# Constructions of Abstract Polytopes with Preassigned Automorphism Group and Number of Flag-Orbits

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If $F$ and $G$ are two faces of $\mathcal{P}$, then they are said to be *incident* if $F \leq G$ or $G \leq F$. If $F \leq G$, then define the *section* $G/F := \{H : F \leq H \leq G\}$.

**3) For $i = 0, \ldots n-1$, if $F$ is a face of $\mathcal{P}$ of rank $i-1$ and $G$ is a face of $\mathcal{P}$ of rank $i+1$ incident to $F$, then the section $G/F$ contains exactly two faces of $\mathcal{P}$ of rank $i$.**

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**4)** $\mathcal{P}$ **is strongly connected. Meaning, given any section of** $\mathcal{P}$**, of rank greater than or equal to 2, and two proper faces** $F$ **and** $G$ **of the section there is a sequence of proper faces of the section from** $F$ **to** $G$ **such that successive faces in the sequence are incident.**

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Every abstract regular $n$-polytope $\mathcal{P}$ has a *Schläfli type* $\{p_1, \ldots p_{n-1}\}$ where $p_i$ is the number of $i$-faces in any section $F/G$ of $\mathcal{P}$, where $F$ is an $(i+1)$-face and $G$ is an $(i-2)$-face of $\mathcal{P}$.

A *string C-group* $\Gamma$, of rank $n$, is a group with $n$ distinguished generators which is a quotient of a group with presentation

$$\langle \rho_0, \rho_1, \ldots, \rho_{n-1} | \rho_i \rho_j^{p_{ij}} \rangle,$$

where $p_{ij} = 2$ unless $i = j - 1$, which satisfies the intersection condition:

$$\Gamma_I \cap \Gamma_J = \Gamma_{I \cap J}$$

where $I, J \subset \{0, 1, \ldots, n-1\}$ and $\Gamma_I = \langle \rho_i | i \in I \rangle$.

# Main Question

**Question**

Given a string C-group, $\Gamma$, and a number $j \geq 1$, is it always the case that there is polytope $\mathcal{P}$ with $j$ flag-orbits such that $\Gamma(\mathcal{P}) \cong \Gamma$?

Let $\mathcal{P}$ be a polytope of rank $n \geq 3$ and let $\mathcal{P}_i$ be the set of $i$-faces of $\mathcal{P}$. For $0 \leq k \leq n-1$, define $[\mathcal{P}]_k$ to be the ranked poset, of rank $n$, whose faces are as follows:

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2. The set of $k$-faces of $[\mathcal{P}]_k$ is

$$\{(F, G) | F \in \mathcal{P}_k, G \in \mathcal{P}_{k+1}, F \leq_{\mathcal{P}} G\}.$$

# Construction

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3. For $k+1 \leq i \leq n-1$, the set of $i$-faces of $[\mathcal{P}]_k$ is:

$$\mathcal{P}_i \cup \{(F, G) | F \in \mathcal{P}_k, G \in \mathcal{P}_{i+1}, F \leq_{\mathcal{P}} G\}.$$

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2. For $F \in \mathcal{P}_{k-1},\ G \in \mathcal{P}_k,\ H \in \mathcal{P}_{k+1}$:
   $F \leq_{[\mathcal{P}]_k} (G, H)$ if and only if $F \leq_{\mathcal{P}} G$; (and hence $F \leq_{\mathcal{P}} G \leq_{\mathcal{P}} H$);

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3. For $k+1 \leq i \leq n-1,\ F, F' \in \mathcal{P}_k,\ G, G' \in \mathcal{P}_i,\ H \in \mathcal{P}_{i+1}$:
   a. $(F, G) \leq (F', H)$ if and only if $F = F'$ and $G \leq_{\mathcal{P}} H'$;
   b. $(F, G) \leq G'$ if and only if $G = G'$;

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   a. $(F, G) \leq (F', H)$ if and only if $F = F'$ and $G \leq_{\mathcal{P}} H'$;
   b. $(F, G) \leq G'$ if and only if $G = G'$;
4. For
   $k+2 \leq i \leq n-1, F \in \mathcal{P}_k,\ H' \in \mathcal{P}_{i-1},\ G \in \mathcal{P}_i,\ H \in \mathcal{P}_{i+1}$:
   a. $H' \leq_{[\mathcal{P}]_k} G$ if and only if $H' \leq_{\mathcal{P}} G$;
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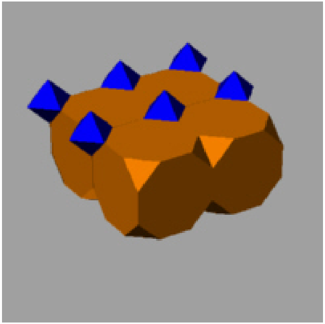
   $k+2 \le i \le n-1, F \in \mathcal{P}_k,\ H' \in \mathcal{P}_{i-1},\ G \in \mathcal{P}_i,\ H \in \mathcal{P}_{i+1}$:
   a. $H' \le_{[\mathcal{P}]_k} G$ if and only if $H' \le_{\mathcal{P}} G$;
   b. $H'$ is never incident to $(F, H)$;
5. The $n$-face (which is unique) is incident to every $(n-1)$-face.



**Theorem**

$[\mathcal{P}]_k$ *is an n-polytope, for* $k = 0, \ldots, n-1$.

**Theorem**

*Let $1 \leq k \leq n-1$. If $\mathcal{P}$ is a regular polytope of rank $n \geq 3$ of Schläfli type $\{p_1, p_2, \ldots, p_{n-1}\}$ with $p_k \neq 2$, then $\Gamma(\mathcal{P}) \cong \Gamma([\mathcal{P}]_k)$.*

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**Theorem**

*Let $\mathcal{P}$ be a regular $n$-polytope and let $0 \leq k \leq n-1$, and assume that $\Gamma(\mathcal{P}) \cong \Gamma([\mathcal{P}]_k)$. Then $[\mathcal{P}]_k$ is an $(n-k)$-orbit polytope. In particular, under the conditions of the previous theorem, $[\mathcal{P}]_k$ is an $(n-k)$-orbit polytope.*

# Strengths and Weaknesses

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Weakness

- Can only use this construction to generate polytopes with up to $n$ flag-orbits, where $n$ is the rank of $\mathcal{P}$.

**Theorem**

*Given a regular n-polytope $\mathcal{P}$ of Schläfli type $\{p_1, \ldots p_{n-1}\}$ with $p_2$ and $p_{n-2}$ not both 2, and automorphism group $\Gamma = \Gamma(\mathcal{P})$. Then for every integer, $j \geq 1$ there is a j-orbit polytope with automorphism group $\Gamma$.*

### Theorem

*For any rank $n \geq 2$, there is no two-orbit n-polytope whose group of automorphisms is isomorphic to the Coxeter group*

$$[\overbrace{2,\ldots,2}^{n-1}] \cong \overbrace{C_2 \times C_2 \times \cdots \times C_2}^{n}.$$

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There are, however, polytopes with any number of flag orbits greater than 3 which have this group as an automorphism group.

- In any odd rank the previous example is the *only* counterexample.

# Final Summary

- In any odd rank the previous example is the *only* counterexample.
- In any even rank $n$, if the $\Gamma$ has Schläfli type is not $\{2, 2, \ldots, 2\}$ or $\{p_1, 2, p_3, 2, \ldots, p_{n-3}, 2, p_{n-1}\}$ with $p_i \neq 2$ for $i$ odd, then there are polytopes with any positive number of orbits.

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- Conjecture: If $\Gamma$ has Schläfli type $\{p_1, 2, p_3, 2, \ldots, p_{n-3}, 2, p_{n-1}\}$ with $p_i \neq 2$ for $i$ odd, then there is no two-orbit polytope which has $\Gamma$ as an automorphism group, but there are $j$-orbit polytopes which have $\Gamma$ as an automorphism group for $j = 3, 4, \ldots$