Discrete groups, highly symmetrical maps and 2-dimensional orbifolds

Roman Nedela and Ján Karabáš

Matej Bel University, Banská Bystrica

Toronto, October 2011

Two sets of problems

Problem A. Given genus g classify maps with high degree of symmetry (few orbits of $Aut^+(M)$ or of $Aut(M)$ on vertices, edges, faces, darts, flags,. . .).

Problem B. Given genus g, enumerate maps with e edges (e edges and v-vertices).

The considered class of maps may satisfy certain constraints (non-degenerate, simple graphs, polytopal, polyhedral, triangular, bipartite,...

Two sets of problems

Problem A. Given genus g classify maps with high degree of symmetry (few orbits of $Aut^+(M)$ or of $Aut(M)$ on vertices, edges, faces, darts, flags,...).

Problem B. Given genus g, enumerate maps with e edges (e edges and v-vertices).

The considered class of maps may satisfy certain constraints (non-degenerate, simple graphs, polytopal, polyhedral, triangular, bipartite,...

Two sets of problems

Problem A. Given genus g classify maps with high degree of symmetry (few orbits of $Aut^+(M)$ or of $Aut(M)$ on vertices, edges, faces, darts, flags,...).

Problem B. Given genus g, enumerate maps with e edges (e edges and v-vertices).

The considered class of maps may satisfy certain constraints (non-degenerate, simple graphs, polytopal, polyhedral, triangular, bipartite,...

Why a classification of discrete groups is of interest?

Two examples:

Problems of type A are equivalent to the classification of discrete groups of genus g of a restricted orbifold type.

Problems of type B require to determine g-admissible cyclic actions. But it is not enough!

Surfaces and discrete groups

Surface a 2-dimensional manifold $\mathcal{S}$, connected; preferably compact, without boundary and orientable;

Classification of compact surfaces by *orientability* and *genus* – number of handles (or crosscaps) attached;

Discrete group a group of self-homeomorphisms of $\mathcal{S}$, s.t. each orbit forms a discrete set;

Discrete group of a compact connected surface is a finite group;

Two kinds of elements orientation preserving and orientation reversing

Surfaces and discrete groups

Surface a 2-dimensional manifold $\mathcal{S}$, connected; preferably compact, without boundary and orientable;

Classification of compact surfaces by *orientability* and *genus* – number of handles (or crosscaps) attached;

Discrete group a group of self-homeomorphisms of $\mathcal{S}$, s.t. each orbit forms a discrete set;

Discrete group of a compact connected surface is a finite group;

Two kinds of elements orientation preserving and orientation reversing

Surfaces and discrete groups

Surface a 2-dimensional manifold $\mathcal{S}$, connected; preferably compact, without boundary and orientable;

Classification of compact surfaces by *orientability* and *genus* – number of handles (or crosscaps) attached;

Discrete group a group of self-homeomorphisms of $\mathcal{S}$, s.t. each orbit forms a discrete set;

Discrete group of a compact connected surface is a finite group;

Two kinds of elements orientation preserving and orientation reversing

Surfaces and discrete groups

Surface a 2-dimensional manifold $\mathcal{S}$, connected; preferably compact, without boundary and orientable;

Classification of compact surfaces by *orientability* and *genus* – number of handles (or crosscaps) attached;

Discrete group a group of self-homeomorphisms of $\mathcal{S}$, s.t. each orbit forms a discrete set;

Discrete group of a compact connected surface is a finite group;

Two kinds of elements orientation preserving and orientation reversing

Surfaces and discrete groups

Surface a 2-dimensional manifold $\mathcal{S}$, connected; preferably compact, without boundary and orientable;

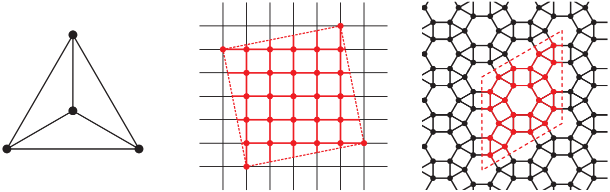
Classification of compact surfaces by *orientability* and *genus* – number of handles (or crosscaps) attached;

Discrete group a group of self-homeomorphisms of $\mathcal{S}$, s.t. each orbit forms a discrete set;

Discrete group of a compact connected surface is a finite group;

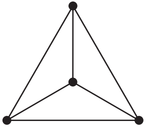
Two kinds of elements orientation preserving and orientation reversing

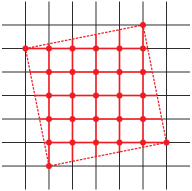
Discrete groups are groups of symmetries of Maps

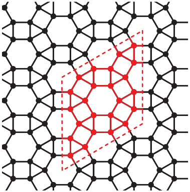


1. An automorphism of a vertex-transitive map M on a surface $\mathcal{S}$ extends to a self-homeomorphism of $\mathcal{S}$;
2. Every finite group of automorphisms of a surface $\mathcal{S}$ is a group of automorphisms of a (Cayley) vertex-transitive map on $\mathcal{S}$;
3. Every finite group appears as a discrete group of automorphisms of $\mathcal{S}$ (compact, closed);
4. Not all actions of finite groups can be realized;
5. **A point stabilizer is a subgroup of a dihedral group**;
6. Vertex-transitive maps are (the best) models for investigation of discrete groups

Discrete groups are groups of symmetries of Maps

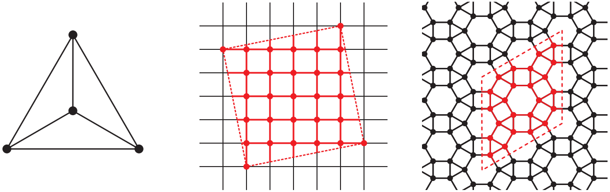






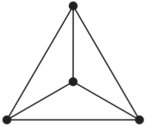
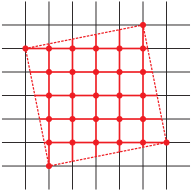
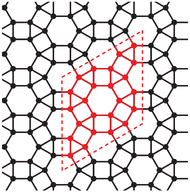
1. An automorphism of a vertex-transitive map $\mathbf{M}$ on a surface $\mathcal{S}$ extends to a self-homeomorphism of $\mathcal{S}$;
2. Every finite group of automorphisms of a surface $\mathcal{S}$ is a group of automorphisms of a (Cayley) vertex-transitive map on $\mathcal{S}$;
3. Every finite group appears as a discrete group of automorphisms of $\mathcal{S}$ (compact, closed);
4. Not all actions of finite groups can be realized;
5. **A point stabilizer is a subgroup of a dihedral group**;
6. Vertex-transitive maps are (the best) models for investigation of discrete groups

Discrete groups are groups of symmetries of Maps



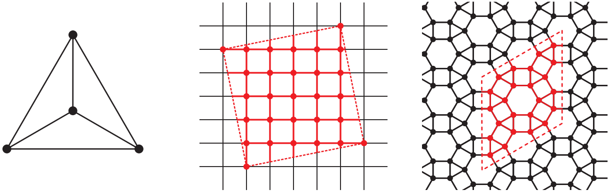
1. An automorphism of a vertex-transitive map $\mathbf{M}$ on a surface $\mathcal{S}$ extends to a self-homeomorphism of $\mathcal{S}$;
2. Every finite group of automorphisms of a surface $\mathcal{S}$ is a group of automorphisms of a (Cayley) vertex-transitive map on $\mathcal{S}$;
3. Every finite group appears as a discrete group of automorphisms of $\mathcal{S}$ (compact, closed);
4. Not all actions of finite groups can be realized;
5. **A point stabilizer is a subgroup of a dihedral group**;
6. Vertex-transitive maps are (the best) models for investigation of discrete groups

Discrete groups are groups of symmetries of Maps

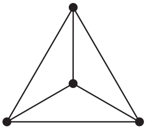
1. An automorphism of a vertex-transitive map $\mathbf{M}$ on a surface $\mathcal{S}$ extends to a self-homeomorphism of $\mathcal{S}$;
2. Every finite group of automorphisms of a surface $\mathcal{S}$ is a group of automorphisms of a (Cayley) vertex-transitive map on $\mathcal{S}$;
3. Every finite group appears as a discrete group of automorphisms of $\mathcal{S}$ (compact, closed);
4. Not all actions of finite groups can be realized;
5. **A point stabilizer is a subgroup of a dihedral group**;
6. Vertex-transitive maps are (the best) models for investigation of discrete groups

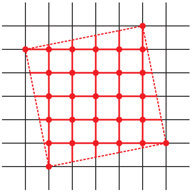
Discrete groups are groups of symmetries of Maps

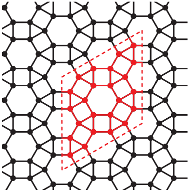


1. An automorphism of a vertex-transitive map $\mathbf{M}$ on a surface $\mathcal{S}$ extends to a self-homeomorphism of $\mathcal{S}$;
2. Every finite group of automorphisms of a surface $\mathcal{S}$ is a group of automorphisms of a (Cayley) vertex-transitive map on $\mathcal{S}$;
3. Every finite group appears as a discrete group of automorphisms of $\mathcal{S}$ (compact, closed);
4. Not all actions of finite groups can be realized;
5. **A point stabilizer is a subgroup of a dihedral group**;
6. Vertex-transitive maps are (the best) models for investigation of discrete groups

Discrete groups are groups of symmetries of Maps

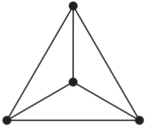
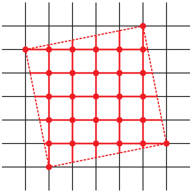
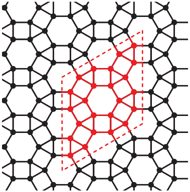






1. An automorphism of a vertex-transitive map $\mathbf{M}$ on a surface $\mathcal{S}$ extends to a self-homeomorphism of $\mathcal{S}$;
2. Every finite group of automorphisms of a surface $\mathcal{S}$ is a group of automorphisms of a (Cayley) vertex-transitive map on $\mathcal{S}$;
3. Every finite group appears as a discrete group of automorphisms of $\mathcal{S}$ (compact, closed);
4. Not all actions of finite groups can be realized;
5. **A point stabilizer is a subgroup of a dihedral group**;
6. Vertex-transitive maps are (the best) models for investigation of discrete groups

Discrete groups are groups of symmetries of Maps

1. An automorphism of a vertex-transitive map $\mathbf{M}$ on a surface $\mathcal{S}$ extends to a self-homeomorphism of $\mathcal{S}$;
2. Every finite group of automorphisms of a surface $\mathcal{S}$ is a group of automorphisms of a (Cayley) vertex-transitive map on $\mathcal{S}$;
3. Every finite group appears as a discrete group of automorphisms of $\mathcal{S}$ (compact, closed);
4. Not all actions of finite groups can be realized;
5. **A point stabilizer is a subgroup of a dihedral group**;
6. Vertex-transitive maps are (the best) models for investigation of discrete groups

Regular branched coverings and quotient spaces

- Euler characteristics

$$\chi = v - e + f = \begin{cases} 2-2g, & \mathcal{S} \text{ orientable} \\ 2-\hat{g}, & \mathcal{S} \text{ non-orientable} \end{cases}$$

- Regular covers given by group actions $\mathcal{S}_g/G = \mathcal{S}_\gamma$;
- !!! Closed orientable surfaces are closed under taking quotients by orientation preserving actions!!!
- How to relate the Euler characteristics of $\mathcal{S}_g$ and a covered surface $\mathcal{S}_\gamma$?
- **Smooth coverings** of orientable surfaces ($|G|$-folded covers)

$$(2-2g) = |G|(2-2\gamma);$$

Regular branched coverings and quotient spaces

- Euler characteristics

$$\chi = v - e + f = \begin{cases} 2-2g, & \mathcal{S} \text{ orientable} \\ 2-\hat{g}, & \mathcal{S} \text{ non-orientable} \end{cases}$$

- Regular covers given by group actions $\mathcal{S}_g/G = \mathcal{S}_\gamma$;
- !!! Closed orientable surfaces are closed under taking quotients by orientation preserving actions!!!
- How to relate the Euler characteristics of $\mathcal{S}_g$ and a covered surface $\mathcal{S}_\gamma$?
- **Smooth coverings** of orientable surfaces ($|G|$-folded covers)

$$(2-2g) = |G|(2-2\gamma);$$

Regular branched coverings and quotient spaces

- Euler characteristics

$$\chi = v - e + f = \begin{cases} 2-2g, & \mathcal{S} \text{ orientable} \\ 2-\hat{g}, & \mathcal{S} \text{ non-orientable} \end{cases}$$

- Regular covers given by group actions $\mathcal{S}_g/G = \mathcal{S}_\gamma$;
- !!! Closed orientable surfaces are closed under taking quotients by orientation preserving actions!!!
- How to relate the Euler characteristics of $\mathcal{S}_g$ and a covered surface $\mathcal{S}_\gamma$?
- **Smooth coverings** of orientable surfaces ($|G|$-folded covers)

$$(2-2g) = |G|(2-2\gamma);$$

Regular branched coverings and quotient spaces

- Euler characteristics

$$\chi = v - e + f = \begin{cases} 2-2g, & \mathcal{S} \text{ orientable} \\ 2-\hat{g}, & \mathcal{S} \text{ non-orientable} \end{cases}$$

- Regular covers given by group actions $\mathcal{S}_g/G = \mathcal{S}_\gamma$;
- !!! Closed orientable surfaces are closed under taking quotients by orientation preserving actions!!!
- How to relate the Euler characteristics of $\mathcal{S}_g$ and a covered surface $\mathcal{S}_\gamma$?
- **Smooth coverings** of orientable surfaces ($|G|$-folded covers)

$$(2-2g) = |G|(2-2\gamma);$$

- Euler characteristics

$$\chi = v - e + f = \begin{cases} 2-2g, & \mathcal{S} \text{ orientable} \\ 2-\hat{g}, & \mathcal{S} \text{ non-orientable} \end{cases}$$

- Regular covers given by group actions $\mathcal{S}_g/G = \mathcal{S}_\gamma$;
- !!! Closed orientable surfaces are closed under taking quotients by orientation preserving actions!!!
- How to relate the Euler characteristics of $\mathcal{S}_g$ and a covered surface $\mathcal{S}_\gamma$?
- **Smooth coverings** of orientable surfaces ($|G|$-folded covers)

$$(2-2g) = |G|(2-2\gamma);$$

Riemann-Hurwitz equation, orientation preserving action

- **Regular branched coverings** between orientable surfaces, Riemann-Hurwitz equation - case G is orientation preserving,

$$2-2g=|G|\left(2-2\gamma-\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right);\ \forall i: m_i\geq 2\in\mathbb{Z};\ m_i|\,|G|\,;$$

- **Quotient Orbifold** is a surface of genus γ, with r points distinguished, each of them is endowed with an integer *branch index* $m_i>1$; singular points of the covering, m_i- the number of wrappings of the neighbourhood centered at x_i
- Orbifold is described by its **signature** $\mathcal{O}(\gamma;\{m_1,m_2\ldots,m_r\})$.

Riemann-Hurwitz equation, orientation preserving action

- **Regular branched coverings** between orientable surfaces, Riemann-Hurwitz equation - case G is orientation preserving,

$$2-2g=|G|\left(2-2\gamma-\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right);\ \forall i: m_i\geq 2\in\mathbb{Z};\ m_i|\,|G|\,;$$

- **Quotient Orbifold** is a surface of genus γ, with r points distinguished, each of them is endowed with an integer *branch index* $m_i > 1$; singular points of the covering, m_i- the number of wrappings of the neighbourhood centered at x_i
- Orbifold is described by its **signature** $\mathcal{O}(\gamma;\{m_1,m_2\ldots,m_r\})$.

Riemann-Hurwitz equation, orientation preserving action

- **Regular branched coverings** between orientable surfaces, Riemann-Hurwitz equation - case G is orientation preserving,

$$2 - 2g = |G| \left(2 - 2\gamma - \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right) ; \; \forall i : m_i \geq 2 \in \mathbb{Z}; \; m_i | \, |G| \, ;$$

- **Quotient Orbifold** is a surface of genus γ, with r points distinguished, each of them is endowed with an integer *branch index* $m_i > 1$; singular points of the covering, m_i- the number of wrappings of the neighbourhood centered at x_i
- Orbifold is described by its **signature** $\mathcal{O}(\gamma; \{m_1, m_2 \ldots, m_r\})$.

Smooth coverings - classical topic in homotopy theory:

Fundamental group $\pi_1(\mathcal{S})$ elements – (eq. classes of) closed curves is $\mathcal{S}$, contractible (closed) curves are identities, operation – composition of curves;

subgroup/covering correspondence a covering determines a subgroup of $\pi_1(\mathcal{S})$, a subgroup $G \leq \pi_1(\mathcal{S})$ determines a regular covering $\tilde{S} \to S$ with $CTP(p) \cong G$,

Universal cover over $\mathcal{S}$ a simply connected surface $\widetilde{\mathcal{S}}$ $(1 \leq \pi_1(\mathcal{S})$ covering all the covers of $\mathcal{S}$;

Fundamental group a surface of genus g

$$\pi_1(\mathcal{S}_g) = \langle a_1, b_1, \ldots, a_\gamma, b_\gamma \mid \prod_{i=1}^{\gamma} [a_i, b_i] \rangle.$$

group - covering duality and universal covers

Smooth coverings - classical topic in homotopy theory:

Fundamental group $\pi_1(\mathcal{S})$ elements – (eq. classes of) closed curves is $\mathcal{S}$, contractible (closed) curves are identities, operation – composition of curves;

subgroup/covering correspondence a covering determines a subgroup of $\pi_1(\mathcal{S})$, a subgroup $G \leq \pi_1(\mathcal{S})$ determines a regular covering $\tilde{S} \to S$ with $CTP(p) \cong G$,

Universal cover over $\mathcal{S}$ a simply connected surface $\widetilde{\mathcal{S}}$ $(1 \leq \pi_1(\mathcal{S})$ covering all the covers of $\mathcal{S}$;

Fundamental group a surface of genus g

$$\pi_1(\mathcal{S}_g) = \langle a_1, b_1, \ldots, a_\gamma, b_\gamma \mid \prod_{i=1}^{\gamma} [a_i, b_i] \rangle.$$

group - covering duality and universal covers

Smooth coverings - classical topic in homotopy theory:

Fundamental group $\pi_1(\mathcal{S})$ elements – (eq. classes of) closed curves is $\mathcal{S}$, contractible (closed) curves are identities, operation – composition of curves;

subgroup/covering correspondence a covering determines a subgroup of $\pi_1(\mathcal{S})$, a subgroup $G \leq \pi_1(\mathcal{S})$ determines a regular covering $\tilde{S} \to S$ with $CTP(p) \cong G$,

Universal cover over $\mathcal{S}$ a simply connected surface $\widetilde{\mathcal{S}}$ $(1 \leq \pi_1(\mathcal{S})$ covering all the covers of $\mathcal{S}$;

Fundamental group a surface of genus g

$$\pi_1(\mathcal{S}_g) = \langle a_1, b_1, \ldots, a_\gamma, b_\gamma \mid \prod_{i=1}^{\gamma} [a_i, b_i] \rangle.$$

group - covering duality and universal covers

Smooth coverings - classical topic in homotopy theory:

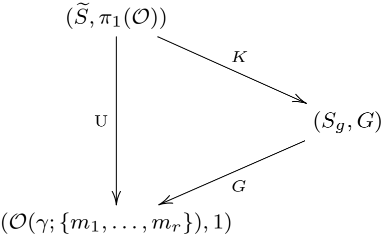
Fundamental group $\pi_1(\mathcal{S})$ elements – (eq. classes of) closed curves is $\mathcal{S}$, contractible (closed) curves are identities, operation – composition of curves;

subgroup/covering correspondence a covering determines a subgroup of $\pi_1(\mathcal{S})$, a subgroup $G \leq \pi_1(\mathcal{S})$ determines a regular covering $\tilde{S} \to S$ with $CTP(p) \cong G$,

Universal cover over $\mathcal{S}$ a simply connected surface $\widetilde{\mathcal{S}}$ ($1 \leq \pi_1(\mathcal{S})$ covering all the covers of $\mathcal{S}$;

Fundamental group a surface of genus g

$$\pi_1(\mathcal{S}_g) = \langle a_1, b_1, \ldots, a_\gamma, b_\gamma \mid \prod_{i=1}^{\gamma} [a_i, b_i] \rangle.$$



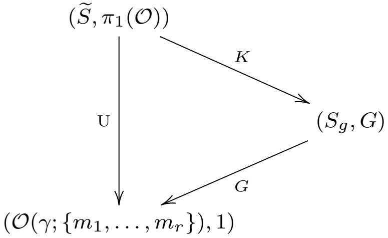
A g-admissible group G is an epimorphic image $G \cong \pi_1(\mathcal{O})/K$ for a quotient orbifold $S_g/G = \mathcal{O}(\gamma; \{m_1, m_2, \ldots, m_r\})$.

Fuchsian group $F(\gamma; \{m_1, m_2, \ldots, m_r\})$

$$\langle x_1, \ldots, x_r, a_1, b_1, \ldots, a_\gamma, b_\gamma \mid x_1^{m_1} = x_2^{m_2} = \ldots = x_r^{m_r} = 1, \prod_{i=1}^{\gamma} [a_i, b_i] \prod_{j=1}^{r} x_j = 1 \rangle.$$

is the orbifold fundamental group $\pi_1(\mathcal{O})$.

Generalization to orbifolds: Monodromy/automorphism duality

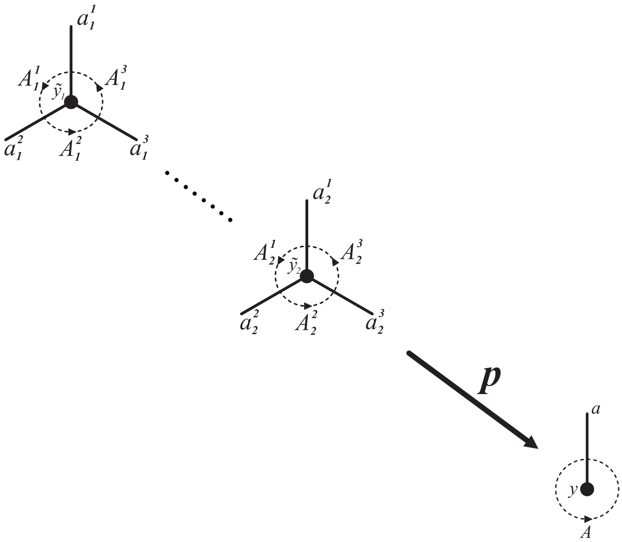


A g-admissible group G is an epimorphic image $G \cong \pi_1(\mathcal{O})/K$ for a quotient orbifold $S_g/G = \mathcal{O}(\gamma; \{m_1, m_2, \ldots, m_r\})$.

Fuchsian group $F(\gamma; \{m_1, m_2, \ldots, m_r\})$

$$\langle x_1, \ldots, x_r, a_1, b_1, \ldots, a_\gamma, b_\gamma \mid x_1^{m_1} = x_2^{m_2} = \ldots = x_r^{m_r} = 1,$$
$$\prod_{i=1}^{\gamma} [a_i, b_i] \prod_{j=1}^{r} x_j = 1\rangle.$$

is the orbifold fundamental group $\pi_1(\mathcal{O})$.



a_1^1
A_1^1
A_1^3
$\tilde{y}_1$
a_1^2
A_1^2
a_1^3
a_2^1
A_2^1
A_2^3
$\tilde{y}_2$
a_2^2
A_2^2
a_2^3
p
a
y
A

Classification, first step: numeric solution of R-H

Rieman-Hurwitz equation

$$2-2g=|\mathrm{G}|\left(2-2\gamma-\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right)$$

We have to meet the following criteria:

1. $\gamma \leq g$,
2. $r \leq 2g+2$,
3. $\forall i : m_i \geq 2 \in \mathbb{Z}$,
4. $\forall i : m_i$ is a divisor of $|\mathrm{G}|$,
5. $|\mathrm{G}| \leq 84(g-1)$.

We obtain the list of (possible!!!) signatures of orbifolds

$$(\gamma; \{m_1, \ldots, m_r\}).$$

Not every signature is g-admissible – RHE holds, but an action of G does not exist.

Classification, first step: numeric solution of R-H

Rieman-Hurwitz equation

$$2 - 2g = |\mathrm{G}| \left(2 - 2\gamma - \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)\right)$$

We have to meet the following criteria:

1. $\gamma \leq g$,
2. $r \leq 2g + 2$,
3. $\forall i : m_i \geq 2 \in \mathbb{Z}$,
4. $\forall i : m_i$ is a divisor of $|\mathrm{G}|$,
5. $|\mathrm{G}| \leq 84(g - 1)$.

We obtain the list of (possible!!!) signatures of orbifolds

$$(\gamma; \{m_1, \ldots, m_r\}).$$

Not every signature is g-admissible – RHE holds, but an action of G does not exist.

Classification, first step: numeric solution of R-H

Rieman-Hurwitz equation

$$2-2g=|\mathrm{G}|\left(2-2\gamma-\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right)$$

We have to meet the following criteria:

1. $\gamma \leq g$,
2. $r \leq 2g+2$,
3. $\forall i : m_i \geq 2 \in \mathbb{Z}$,
4. $\forall i : m_i$ is a divisor of $|\mathrm{G}|$,
5. $|\mathrm{G}| \leq 84(g-1)$.

We obtain the list of (possible!!!) signatures of orbifolds

$$(\gamma; \{m_1, \ldots, m_r\}).$$

Not every signature is g-admissible – RHE holds, but an action of G does not exist.

Arithmetics vs. Group Theory, genus 2 actions

$\lvert G\rvert$	Orbifold	Actions	$\lvert G\rvert$	Orbifold	Actions
1	$(2;\{\})$	1	12	$(0;\{3,2,2,2\})$	D_{12}
2	$(1;\{2,2\})$	C_2	12	$(0;\{4,4,3\})$	$C_3 : C_4$
2	$(0;\{2,2,2,2,2,2\})$	C_2	12	$(0;\{6,3,3\})$	—
3	$(1;\{3\})$	—	12	$(0;\{6,6,2\})$	$C_6 \times C_2$
3	$(0;\{3,3,3,3\})$	C_3	12	$(0;\{12,4,2\})$	—
4	$(1;\{2\})$	—	15	$(0;\{5,3,3\})$	—
4	$(0;\{2,2,2,2,2\})$	$C_2 \times C_2$	16	$(0;\{8,4,2\})$	QD_{16}
4	$(0;\{4,4,2,2\})$	C_4	18	$(0;\{18,3,2\})$	—
5	$(0;\{5,5,5\})$	C_5	20	$(0;\{5,5,2\})$	—
6	$(0;\{3,3,2,2\})$	$C_6,\ S_3$	24	$(0;\{4,3,3\})$	$SL(2,3)$
6	$(0;\{6,2,2,2\})$	—	24	$(0;\{6,4,2\})$	$(C_6 \times C_2) : C_2$
6	$(0;\{6,6,3\})$	C_6	24	$(0;\{12,3,2\})$	—
8	$(0;\{4,2,2,2\})$	D_8	30	$(0;\{10,3,2\})$	—
8	$(0;\{4,4,4\})$	Q_8	36	$(0;\{9,3,2\})$	—
8	$(0;\{8,8,2\})$	C_8	40	$(0;\{5,4,2\})$	—
9	$(0;\{9,3,3\})$	—	48	$(0;\{8,3,2\})$	$GL(2,3)$
10	$(0;\{10,5,2\})$	C_{10}	84	$(0;\{7,3,2\})$	—

How to determine a discrete action?

A presentation of G:

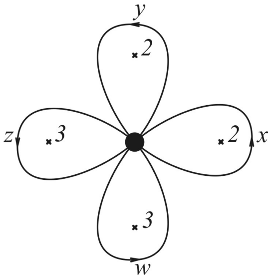
$$G = \langle x_1, \ldots, x_r, a_1, b_1, \ldots, a_\gamma, b_\gamma \mid x_1^{m_1} = x_2^{m_2} = \ldots = x_r^{m_r} = 1, \prod_{i=1}^{\gamma} [a_i, b_i] \prod_{j=1}^{r} x_j = 1, \ldots \rangle.$$

The "dots" make the group finite!!! Then action is then determined by the canonical Cayley map.

Canonical one-vertex map

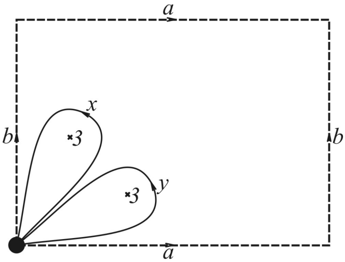
1. canonical quotient map $\bar{\mathbf{M}}$ is a bouquet of r loops,
2. every loop is the boundary of a face containing exactly one branch point with respective branch index m_i,
3. outer face of the map is an r-gon containing no branch point.

$\mathcal{O}(0; \{2, 2, 3, 3\})$



$\langle x, y, z, w \mid x^2 = y^2 = z^3 = w^3 = xyzw = 1\rangle$

$\mathcal{O}(1; \{3, 3\})$



$\langle x, y, a, b \mid x^3 = y^3 = [a, b]xy = 1\rangle$

Fundamental domain and dual of Cayley map

1. In general we do not need to describe the action through the canonical base map, but then we need to change the presentation or to compute the voltages in the generators used in presentation,
2. There are infinitely many one-vertex maps on a prescribed orbifold $\mathcal{O}(\gamma; \{m_0, m_1, m_2 \ldots, m_n\})$, one needs to assume at least non-degeneracy, then the valency of the quotient is bounded by a function of g.
3. All these describe the same action of G on S_g.

Step 2: Determining G for given g-admissible signature and $|G|$

1. $|F : K| = |G|$, F is the Fuchsian group of given signature,
2. the epimorphism $F \to G$ with kernel K is order preserving, i.e. no generator is send to identity and the orders of elliptic elements are preserved
3. `LowIndexNormalSubgroups` (Magma, D. Holt) procedure is the tool to determine all the normal subgroups of given order,
4. As a result we get the $Epi_0(F, G)$ kernels determining the g-admissible actions of prescribed orbifold type.
5. AT PRESENT WE HAVE COMPLETED THE LIST OF ACTIONS UP TO GENUS 8, see `http://www.savbb.sk/~karabas/science.html#rhsu`
6. Abstract structure of groups and g-admissible orbifold types were dermined up to genus 24, for large groups much further (see Conder's web page)
7. Small 9-admissible troublemakers with more than 10^5 kernels: $C_2 \times C_2$ of types $(1; 2^8)$ or $(0; 2^{12})$, $C_2 \times C_2 \times C_2$ of types $(1; 4^2, 2)$ or $(0; 2^8)$, D_8 of type $(0; 2^8)$.

Step 2: Determining G for given g-admissible signature and $|G|$

1. $|F : K| = |G|$, F is the Fuchsian group of given signature,
2. the epimorphism $F \to G$ with kernel K is order preserving, i.e. no generator is send to identity and the orders of elliptic elements are preserved
3. `LowIndexNormalSubgroups` (Magma, D. Holt) procedure is the tool to determine all the normal subgroups of given order,
4. As a result we get the $Epi_0(F, G)$ kernels determining the g-admissible actions of prescribed orbifold type.
5. AT PRESENT WE HAVE COMPLETED THE LIST OF ACTIONS UP TO GENUS 8, see `http://www.savbb.sk/~karabas/science.html#rhsu`
6. Abstract structure of groups and g-admissible orbifold types were dermined up to genus 24, for large groups much further (see Conder's web page)
7. Small 9-admissible troublemakers with more than 10^5 kernels: $C_2 \times C_2$ of types $(1; 2^8)$ or $(0; 2^{12})$, $C_2 \times C_2 \times C_2$ of types $(1; 4^2, 2)$ or $(0; 2^8)$, D_8 of type $(0; 2^8)$.

Step 2: Determining G for given g-admissible signature and $|G|$

1. $|F : K| = |G|$, F is the Fuchsian group of given signature,
2. the epimorphism $F \to G$ with kernel K is order preserving, i.e. no generator is send to identity and the orders of elliptic elements are preserved
3. `LowIndexNormalSubgroups` (Magma, D. Holt) procedure is the tool to determine all the normal subgroups of given order,
4. As a result we get the $Epi_0(F, G)$ kernels determining the g-admissible actions of prescribed orbifold type.
5. AT PRESENT WE HAVE COMPLETED THE LIST OF ACTIONS UP TO GENUS 8, see `http://www.savbb.sk/~karabas/science.html#rhsu`
6. Abstract structure of groups and g-admissible orbifold types were dermined up to genus 24, for large groups much further (see Conder's web page)
7. Small 9-admissible troublemakers with more than 10^5 kernels: $C_2 \times C_2$ of types $(1; 2^8)$ or $(0; 2^{12})$, $C_2 \times C_2 \times C_2$ of types $(1; 4^2, 2)$ or $(0; 2^8)$, D_8 of type $(0; 2^8)$.

Step 2: Determining G for given g-admissible signature and $|G|$

1. $|F : K| = |G|$, F is the Fuchsian group of given signature,
2. the epimorphism $F \to G$ with kernel K is order preserving, i.e. no generator is send to identity and the orders of elliptic elements are preserved
3. `LowIndexNormalSubgroups` (Magma, D. Holt) procedure is the tool to determine all the normal subgroups of given order,
4. As a result we get the $Epi_0(F, G)$ kernels determining the g-admissible actions of prescribed orbifold type.
5. AT PRESENT WE HAVE COMPLETED THE LIST OF ACTIONS UP TO GENUS 8, see `http://www.savbb.sk/~karabas/science.html#rhsu`
6. Abstract structure of groups and g-admissible orbifold types were dermined up to genus 24, for large groups much further (see Conder's web page)
7. Small 9-admissible troublemakers with more than 10^5 kernels: $C_2 \times C_2$ of types $(1; 2^8)$ or $(0; 2^{12})$, $C_2 \times C_2 \times C_2$ of types $(1; 4^2, 2)$ or $(0; 2^8)$, D_8 of type $(0; 2^8)$.

Step 2: Determining G for given g-admissible signature and $|G|$

1. $|F : K| = |G|$, F is the Fuchsian group of given signature,
2. the epimorphism $F \to G$ with kernel K is order preserving, i.e. no generator is send to identity and the orders of elliptic elements are preserved
3. `LowIndexNormalSubgroups` (Magma, D. Holt) procedure is the tool to determine all the normal subgroups of given order,
4. As a result we get the $Epi_0(F, G)$ kernels determining the g-admissible actions of prescribed orbifold type.
5. AT PRESENT WE HAVE COMPLETED THE LIST OF ACTIONS UP TO GENUS 8, see `http://www.savbb.sk/~karabas/science.html#rhsu`
6. Abstract structure of groups and g-admissible orbifold types were dermined up to genus 24, for large groups much further (see Conder's web page)
7. Small 9-admissible troublemakers with more than 10^5 kernels: $C_2 \times C_2$ of types $(1; 2^8)$ or $(0; 2^{12})$, $C_2 \times C_2 \times C_2$ of types $(1; 4^2, 2)$ or $(0; 2^8)$, D_8 of type $(0; 2^8)$.

Step 2: Determining G for given g-admissible signature and $|G|$

1. $|F : K| = |G|$, F is the Fuchsian group of given signature,
2. the epimorphism $F \to G$ with kernel K is order preserving, i.e. no generator is send to identity and the orders of elliptic elements are preserved
3. `LowIndexNormalSubgroups` (Magma, D. Holt) procedure is the tool to determine all the normal subgroups of given order,
4. As a result we get the $Epi_0(F, G)$ kernels determining the g-admissible actions of prescribed orbifold type.
5. AT PRESENT WE HAVE COMPLETED THE LIST OF ACTIONS UP TO GENUS 8, see `http://www.savbb.sk/~karabas/science.html#rhsu`
6. Abstract structure of groups and g-admissible orbifold types were dermined up to genus 24, for large groups much further (see Conder's web page)
7. Small 9-admissible troublemakers with more than 10^5 kernels: $C_2 \times C_2$ of types $(1; 2^8)$ or $(0; 2^{12})$, $C_2 \times C_2 \times C_2$ of types $(1; 4^2, 2)$ or $(0; 2^8)$, D_8 of type $(0; 2^8)$.

Step 2: Determining G for given g-admissible signature and $|G|$

1. $|F:K| = |G|$, F is the Fuchsian group of given signature,
2. the epimorphism $F \to G$ with kernel K is order preserving, i.e. no generator is send to identity and the orders of elliptic elements are preserved
3. `LowIndexNormalSubgroups` (Magma, D. Holt) procedure is the tool to determine all the normal subgroups of given order,
4. As a result we get the $Epi_0(F,G)$ kernels determining the g-admissible actions of prescribed orbifold type.
5. AT PRESENT WE HAVE COMPLETED THE LIST OF ACTIONS UP TO GENUS 8, see `http://www.savbb.sk/~karabas/science.html#rhsu`
6. Abstract structure of groups and g-admissible orbifold types were dermined up to genus 24, for large groups much further (see Conder's web page)
7. Small 9-admissible troublemakers with more than 10^5 kernels: $C_2 \times C_2$ of types $(1; 2^8)$ or $(0; 2^{12})$, $C_2 \times C_2 \times C_2$ of types $(1; 4^2, 2)$ or $(0; 2^8)$, D_8 of type $(0; 2^8)$.

A note on the equivalence classes of the actions (coverings)

1. The above procedure describe basic equivalence classes enumerated by $Epi_0(F, G)$
2. Outer automorphisms of F or G can be used to identify some of the actions, the chosen equivalence depends on the context

A note on the equivalence classes of the actions (coverings)

1. The above procedure describe basic equivalence classes enumerated by $Epi_0(F,G)$
2. Outer automorphisms of F or G can be used to identify some of the actions, the chosen equivalence depends on the context

A sample of results: Maximal actions

g	$\lvert G \rvert$	Orbifold	$\mathrm{Epi}_{\mathcal{O}}(\mathcal{S}_g, \mathrm{G})$	G
2	48	$(0, \{8,3,2\})$	2	$GL(2,3)$
3	168	$(0, \{7,3,2\})$	2	$PSL(3,2)$
4	120	$(0, \{5,4,2\})$	1	S_5
5	192	$(0, \{8,3,2\})$	4	$(((C_4 \times C_2) : C_4) : C_3) : C_2$
6	150	$(0, \{10,3,2\})$	4	$((C_5 \times C_5) : C_3) : C_2$
7	504	$(0, \{7,3,2\})$	3	$PSL(2,8)$
8	336	$(0, \{8,3,2\})$	2	$PSL(3,2) : C_2$
9	320	$(0, \{5,4,2\})$	4	$(((C_2 \times Q_8) : C_2) : C_5) : C_2$
10	432	$(0, \{8,3,2\})$	2	$(((C_3 \times C_3) : Q_8) : C_3) : C_2$
11	240	$(0, \{6,4,2\})$	2	$C_2 \times S5$
12	120	$(0, \{15,4,2\})$	4	$(C_5 \times A_4) : C_2$
13	360	$(0, \{10,3,2\})$	2	$A_5 \times S_3$
14	1092	$(0, \{7,3,2\})$	6	$PSL(2,13)$
15	504	$(0, \{9,3,2\})$	3	$PSL(2,8)$

Maximal actions – continued

g	$\lvert G \rvert$	Orbifold	$\mathrm{Epi}_{\mathcal{O}}(\mathcal{S}_g, \mathrm{G})$	G
16	720	$(0, \{8,3,2\})$	2	$A_6 : C_2$
17	1344	$(0, \{7,3,2\})$	2	$(C_2 \times C_2 \times C_2).PSL(3,2)$
18	168	$(0, \{21,4,2\})$	6	$(C_7 \times A_4) : C_2$
19	720	$(0, \{5,4,2\})$	4	$C_2 \times A_6$
20	228	$(0, \{6,6,2\})$	24	$C_2 \times ((C_{19} : C_3) : C_2)$
21	480	$(0, \{6,4,2\})$	2	$(C_2 \times C_2 \times A_5) : C_2$
22	1008	$(0, \{8,3,2\})$	4	$(C_3 \times PSL(3,2)) : C_2$
23	192	$(0, \{48,4,2\})$	8	$(C_3 \times (C_{16} : C_2)) : C_2$
24	216	$(0, \{27,4,2\})$	9	$((C_2 \times C_2) : C_{27}) : C_2$

A sample of classification: small discrete groups

g	Signature	$\#C_2$	$\#C_4 \times C_2$	g	Signature	$\#C_2$	$\#C_4 \times C_2$
2	$(1, \{2^2\})$	4		6	$(2, \{2^6\})$	16	
2	$(0, \{2^6\})$	1		6	$(1, \{2^{10}\})$	4	
3	$(2, \{\})$	15		6	$(0, \{2^{14}\})$	1	
3	$(1, \{2^4\})$	4		7	$(4, \{\})$	255	
3	$(0, \{2^8\})$	1		7	$(3, \{2^4\})$	64	
3	$(0, \{4^2, 2^2\})$		32	7	$(2, \{2^8\})$	16	
4	$(2, \{2^2\})$	16		7	$(1, \{2^{12}\})$	4	
4	$(1, \{2^6\})$	4		7	$(0, \{2^{16}\})$	1	
4	$(0, \{2^{10}\})$	1		7	$(1, \{2^3\})$		288
5	$(3, \{\})$	63		7	$(1, \{4^2\})$		192
5	$(2, \{2^4\})$	16		7	$(0, \{4^2, 2^4\})$		320
5	$(1, \{2^8\})$	4		7	$(0, \{4^4, 2\})$		176
5	$(0, \{2^{12}\})$	1		8	$(4, \{2^2\})$	256	
5	$(1, \{2^2\})$		120	8	$(3, \{2^6\})$	64	
5	$(0, \{4^2, 2^3\})$		104	8	$(2, \{2^{10}\})$	16	
5	$(0, \{4^4\})$		48	8	$(1, \{2^{14}\})$	4	
6	$(3, \{2^2\})$	64		8	$(0, \{2^{18}\})$	1	

Back to PROBLEM A.

Classification (of a subclass) of non-degenerate vertex-transitive maps of genus g

1. Observation, $\bar{M} = M/Aut^+(M)$ is a one-vertex or a two-vertex map on a g-admissible orbifold,
2. By non-degeneracy the valency of $\bar{M}$ bounded by $3 + \sqrt{12g - 2}$,
3. For each g-admissible orbifold O there are just finitely many such base maps,
4. The lifts over $\bar{M}$ can be described using voltage assignments defined in the discrete group G giving the quotient orbifold $O = S_g/G$. (a modified Gross-Tucker)

Back to PROBLEM A.

Classification (of a subclass) of non-degenerate vertex-transitive maps of genus g

1. Observation, $\bar{M} = M/Aut^+(M)$ is a one-vertex or a two-vertex map on a g-admissible orbifold,
2. By non-degeneracy the valency of $\bar{M}$ bounded by $3 + \sqrt{12g - 2}$,
3. For each g-admissible orbifold O there are just finitely many such base maps,
4. The lifts over $\bar{M}$ can be described using voltage assignments defined in the discrete group G giving the quotient orbifold $O = S_g/G$. (a modified Gross-Tucker)

Back to PROBLEM A.

Classification (of a subclass) of non-degenerate vertex-transitive maps of genus g

1. Observation, $\bar{M} = M/Aut^+(M)$ is a one-vertex or a two-vertex map on a g-admissible orbifold,
2. By non-degeneracy the valency of $\bar{M}$ bounded by $3 + \sqrt{12g - 2}$,
3. For each g-admissible orbifold O there are just finitely many such base maps,
4. The lifts over $\bar{M}$ can be described using voltage assignments defined in the discrete group G giving the quotient orbifold $O = S_g/G$. (a modified Gross-Tucker)

Back to PROBLEM A.

Classification (of a subclass) of non-degenerate vertex-transitive maps of genus g

1. Observation, $\bar{M} = M/Aut^+(M)$ is a one-vertex or a two-vertex map on a g-admissible orbifold,
2. By non-degeneracy the valency of $\bar{M}$ bounded by $3 + \sqrt{12g-2}$,
3. For each g-admissible orbifold O there are just finitely many such base maps,
4. The lifts over $\bar{M}$ can be described using voltage assignments defined in the discrete group G giving the quotient orbifold $O = S_g/G$. (a modified Gross-Tucker)

Reconstruction of maps by voltage assignments

By a T-reduced voltage assignment on N we mean a mapping $\xi : V \cup D \to G$ taking values in a group G satisfying the following conditions:

1. all darts on the rooted spanning tree (T, x_0) receive trivial voltages,
2. $\xi_{xL} = \xi_x^{-1}$ for all $x \in D$,
3. $G = \langle \{\xi_x : x \in D \cup V\} \rangle$.

The derived map $M = N^\xi = (D^\xi; R^\xi, L^\xi)$ is defined as follows. Then $D^\xi = D \times G$ and

$$(x,g)R^\xi = \begin{cases} (xR, g \cdot \xi_v), & x \in D^+(T) \cup \{x_0\}, \\ (xR, g), & \text{otherwise} \end{cases}$$

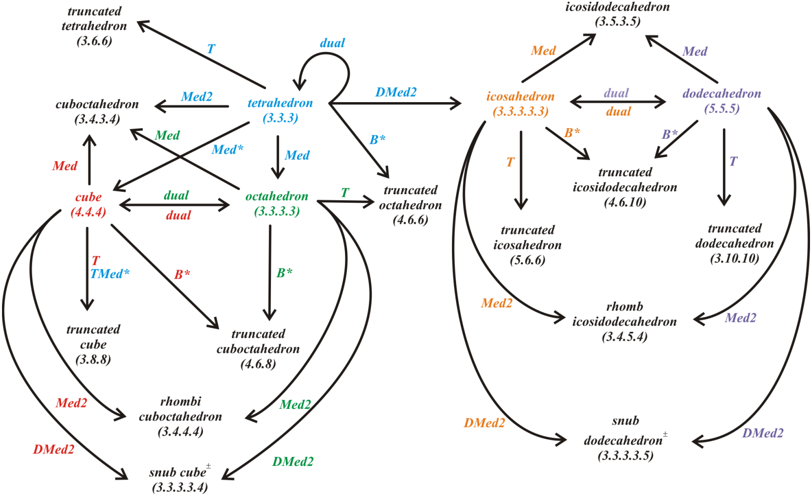
$$(x,g)L^\xi = (xL, g \cdot \xi_x)$$

If $\xi|_V = id$, then N^ξ coincides with the classic construction by Gross & Tucker (1987).

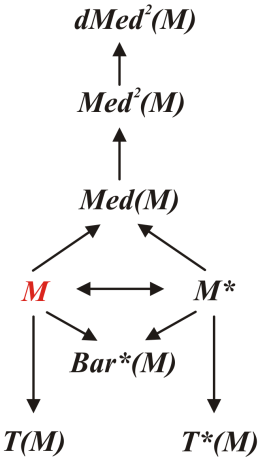
Classification of particular families of vertex- or edge-transitive maps

1. Regular maps and hypermaps - actions of type $(0;k,m,n)$, Conder for genus $g \leq 101$, some infinite families $\chi = -2p$,
2. V.T. polyhedral maps up to genus 4, K+N, Math. Comp.2012,
3. edge-transitive maps up to genus 4, Orbanic, Pisanski,... 2011
4. using the Conder's lists of g-admissible actions of types $(0;k,m,l)$, $(0;k,m,n,l)$, $(1;k)$ and $(1;k,m)$ we get the edge-transitive maps up to genus 100, more in the following talk by J.K.,
5. v-t maps of type $(0;k,m,2)$

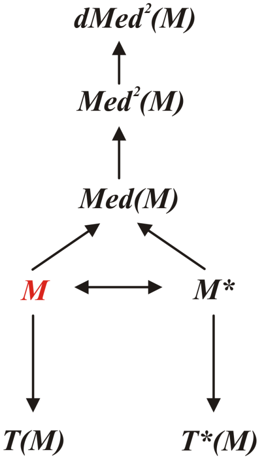
Spherical case: all v-t maps (Archimedean solids) come from Platonic maps



Operations over not self-dual maps

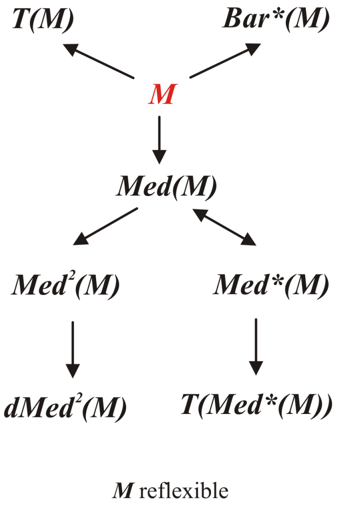


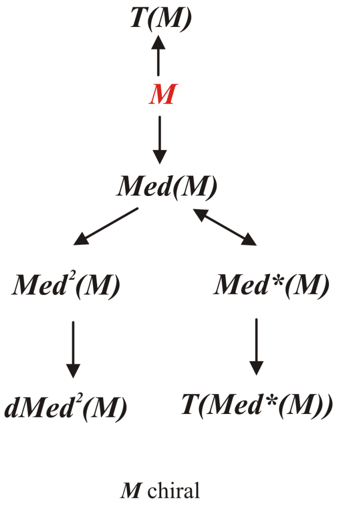
M reflexible



M chiral

Operations over self-dual maps





PROBLEM B: Back to map enumeration - Dictionary:

Rooted ormap $=$ a torsion-free subgroup of $Z * Z_2$ of finite index,

Rooted map $=$ a torsion-free subgroup of $Z_2 * (Z_2 \times Z_2)$ of finite index,

Labelled ormap $=$ a transitive homomorphism $Z * Z_2 \to S_n$,

Labelled map $=$ a transitive homomorphism $Z * (Z_2 \times Z_2) \to S_n$,

Isoclass of an ormap $=$ conjugacy class of a torsion-free subgroup of $Z * Z_2$,

Isoclass of a map $=$ conjugacy class of a torsion-free subgroup of $Z * (Z_2 \times Z_2)$

WHY ROOTED MAPS? WHY LABELLED MAPS?
!!! **Action of $\mathrm{G}(M)$ is semiregular** !!!

Mednykh s lemma: From subgroups to conjugacy classes

RECALL the dictionary: ISOCLASS of (OR)MAPS = **conjugacy class of subgroups** in the universal group

Theorem (Mednykh)

Let Γ be a finitely generated group. Let $\mathcal{P}$ be a set of subgroups of Γ closed under conjugation. Then the number of conjugacy classes of subgroups of index n in $\mathcal{P}$ is given by the formula

$$N_{\Gamma}^{\mathcal{P}}(n) = \frac{1}{n} \sum_{\substack{\ell \mid n \\ \ell m = n}} \sum_{\substack{K < \Gamma \\ [\Gamma : K] = m}} Epi_{\mathcal{P}}(K, Z_{\ell}).$$

$Epi_{\mathcal{P}}(K, Z_{\ell})$ - number of order preserving epimorphisms $\Gamma \to Z_{\ell}$ s. t. the kernel $\in \mathcal{P}$.

Mednykh Lemma for $\Gamma = \Delta(\infty, \infty, 2) \cong Z * Z_2$

$$\Theta_g(e) = \frac{1}{2e} \sum_{\ell | e} \sum_{\substack{O \in Orb(S_g/Z_\ell) \\ O = [g; 2^{q_2}, 3^{q_3} \ldots, \ell^{q_\ell}]}} Epi_0(\pi_1(O), Z_\ell) \cdot \mu_O(m).$$

The number $Epi_0(\pi_1(O), Z_\ell)$ enumerates cyclic actions on S_g,

the number $\mu_O(m)$ enumerates the number of rooted maps on the quotient orbifold $O = S_g/Z_\ell$ with m darts.

Counting Rooted Ormaps (subgroups of given index) on closed orientable orbifolds

PROPOSITION M+N: Let $O = O[g; 2^{q_2}, \dots, \ell^{q_\ell}]$ be an orbifold, $q_i \geq 0$ for $i = 2, \dots, \ell$. Then the number of rooted maps $\nu_O(m)$ with m darts on the orbifold O is

$$\nu_O(m) = \sum_{s=0}^{q_2} \binom{m}{s} \binom{\frac{m-s}{2}+2-2g}{q_2-s, q_3, \dots, q_\ell} \mathcal{N}_g((m-s)/2),$$

with a convention that $\mathcal{N}_g(n) = 0$ if n is not an integer,
$\mathcal{N}_g(n)$ is the number of ordinary maps of genus g with n edges.

Open problems

1. Enumeration of the number of abelian g-admissible actions
2. Enumeration of reflexible maps, open even for the sphere
3. Classification of all actions for given surfaces of small $|\chi(S)|$
4. Which equivalences of the actions are of interest?
5. Assymtotic analysis and consequences
6. What can be done for higher dimensions?

Open problems

1. Enumeration of the number of abelian g-admissible actions
2. Enumeration of reflexible maps, open even for the sphere
3. Classification of all actions for given surfaces of small $|\chi(S)|$
4. Which equivalences of the actions are of interest?
5. Assymtotic analysis and consequences
6. What can be done for higher dimensions?

Open problems

1. Enumeration of the number of abelian g-admissible actions
2. Enumeration of reflexible maps, open even for the sphere
3. Classification of all actions for given surfaces of small $|\chi(S)|$
4. Which equivalences of the actions are of interest?
5. Assymtotic analysis and consequences
6. What can be done for higher dimensions?

Open problems

1. Enumeration of the number of abelian g-admissible actions
2. Enumeration of reflexible maps, open even for the sphere
3. Classification of all actions for given surfaces of small $|\chi(S)|$
4. Which equivalences of the actions are of interest?
5. Assymtotic analysis and consequences
6. What can be done for higher dimensions?

Open problems

1. Enumeration of the number of abelian g-admissible actions
2. Enumeration of reflexible maps, open even for the sphere
3. Classification of all actions for given surfaces of small $|\chi(S)|$
4. Which equivalences of the actions are of interest?
5. Assymtotic analysis and consequences
6. What can be done for higher dimensions?

Open problems

1. Enumeration of the number of abelian g-admissible actions
2. Enumeration of reflexible maps, open even for the sphere
3. Classification of all actions for given surfaces of small $|\chi(S)|$
4. Which equivalences of the actions are of interest?
5. Assymtotic analysis and consequences
6. What can be done for higher dimensions?