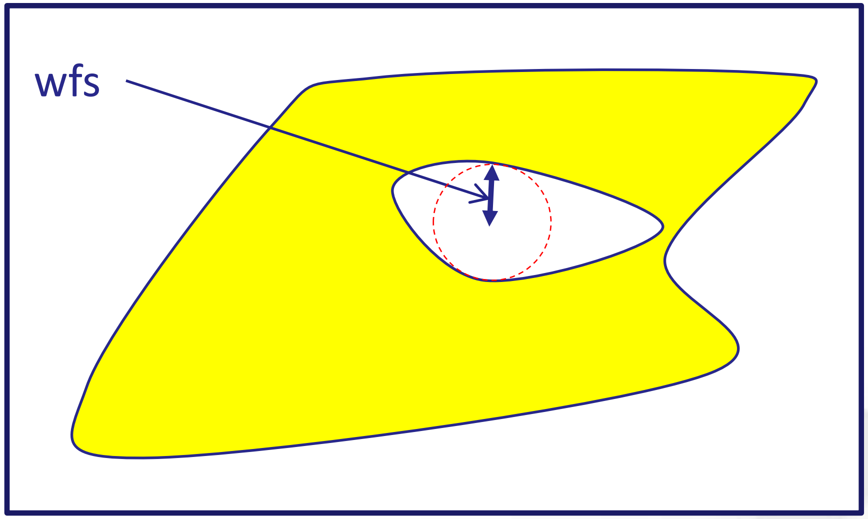
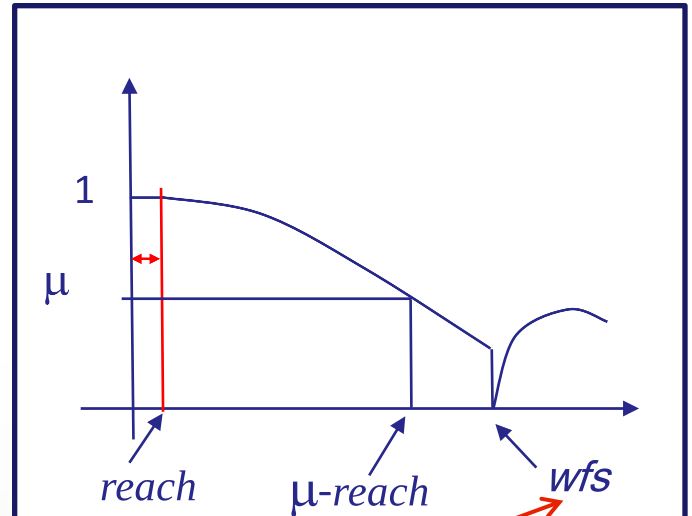
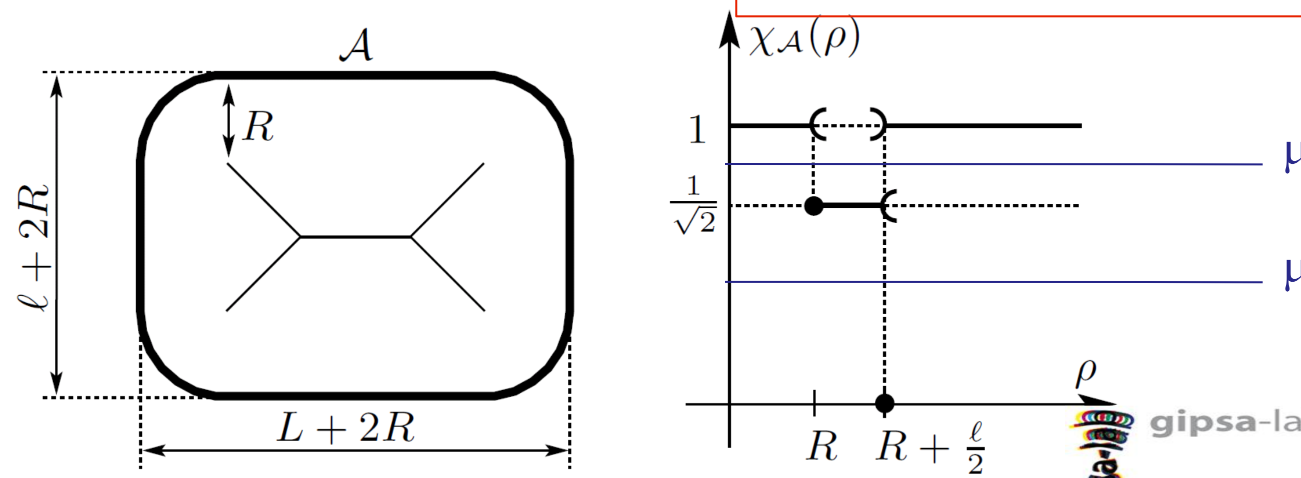
Beyond the reach : WFS and μ-reach





First critical value of distance function



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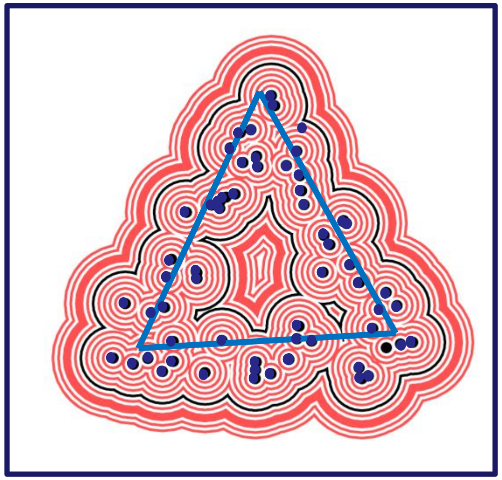
Previous best known result for faithful reconstruction of set with positive m-reach

(Chazal, Cohen-Steiner,Lieutier 2006)

Theorem 2. *Let $\mathcal{A}$ and $\mathcal{S}$ be compact subsets of $\mathbb{R}^n$ and a real number $\alpha > 0$ such that*

$$d_H(\mathcal{S}, \mathcal{A}) < \alpha < \frac{\mu^2}{5\mu^2 + 12} r_\mu(\mathcal{A}) \qquad (2)$$

Then $\mathcal{S}^{\frac{4\alpha}{\mu^2}}$ is a faithful reconsruction of $\mathcal{A}$.



Under the conditions of the theorem, a simple offset of the sample is a faithful reconstruction

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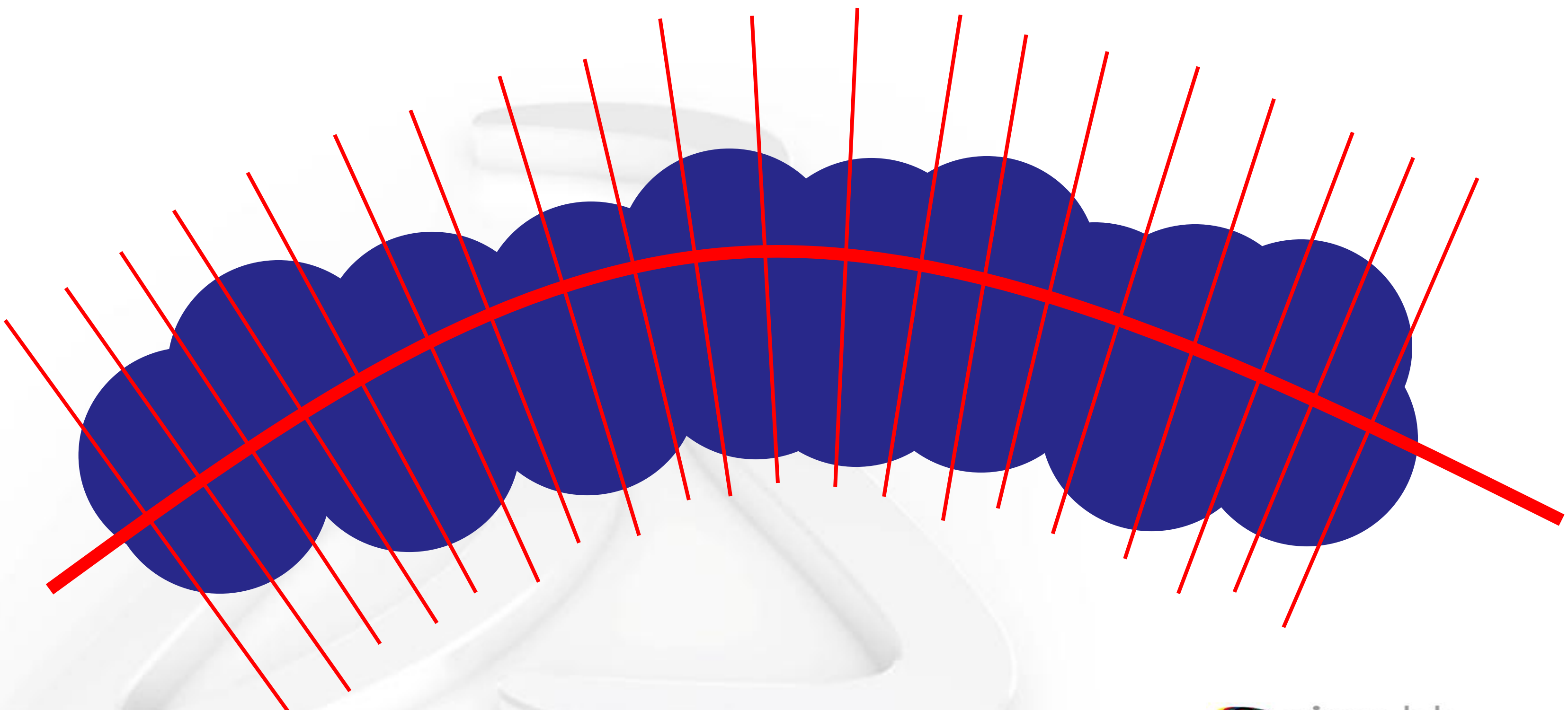
Proofs technics

Proof for Cech / α-complexes for **positive reach** (i.e.smooth) case (NSW04, CL06) relies upon :

Transversality of **normal fibers** to **union of balls boundary**



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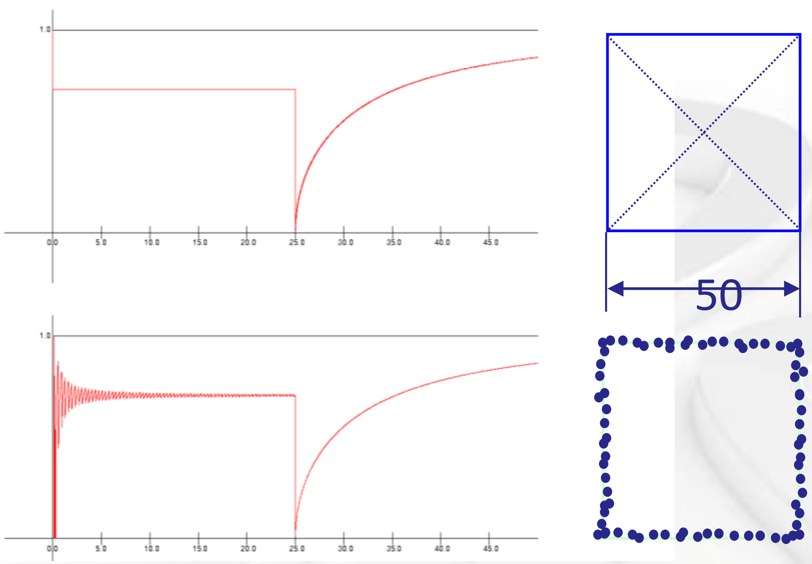
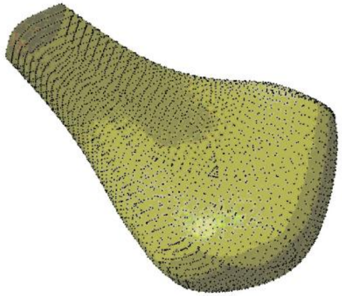
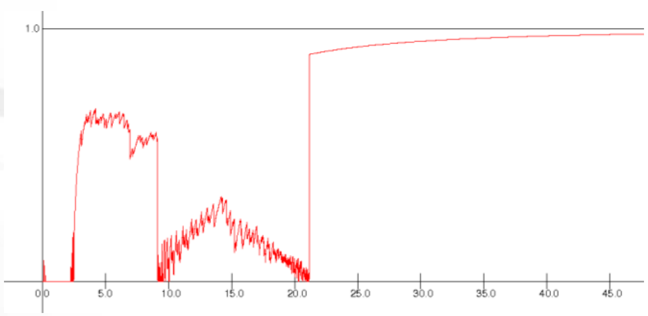
Proofs technics

Proof for **previous result (C.C.-S.L. 06)** Cech / α-complexes for positive μ-reach case (**non smooth**) relies upon:

Hausdorff stability of critical function







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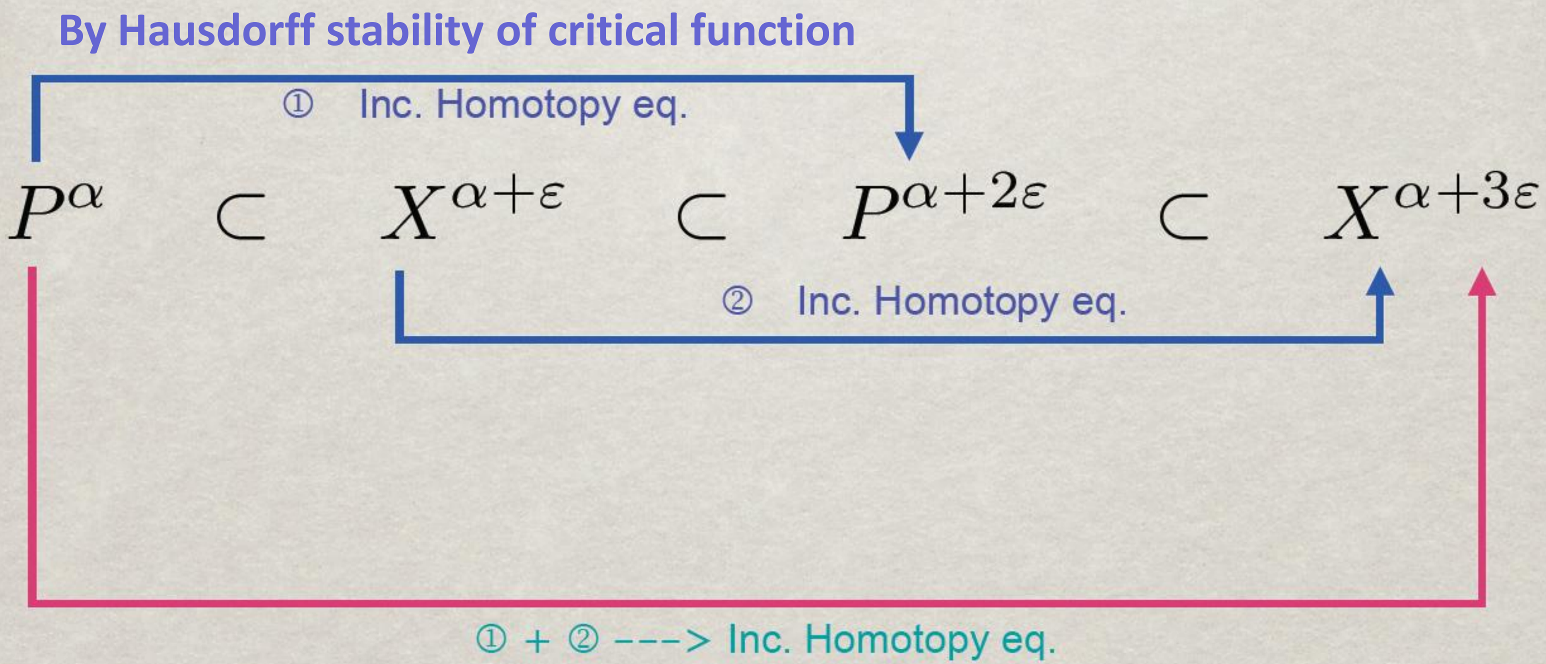
Proofs technics

$d_H(X,P) \leq \varepsilon$



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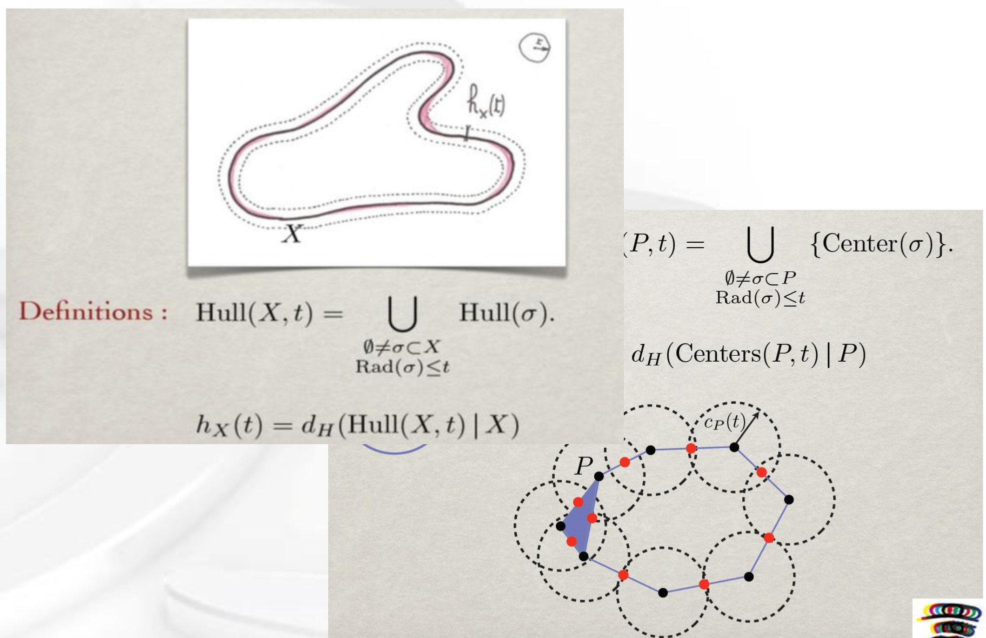
Proofs technics

New Proof Technics (socg 2011) that both:

- **improves** over **previous result (C.C.-S.L. 06)** Cech / α-complexes for **positive μ-reach case** (i.e. non smooth case)
- **Extends** reconstruction theorem to **Rips complex**

Relies upon the new notion of **convexity defect functions :**



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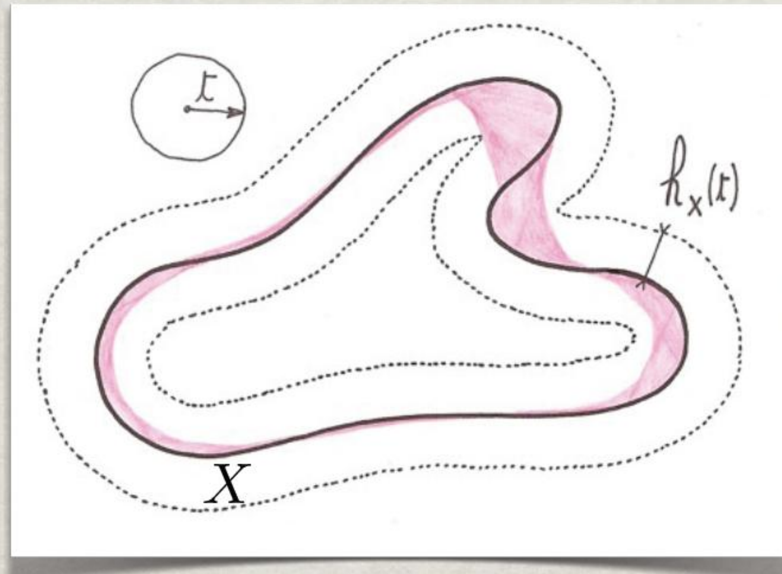
Convexity defects function

h_X : (convex **H**ull) convexity defect function



Definitions :

$$\mathrm{Hull}(X,t) = \bigcup_{\substack{\emptyset \neq \sigma \subset X \\ \mathrm{Rad}(\sigma) \leq t}} \mathrm{Hull}(\sigma).$$

$$h_X(t) = d_H(\mathrm{Hull}(X,t) \mid X)$$

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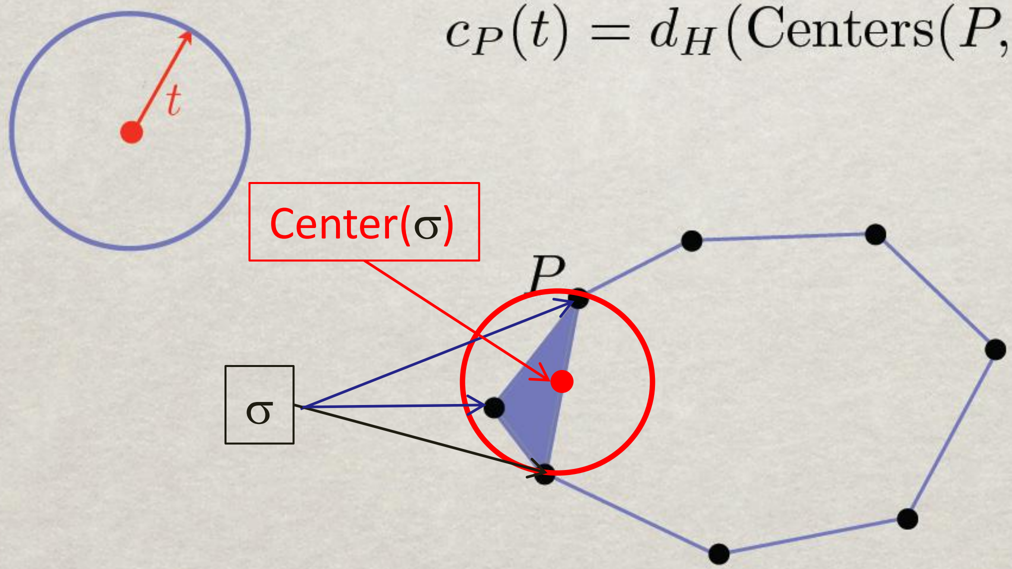
Convexity defects function

C_P : (**C**enter) convexity defect function

Definitions:

$$\text{Centers}(P, t) = \bigcup_{\substack{\emptyset \neq \sigma \subset P \\ \text{Rad}(\sigma) \leq t}} \{\text{Center}(\sigma)\}.$$

$$c_P(t) = d_H(\text{Centers}(P, t) \mid P)$$



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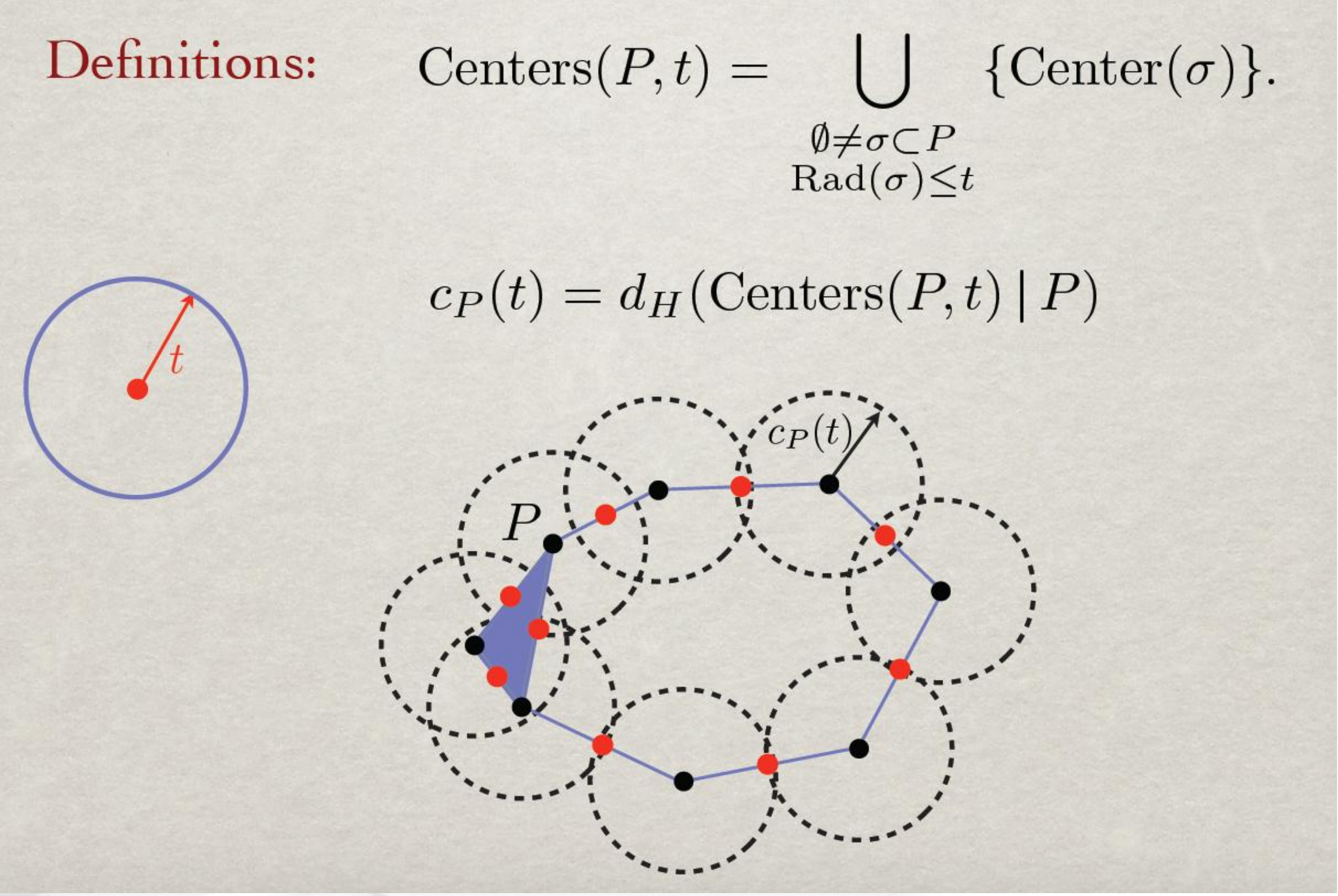
Convexity defects function

C_P : (**C**enter) convexity defect function



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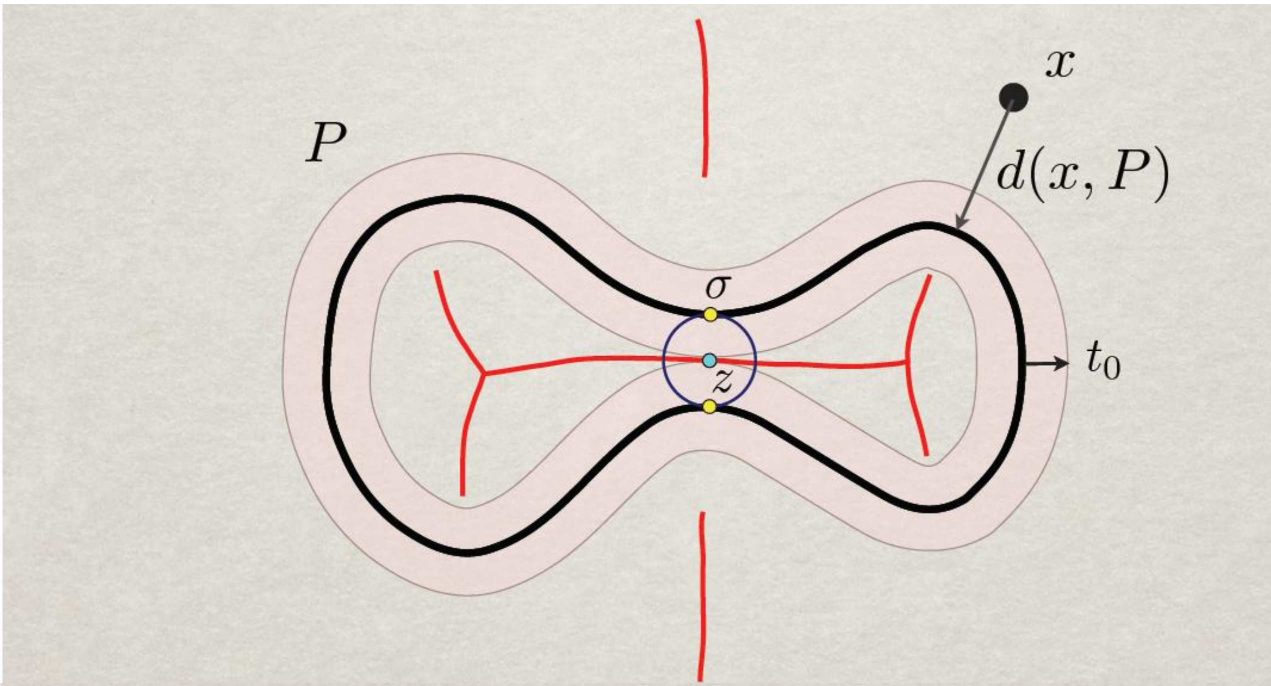
Convexity defects function

h_X : (convex **H**ull) convexity defect function



✸ t_0 critical value $\iff$ $c_P(t_0) = t_0$

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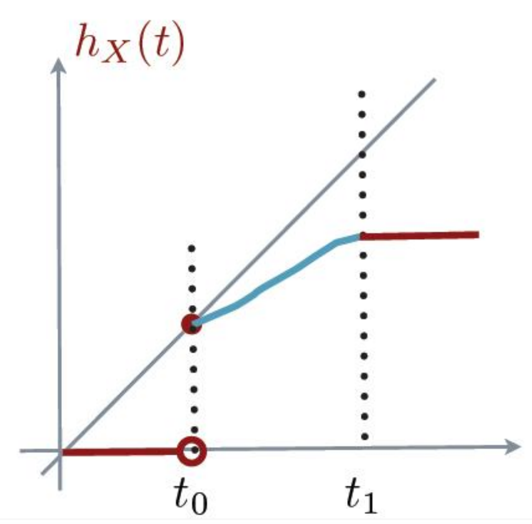
Convexity defects function

Properties:

- If X compact, then X convex $\iff h_X = 0$
- h_X non decreasing
- $c_X(t) \leq h_X(t) \leq t$
- $h_X(t) = t$ iff t critical value of $d(\cdot, X)$
- $d_H(X, P) \leq \varepsilon \Rightarrow h_P(t) \leq h_X(t+\varepsilon) + 2\varepsilon$



Hausdorff stability of defects of convexity

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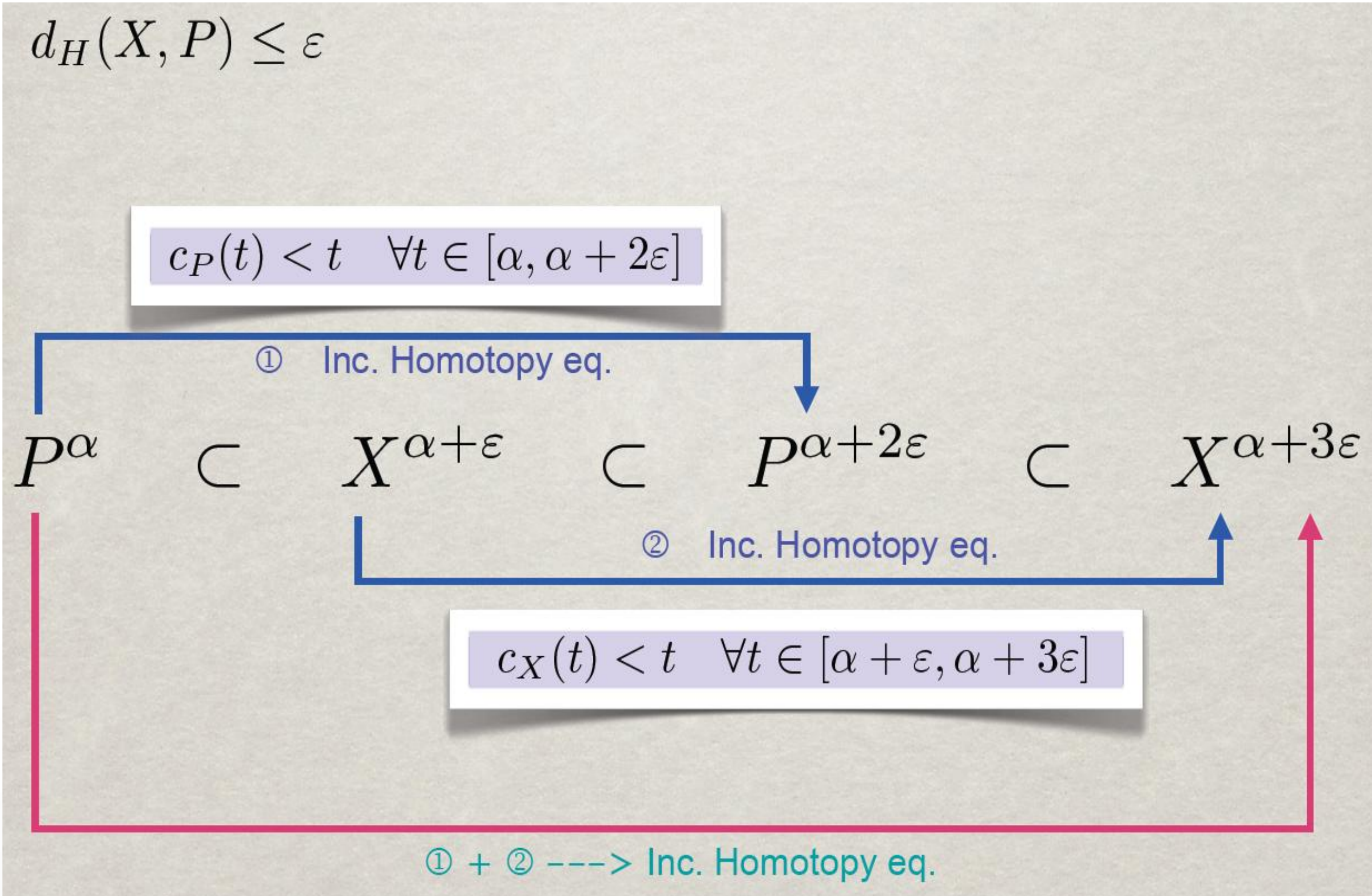
Convexity defects function



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Convexity defects function

$d_H(X,P) \leq \varepsilon$

h_P $\quad \cancel{c}_P(t) < t \quad \forall t \in [\alpha, \alpha + 2\varepsilon]$

① Inc. Homotopy eq.

$$P^{\alpha} \subset X^{\alpha+\varepsilon} \subset P^{\alpha+2\varepsilon} \subset X^{\alpha+3\varepsilon}$$

② Inc. Homotopy eq.

h_X $\quad \cancel{c}_X(t) < t \quad \forall t \in [\alpha+\varepsilon, \alpha + 3\varepsilon]$

① + ② ---> Inc. Homotopy eq.

$h_X(t) < t - 3\varepsilon \quad \forall t \in [\alpha+\varepsilon, \alpha + 3\varepsilon$

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Large μ-reach => small convexity defect functions

LEMMA 6. *Consider two real numbers* $\mu \in (0,1]$ *and* $R \geq 0$. *Let* $X \subset \mathbb{R}^n$ *be a compact set such that* $\chi_X(t) \geq \mu$ *for all* $t \in (0,R)$. *Then, for all* $0 \leq t \leq R$, *one has:*

$$h_X(t) \leq \frac{1 + \mu(1-\mu) - \sqrt{1 - \mu(2-\mu)\left(\frac{t}{R}\right)^2}}{\mu(2-\mu)} R.$$

Small convexity defect functions => Large critical function and therefore large μ-reach

LEMMA 5. *For all compact set* $X \subset \mathbb{R}^n$, *all* $0 \leq \mu \leq 1$ *and all* $t \geq 0$, *the following implication holds:*

$$c_X(t) < (1-\mu)t \implies \chi_X(t) > \mu.$$

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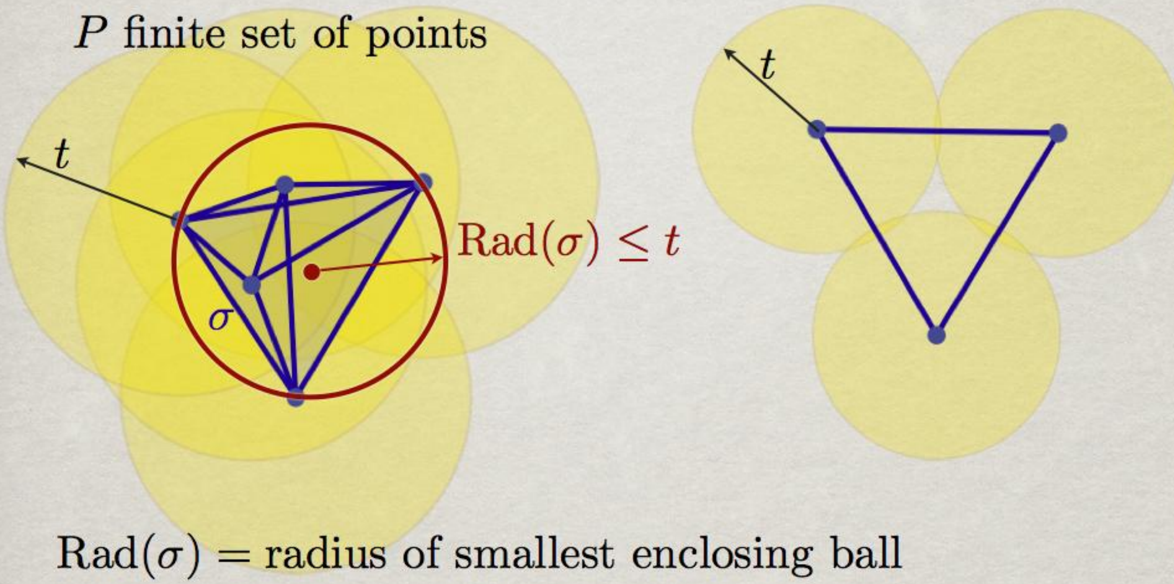
Union of balls ≈ α-complex ≈ Cech complex

CECH COMPLEX

P finite set of points



$\mathrm{Rad}(\sigma)$ = radius of smallest enclosing ball

$\mathcal{C}(P,t) = \{\sigma \mid \emptyset \neq \sigma \subset P, \mathrm{Rad}(\sigma) \leq t\}.$

$\mathcal{C}(P,t) = \mathrm{Nerve}\,\{B(p,t) \mid p \in P\}$

$|\mathcal{C}(P,t)|$ homotopy equivalent to $\bigcup B(p,t) = P^t$

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Convexity defects function

Small defect of convexity

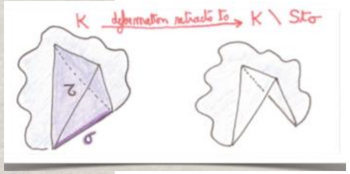
=> **Rips** complexes **collapses** on **Cech** complexes

Theorem 1. *Let $P \subset \mathbb{R}^d$ be a finite set of points in general position. If*

$$c_P(\sqrt{2}\alpha) < 2\alpha - \sqrt{2}\alpha$$

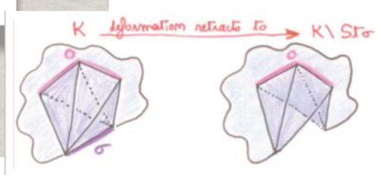
there exists a sequence of collapses from the Rips complex $\mathcal{R}(P,\alpha)$ to the Čech complex $\mathcal{C}(P,\alpha)$.

Proof. (1) sequence of (classical) collapses



$$\mathcal{C}(P,\alpha) \quad \subset \quad \mathcal{R}(P,\alpha) \quad \subset \quad \mathcal{C}(P,\sqrt{2}\alpha)$$

(2) sequence of (exended) collapses



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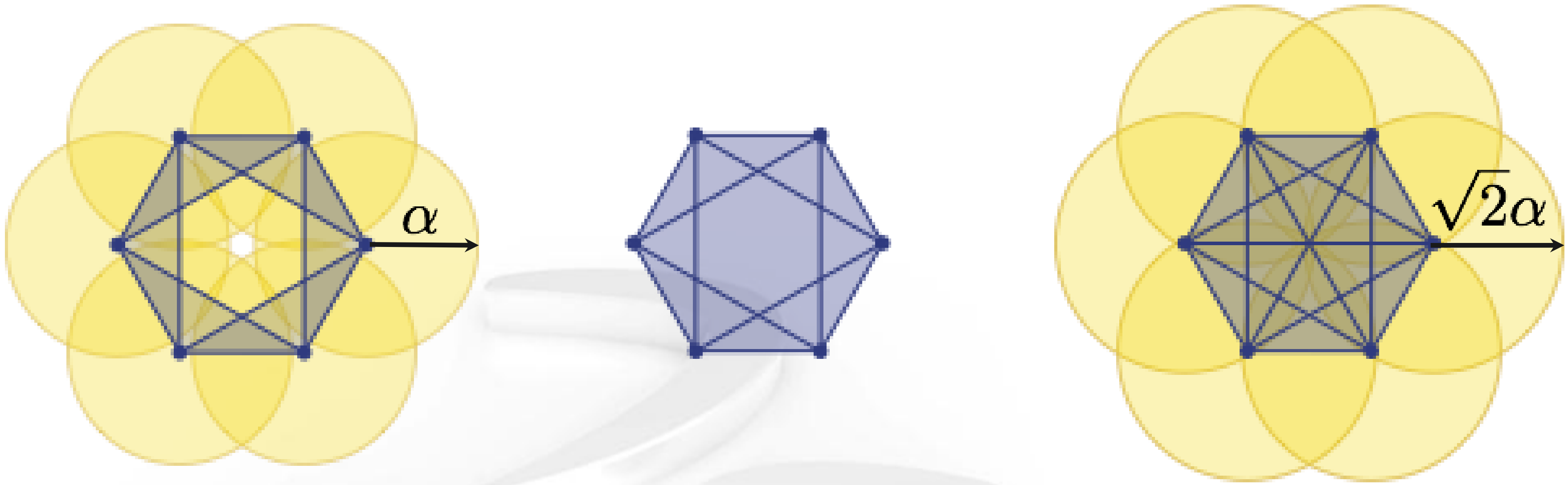
Cech / Rips

$$\mathcal{C}(P,\alpha) \quad \subset \quad \mathcal{R}(P,\alpha) \quad \subset \quad \mathcal{C}(P,\sqrt{2}\alpha)$$



$\simeq$ circle $\qquad \approx$ sphere $\qquad \approx$ ball

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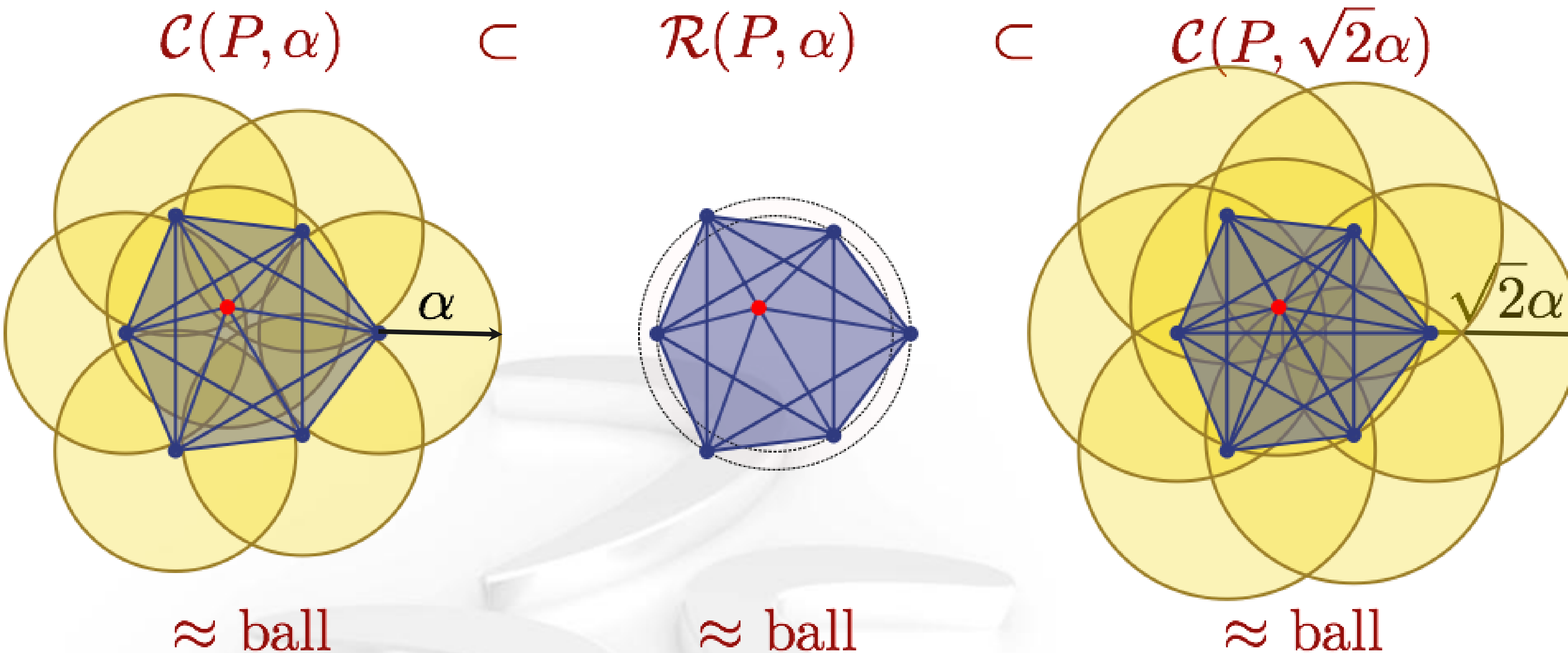
gipsa-lab

cnrs

CATIA

Cech / Rips



Had there been a point close to the center,
it would have distroy spirituous cycles appearing
in the Rips, without changing the Cech.

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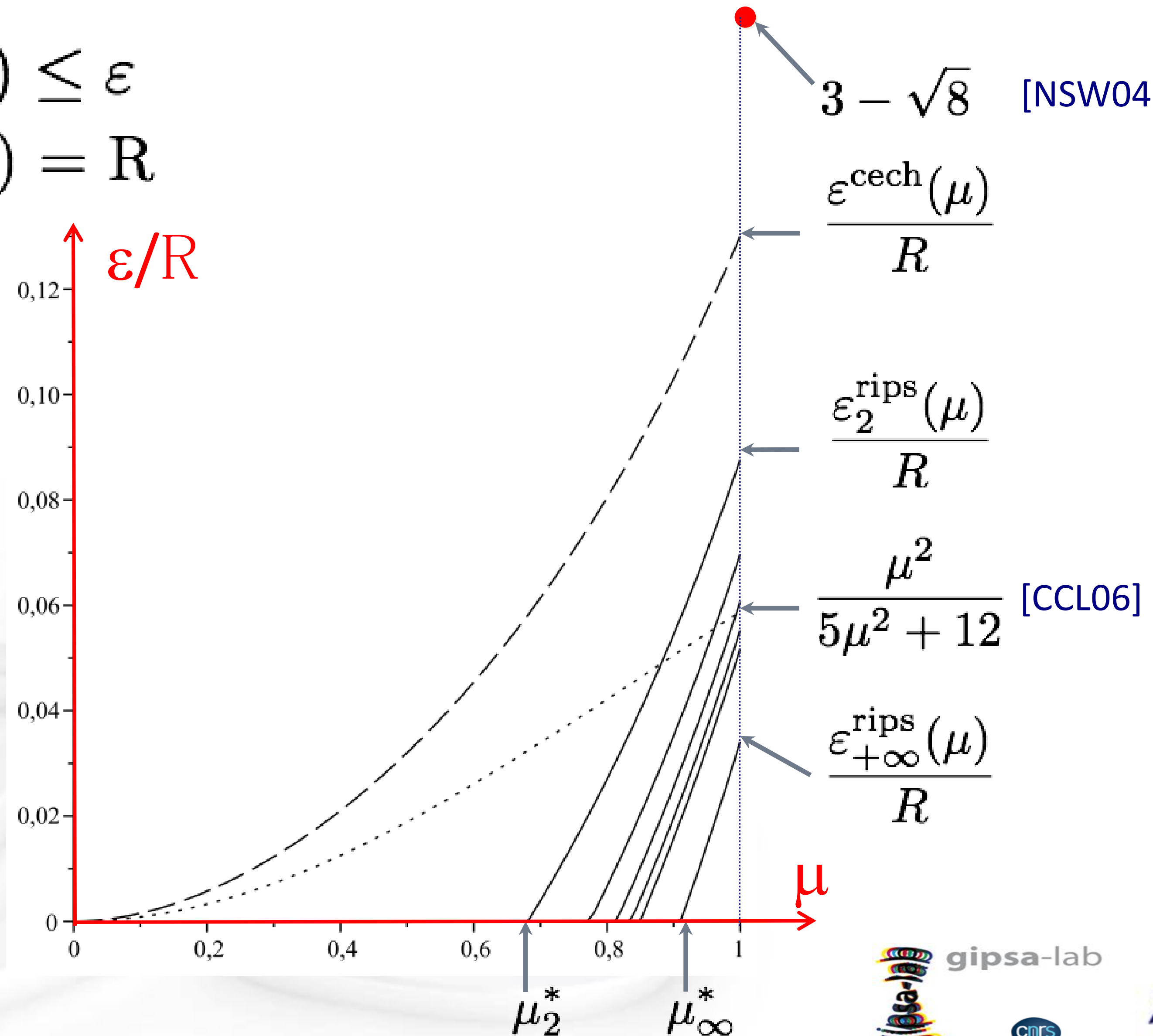
Sampling conditions for Cech and Rips

$$d_H(X, P) \leq \varepsilon$$

$$\mu\text{-reach}(X) = \mathrm{R}$$



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Questions ?

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Questions ?

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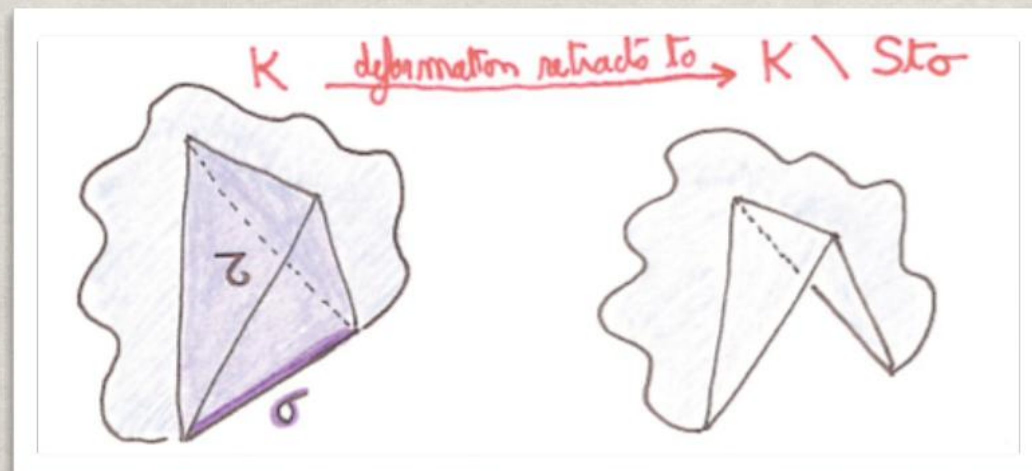
Classical Collapses

$\mathrm{St}_K(\sigma) = $ set of cofaces of σ

K deformation retracts to $K \setminus \mathrm{St}_K(\sigma)$

provided that $\mathrm{St}_K(\sigma)$ has a unique maximal simplex $\tau \neq \sigma$



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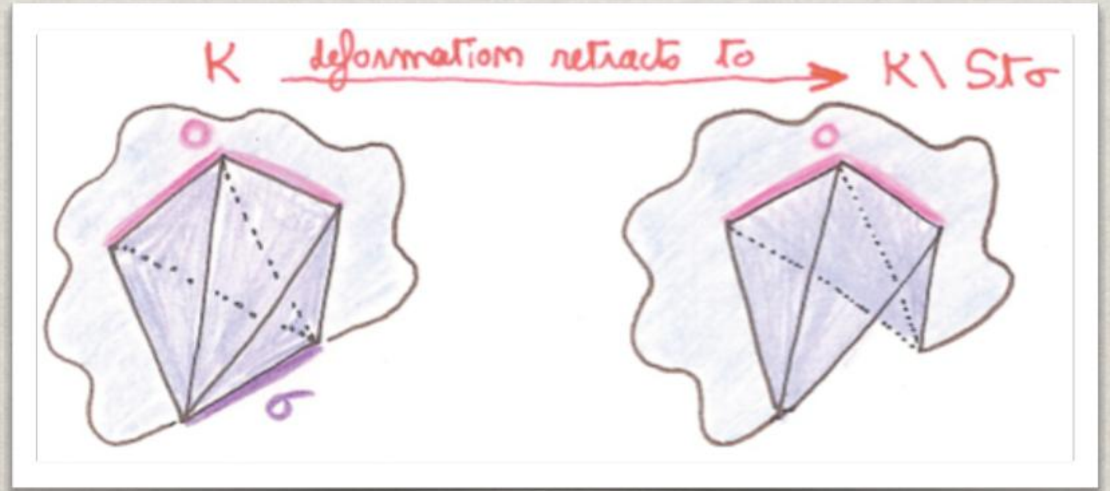
Extended Collapses

$\mathrm{St}_K(\sigma)$ = set of cofaces of σ

K deformation retracts to $K \setminus \mathrm{St}_K(\sigma)$

provided that link of σ is a cone



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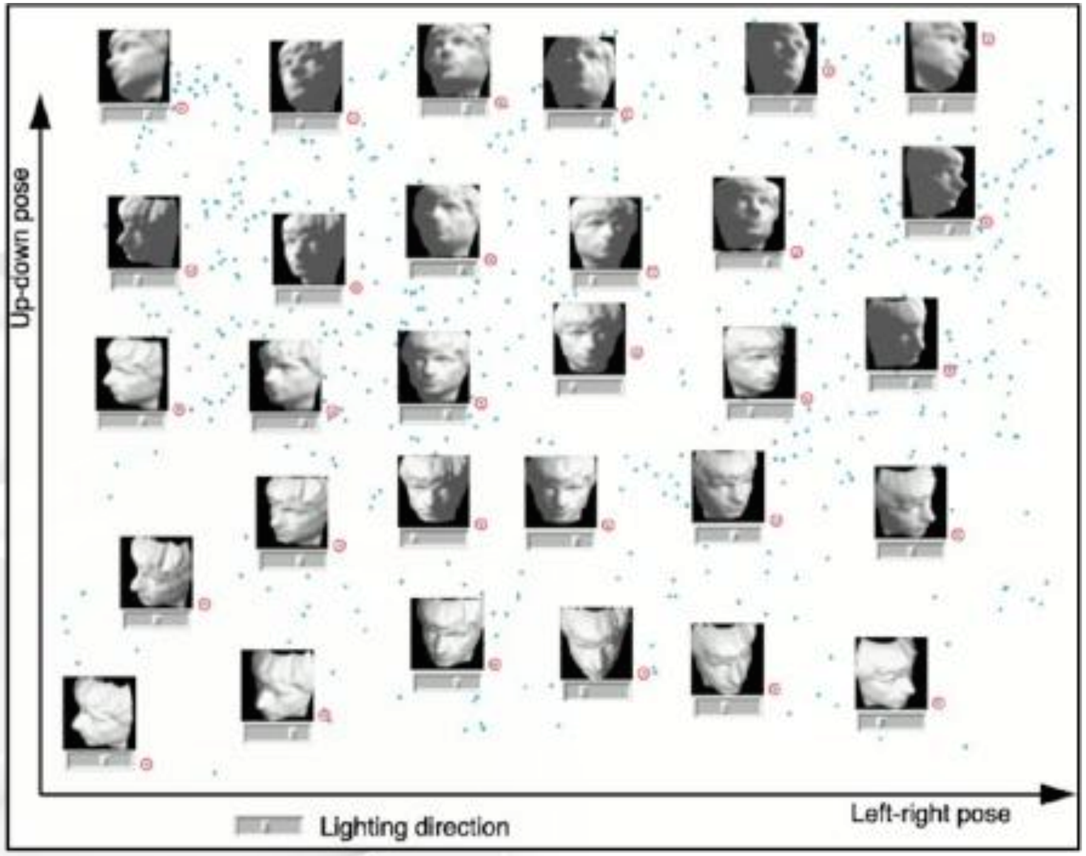
Shape Reconstruction

(or manifold learning)

INPUT

OUTPUT



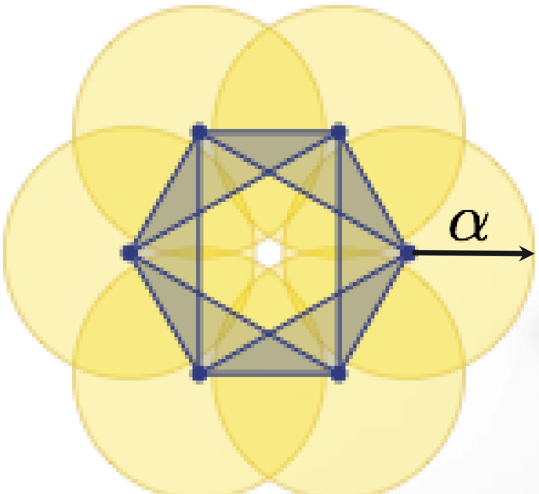
Unordered sequence
of images varying
in pose and lighting

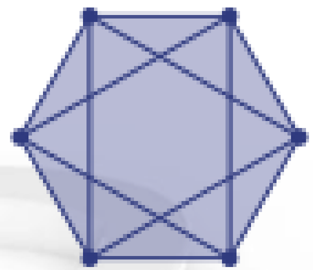
Low-dimensional complex

gipsa-lab
CATIA

Cech / Rips

$$\mathcal{C}(P,\alpha) \quad \neq \quad \mathcal{R}(P,\alpha)$$





$\simeq$ circle $\qquad\qquad \approx$ sphere

Rips and Cech complexes generally don't share **the same topology, but …**

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