Generating vertices of column-row polytopes

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Some background

Algebra $\rightarrow$ Combinatorics $\rightarrow$ Geometry

Some background

The irreducible characters of the symmetric group S_n are indexed by partitions of n.

$$(5, 3, 2) \vdash 10$$

is a partition of 10

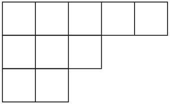
Tensor products of irreducible characters are not irreducible themselves, but they can be decomposed as a direct sum of irreducible characters

$$\chi^\lambda \otimes \chi^\mu = \sum_\nu \kappa(\lambda, \mu, \nu)\chi^\nu$$

The coefficients $\kappa(\lambda, \mu, \nu)$ are known as Kronecker coefficients and are a symmetrical function of λ, μ, ν.

From algebra to combinatorics

To each partition λ there is an associated Young diagram:

$(5, 3, 2) \rightarrow$ 

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A semistandard Young tableaux is a filling of the diagram with natural numbers such that rows are weakly increasing and columns are strictly increasing.

1	1	3	4	4
2	3	5		
5	5			

From algebra to combinatorics

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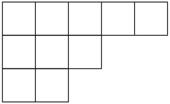
$(5, 3, 2) \rightarrow$ 

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1	1	3	4	4
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5	5			

The tableau has shape $(5, 3, 2)$ and content $(2, 1, 2, 2, 3)$. The number of tableau with shape λ and content α is known as a Kotska number $K_{\alpha\lambda}$.

From algebra to combinatorics

There are several theorems linking products of characters and tableau, we are interested in the case where $\nu = (\nu_1, \nu_2)$ is a two part partition, and in this case, denoting $\bar{\nu} = (\nu_1 + 1, \nu_2 - 1)$:

$$\kappa(\lambda, \mu, \nu) = \mathrm{lr}(\lambda, \mu; \nu) - \mathrm{lr}(\lambda, \mu, \bar{\nu})$$

where $\mathrm{lr}(\lambda, \mu; \nu)$ is the number of pairs of special kind of tableaus known as Littlewood-Richardson multitableau.

Littlewood-Richardson multitableaux

If $\alpha =$ is a partition of n and $\nu = (\nu_1, \nu_2, \ldots, \nu_r)$ a composition of n, a *Littlewood-Richardson multitableaux* of shape α and type ν consists of a sequence $T = (T_1, T_2, \ldots, T_r)$ of tableau such that T_i is a Littlewood-Richardson tableau of shape $\alpha(i)/\alpha(i-1)$ and size ν_i
The shape of the multitableau is the shape of α and its content is the list $(\rho(1), \rho(2), \ldots, \rho(r))$ of contents for each intermediate tableau.

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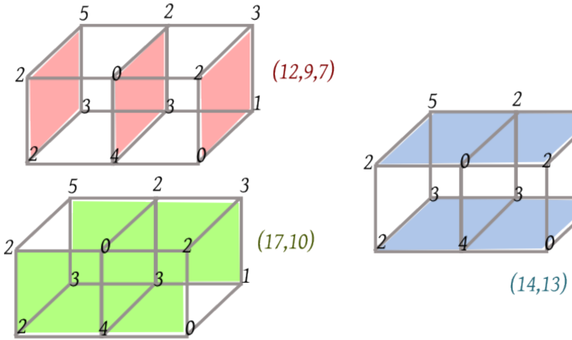
The shape of the multitableau is the shape of α and its content is the list $(\rho(1), \rho(2), \dots, \rho(r))$ of contents for each intermediate tableau.
A Littlewood-Richardson tableau is a skew semistandard tableau whose entries, when read in a certain way become a reverse lattice permutation.

Three dimensional matrices

We will now relate tableaus with three dimensional matrices with fixed 1-margins.



A generalization of RSK correspondence

Using a generalization of the RSK algorithm, E. Vallejo and D. Avella constructed a correspondence between the set $\mathsf{M}(\lambda, \mu, \nu)$ of 3-dimensional matrices with nonnegative entries and 1-margins λ, μ, ν and the set

$$\coprod_{\alpha \trianglerighteq \lambda, \beta \trianglerighteq \mu} \mathsf{K}_{\alpha\lambda} \times \mathsf{K}_{\beta\mu} \times \mathsf{LR}(\alpha, \beta; \nu)$$

where $\mathsf{K}_{\alpha\lambda}$ is the set of semistandard Young tableaux of shape α and content λ and $\mathsf{LR}(\alpha, \beta; \nu)$ is the set of pairs of Littlewood-Richardson multitableaux of respective shapes (α, β), type ν and same content, and we denote by $\mathsf{lr}(\alpha, \beta; \nu)$ the number of such pairs.

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$$M \to (P, Q, (T, S))$$

A generalization of RSK correspondence

When $\nu = (\nu_1, \nu_2)$ is a two-part partition, we recall that Kronecker product $\kappa(\lambda, \mu, \nu)$ can be calculated

$$\langle \chi^\lambda \otimes \chi^\mu, \chi^\nu \rangle = \mathsf{lr}(\lambda, \mu; \nu) - \mathsf{lr}(\lambda, \mu; \tilde{\nu})$$

where $\tilde{\nu} = (\nu_1 + 1, \nu_2 - 1)$.
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where $\tilde{\nu} = (\nu_1 + 1, \nu_2 - 1)$.
And we would like to see the difference not as the number of some tableaus, but as the number of some three dimensional matrices.
Since Kotska numbers of the form $K_{\lambda\lambda}$ are equal to 1, if we look for matrices under the mentioned correspondence such that $\alpha = \lambda$ and $\beta = \mu$ we will have

$$|\mathsf{K}_{\lambda\lambda} \times \mathsf{K}_{\mu\mu} \times \mathsf{LR}(\lambda, \mu; \nu)| = \mathsf{lr}(\lambda, \mu; \nu)$$

The row-column inequalities

Since ν has two parts, we seek $p \times q \times 2$ three dimensional matrices A whose 1-margins are λ, μ, ν, so that under correspondence

$$A \to (P, Q, (T, S))$$

tableaus P, Q have the same shape as content.

We denote by $A^{(1)}$ and $A^{(2)}$ the plane sections having sums ν_1 and ν_2 respectively.
The condition on the entries will in turn become condition on the entries of the two plane matrices

$$A_f = \begin{bmatrix} A^{(1)} \\ A^{(2)} \end{bmatrix}, \qquad A_c = \begin{bmatrix} (A^{(1)})^t \\ (A^{(2)})^t \end{bmatrix}$$

so that when read in certain order, reverse lattice permutations are formed.

The row-column inequalities

We will describe the case where matrices $A^{(1)}$ and $A^{(2)}$ are square $n \times n$. After some simplifications, we obtain that matrices $A^{(1)}$ and $A^{(2)}$ have the following structure

$$A^{(1)} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}, \qquad A^{(2)} = \begin{pmatrix} z_1 & z_2 & \cdots & z_{n-1} & z_n \\ z_2 & z_3 & \cdots & z_n & 0 \\ \vdots & \vdots & & \vdots & \\ z_n & 0 & \cdots & 0 & \end{pmatrix}$$

The row-column inequalities

And therefore the set of sought restrictions can be described on the following array:

$$
\begin{array}{ccccc|c}
x_{11} & x_{12} & x_{13} & \cdots & x_{1n} & z_1 \\
x_{21} & x_{22} & x_{23} & \cdots & x_{2n} & z_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} & z_n \\
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z_1 & z_2 & z_3 & \cdots & z_n &
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x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} & z_n \\
\hline
z_1 & z_2 & z_3 & \cdots & z_n &
\end{array}$$

with the following set of inequalities

$$z_k + \sum_{j=t+1}^{n} x_{k\,j} \geq \sum_{j=t}^{n} x_{k+1\,j}$$

$$z_k + \sum_{i=t+1}^{n} x_{i\,k} \geq \sum_{i=t}^{n} x_{i\,k+1}$$

for $1 \leq k < n$ and $1 \leq t \leq n$ and any x term with a $n+1$ index is ommited.

The row-column inequalities

The inequalities are known as the row-column inequalities since they involve either two partial columns or two partial rows.
Example:

$$\begin{array}{cc|c} x_{11} & x_{12} & z_1 \\ x_{21} & x_{22} & z_2 \\ \hline z_1 & z_2 & \end{array}$$

$$\begin{aligned} z_1 &\geq x_{22}, & z_1 &\geq x_{22} \\ z_1 + x_{21} &\geq x_{22} + x_{12} & z_1 + x_{12} &\geq x_{22} + x_{21} \end{aligned}$$

Example:

$$\begin{array}{ccc|c} 5 & 2 & 3 & 4 \\ 0 & 3 & 4 & 2 \\ 4 & 3 & 1 & 7 \\ \hline 4 & 2 & 7 & \end{array}$$

$$z_1 + x_{31} + x_{21} \geq x_{23} + x_{22} + x_{21}$$

$$\begin{array}{ccc|c} 5 & 2 & 3 & 4 \\ 0 & 3 & 4 & 2 \\ 4 & 3 & 1 & 7 \\ \hline 4 & 2 & 7 & \end{array}$$

$$z_2 + x_{23} \geq x_{33} + x_{32}$$

The row-column polytopes

We can describe $\mathrm{lr}(\lambda, \mu; \nu)$ as the number of integer points contained on a polytope on $\mathbb{R}^{n^2+n}$ defined by the following restrictions:

- The row-column inequalities.
- The nonnegativity of the variables.
- Several sum-restrictions imposed by the fixed 1-margins.

and therefore the Kronecker product is the difference of the number of integer points on two polytopes.

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and therefore the Kronecker product is the difference of the number of integer points on two polytopes.
However, the high dimension and the large number of defining inequalities make the problem very difficult to attack by traditional equation-based approaches.

The row-column polytopes

We turn then to study the polyhedral cone obtained dropping the fixed margins restrictions and leaving only the row-column inequalities, or equivalently, the polytopes described by such inequalities for a given fixed sum of the matrix entries.

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It should be noted that even with the aid of software like POLYMAKE, descriptions are only practical for the smallest cases.

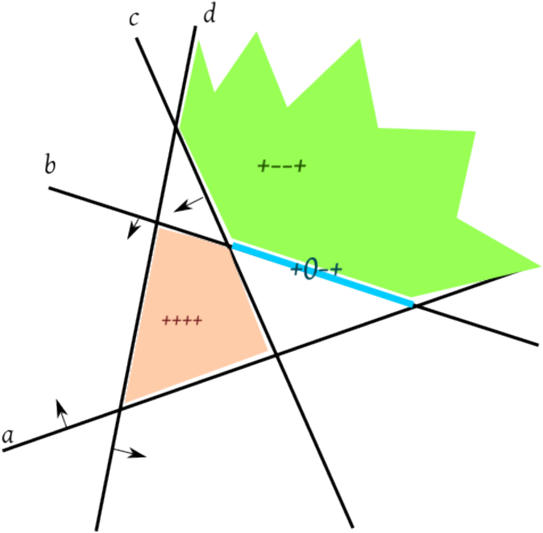
The row-column polytopes

The main result is an algorithm that generates vertices for these polytopes using ideas from the matroid theory.

Oriented matroids and polytopes

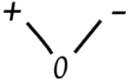
Every polytope determines an hyperplane arrangement given by its facets, and for every arrangement of hyperplanes, there is an associated oriented matroid after choosing a positive half of the ambient space. The vector signs for this matroid are in bijection with the cell decomposition of the space.

Oriented matroids and polytopes



Oriented matroids and polytopes

Moreover, the bijection is also a poset bijection between the space cells, ordered by containment and the induced order on the vector signs by the relation



Oriented matroids and polytopes

Moreover, the bijection is also a poset bijection between the space cells, ordered by containment and the induced order on the vector signs by the relation

So, determining polytope vertices becomes the problem of determining minimal nonnegative vector signs on the matroid poset.

The main result is an algorithm that generates row-column polytope vertices by constructing minimal nonnegative sign vectors.
It should be noted that every vector signs consists of two parts: first one belonging to the row-column inequalities and last one belonging to the nonnegativity conditions (for each x_{ij} and z_n).

Listing vertices: base cases

We first proceed observing that no row-column inequality involves x_{11} and z_n so their vector signs will respectively have $+$ on the entries for $x_{11} > 0$ and $z_n > 0$ and 0 elsewhere, giving minimal vector signs, that is, vertices Observe that these two vertices belong to every other facet for all other inequalities become $0 = 0$ and are thus both pyramidal vertices.

Listing vertices: base cases

Consider now the points having some z_k positive and all other variables equal to zero.
The vector sign must be minimal since it is not possible to remove the only $+$ sign on the nonnegativity part of the vector sign and any $+$ sign on the row-column part can only be removed by adding other positive entries.
We conclude that these are also vertices.
Observe however that unlike the previous vertices, these sign patterns have several $+$ signs and therefore are not pyramidal vertices.

Listing vertices: base cases

Now we take a point such that the positive entries are all equal and fill a diagonal:
Example:

$$\begin{array}{ccc|c} 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ \hline 3 & 0 & 0 & \end{array}$$

All the row-column inequalities become identities and the only $+$ signs on the vector sign are on the nonnegativity part. But we cannot remove any of them (that is, set any entry of the diagonal equal to zero) since it would cause some row-column inequality to become invalid. Therefore vector sign is minimal and the point is a vertex.

Listing vertices: simplest cases

We have then the following list of known vertices, where we indicate only the positive entries in each case:

$$
\begin{aligned}
z_k &> 0, & 1 \le k \le n \\
x_{1\,k} &= x_{2\,k-1} = \cdots = x_{k\,1}, & 1 \le k \le n \\
z_k &= x_{n\,k+1} = x_{2\,k-1} = \cdots = x_{k+1\,n}, & n+1 \le k \le 2n-1
\end{aligned}
$$

Now, in order to obtain a new minimal vector sign from a previous one, there must be at least one sign $+$ becoming 0 and at least one sign 0 becoming $+$. We achieve the latter condition by making positive at least another entry (therefore losing a 0 entry on the non-negativity part of the vector sign) in a way that some row-colum inequality becomes an equality (therefore losing a $+$ sign).
This choice induces an ordering of the vertices so that no vertex appears before a vertex with fewer positive entries than the given one, and also that in order to achieve minimality we need only to minimize the number of strict row-column inequalities for a given set of positive variables.

Generating vertices

We give now a procedure for obtaining a new vertex from a previous one by adding positive entries, called *unfolding* a positive entry.

In order to describe it, we first enumerate the diagonals of the array as follows:

$$
\begin{array}{ccccc|c}
1 & 2 & 3 & \cdots & n & n+1 \\
2 & 3 & 4 & \cdots & n+1 & n+2 \\
3 & 4 & 5 & \cdots & n+2 & n+3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n & n+1 & n+2 & \cdots & 2n-1 & 2n \\
\hline
n+1 & n+2 & n+3 & \cdots & 2n &
\end{array}
$$

Vertex unfolding: first phase

We give now the procedure for obtaining a new vertex from a previous one by adding positive entries, called *unfolding* a positive entry. It has two phases, the first one being:

1. Locate the first diagonal containing positive entries, and verify if any of them appears on a strict row-column inequality I.
2. If no such entry exists, the vertex can no longer be unfolded. Assume otherwise and that $x_{a\,b}$ is such entry and assume without loss of generality it belongs to an strict column[1] inequality.
3. All the entries $x_{i\,b+1}$ with $i < a$ must be zero and find t with $t < a$ such that $x_{t\,b+1}$ belongs to a strict row inequality and also allow $t = 1$.
4. Set $x_{t\,b+1}$ to a positive value so that inequality I becomes an equality.

The above procedure removes a 0 from the original vertex (by making $x_{t\,b+1} > 0$) and removes a $+$ sign (by removing a row-column strict inequality).

[1] The same rules apply exchanging column with rows exchanging indices.

Unfolding example: first phase

Suppose $n = 3$. Earlier we established that $z_2 = 1$ and everything else is a vertex (equivalently, cone ray). It's diagram

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & \end{array}\right)$$

has a single positive entry[2] on the fifth diagonal. It belongs to a strict column inequality I ($z_2 > x_{33}$ among others).
We locate entries on third column belonging to strict row inequalities, in this case x_{33} and we allow also x_{13}
Finally, we make $x_{33} = 1$ or $x_{13} = 1$ so each of the following inequalities becomes an equality:

$$z_2 > z_{33}, \qquad \text{or} \qquad z_2 + x_{32} + x_{22} > x_{33} + x_{23} + x_{13}$$

[2]The bottom row consist of exactly the same variables as rightmost column.

Unfolding example: first phase

Examples:

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \mathbf{1} & 0 \\ \hline 0 & 1 & 0 & \end{array}\right)$$

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where last unfolding was done on a strict row inequality. Observe that in the first example, the result will not be able to unfold again.

Vertex unfolding: first phase

During the first phase we located an entry $x_{a\,b}$ belonging to an strict (column) inequality I and we made positive a new variable $x_{t\,b+1}$ so the $+$ sign corresponding to I turns into zero.
The new sign vector must be minimal for if we could remove any other $+$ sign by modifying other positive entries, the same modification could have been performed on the original vertex and therefore its vector sign would not be minimal, which is a contradiction.

Vertex unfolding: second phase

The previous operation can be modified to yield even more vertices.

1. Locate the first diagonal containing positive entries, and verify if any of them appears on a strict row-column inequality I.
2. If no such entry exists, the vertex can no longer be unfolded. Assume otherwise and that $x_{a\,b}$ is such entry and assume without loss of generality it belongs to an strict column inequality.
3. Locate a row t so that $x_{t\,b+1}$ appears on a strict row colum, or take $t = 1$. Then instead of changing $x_{t\,b+1}$ to turn I into an equality, change $x_{t+1\,b+1}$ with a suitable value. However, a new plus sign appeared this time between columns $b+1$ and $b+2$. To correct this, we change $x_{t,b+2}$ to the same value, and obtaining again the lost 0 on the vector sign.

Vertex unfolding: second phase

The above operation inserts two new positive entries, minimizing the number of $+$ signs being added. Likewise, we can replace the $x_{t\,b+1}$ with three new entries in the same way and so on, in other words, the entry $x_{t\,b+1}$ from the first phase is being unfolded into several positive entries.

Unfolding example: second phase

The following first-phase change on the vertex $z_1 = 1$ (and everything else zero)

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & \end{array}\right)$$

can be further unfolded as

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 0 & 0 & \mathbf{1} & 1 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & \end{array}\right)$$

Here we present a chain of unfoldings (both phases):

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 1 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & \end{array}\right) \qquad \text{unfolding } z_2 \text{ by column}$$

$$\mapsto \left(\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ \mathbf{2} & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & \end{array}\right) \qquad \text{unfolding} x_{23} \text{ by row}$$

$$\mapsto \left(\begin{array}{ccc|c} 0 & \mathbf{3} & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & \end{array}\right) \qquad \text{unfolding} x_{31} \text{ by column}$$

Unfolding: extra conditions

There are some extra techcnical restrictions on the unfolding procedure to yield new vertices, not mentioned earlier in order to have a cleaner description, of the algorithm.

1. If a vertex contains a full diagonal of positive entries, then it must not contain any other positive entry, and all entries on such diagonal must have the same value.
2. During second phase, the positive entries cannot belong to a later diagonal than the one containing the unfolded entry.

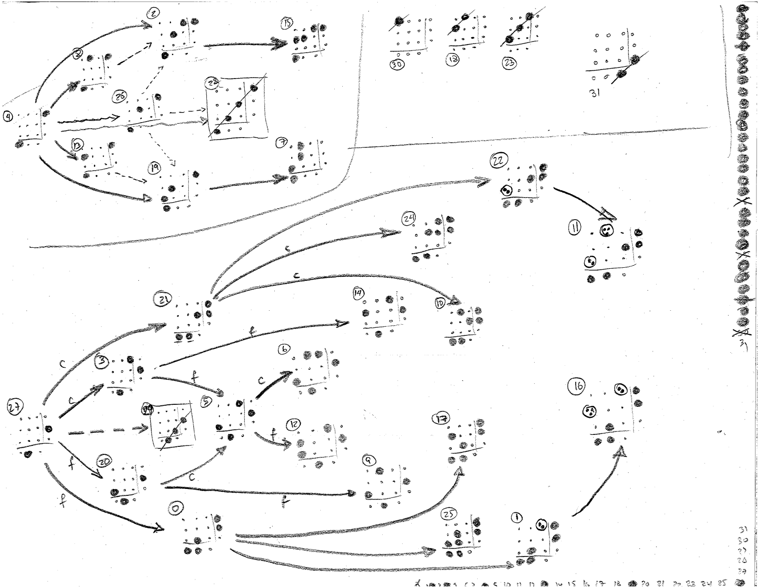
For example, the first restriction doesn't allow us to continue unfolding $x_{12} = 3$ since it would cause x_{21} becoming positive but no vertex can contain a full diagonal of positive entries unless such diagonal is the vertex itself.

Generating vertices: $n = 3$

As example, we consider again the small case where $n = 3$. Here we have a 11-dimensional polytope in $\mathbb{R}^{12}$ described by a set of 20 essential inequalities.

There are 6 base cases, and the unfolding algorithm can be applied on two of them, generating the following diagram listing the 35 vertices.

Generating vertices: $n = 3$.



What next?

The approach using vector signs for the row-column polytope allowed to describe the cone given by the row-column inequalities. We hope that the same approach can be used to describe the more complex polytope obtained adding the margin conditions.

Thank you very much.