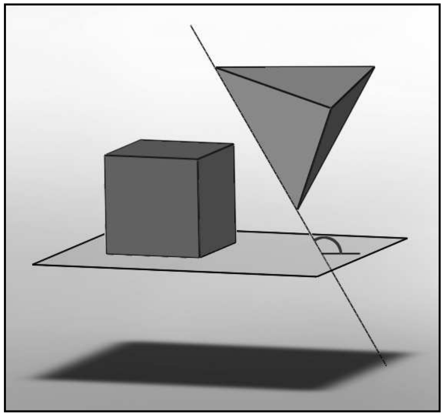
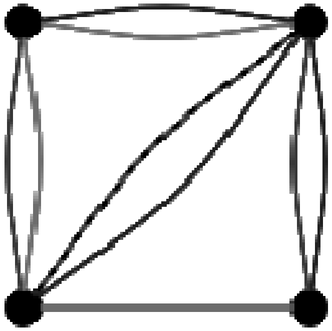
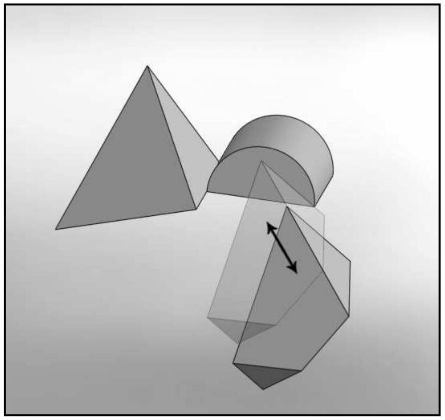






Body-and-cad Rigidity Theory

Audrey Lee-St.John

Joint work with:
Kirk Haller, Meera Sitharam, Ileana Streinu, Neil White
Jessica Sidman

Workshop on Rigidity
Fields Institute

Audrey Lee-St.John
Mount Holyoke College

October 14, 2011
Toronto, ON

Motivation: CAD

- In SolidWorks 3D assembly environment
  - Atomic elements are “sets” or rigid bodies
  - Constraints are called “mates” or “dimensions”
    - Very rich “language” to describe complicated systems

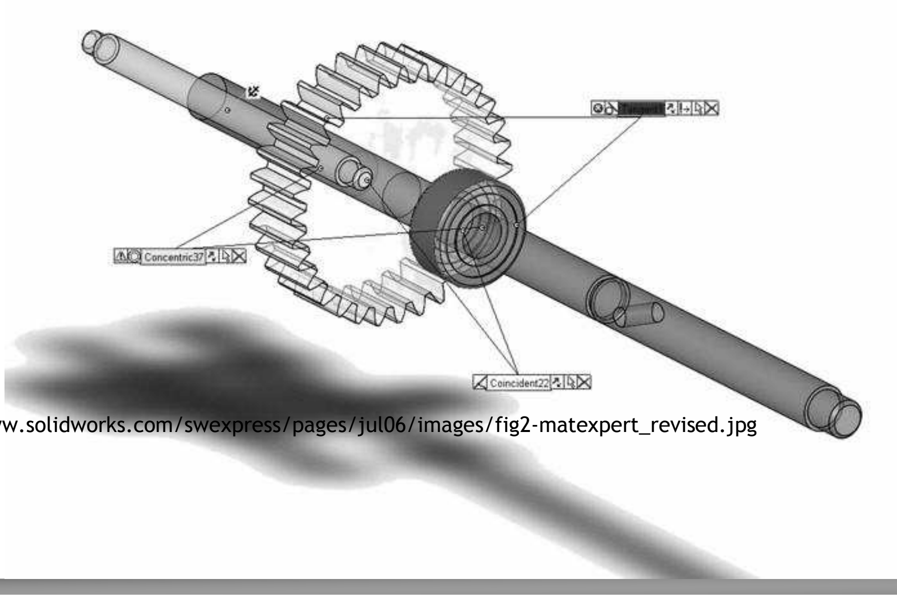


Image from http://www.solidworks.com/swexpress/pages/jul06/images/fig2-matexpert_revised.jpg

CAD constraints

- Constraints are placed by identifying *primitive geometries* on sets
  - E.g., points, lines, planes, curves
- Example constraints
  - Symmetry, angle, equality
- Current rigidity models **do not capture** such systems

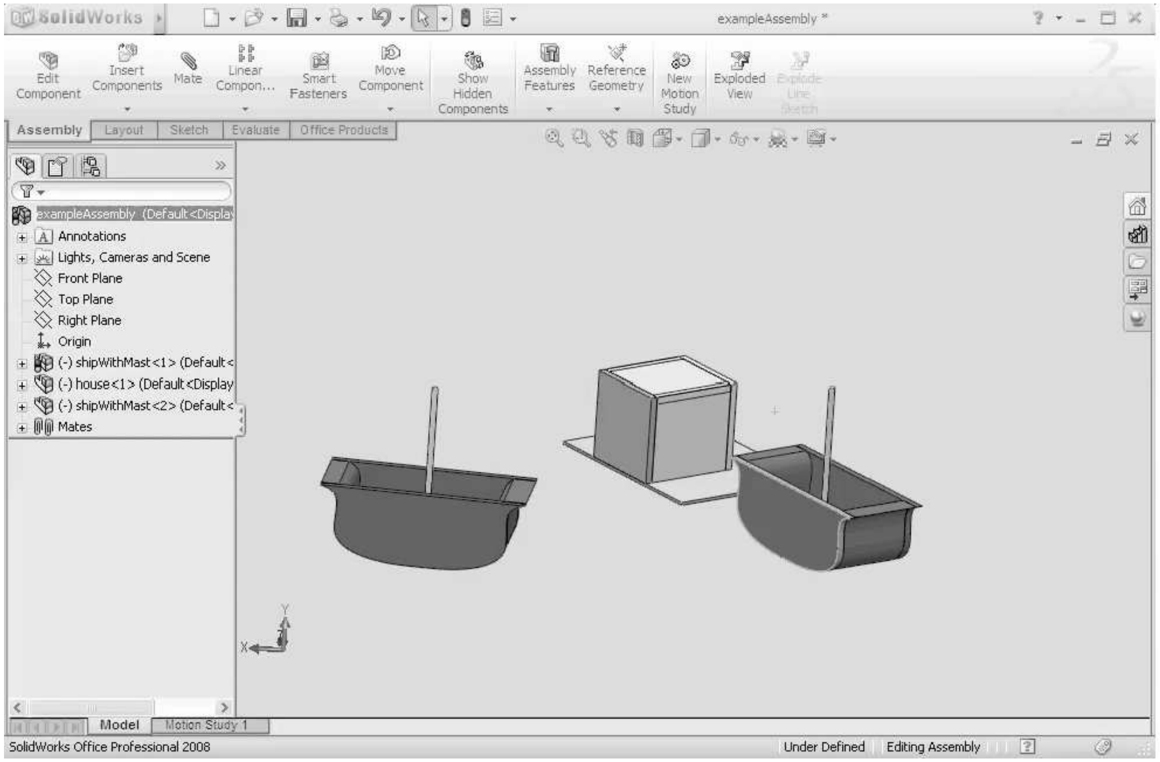
Example

- 2 rigid bodies

- Plane-plane ||
- Plane-plane coincident
- Line-plane perpendicular
- Point-point distance (bar)
- Point-point distance (bar)

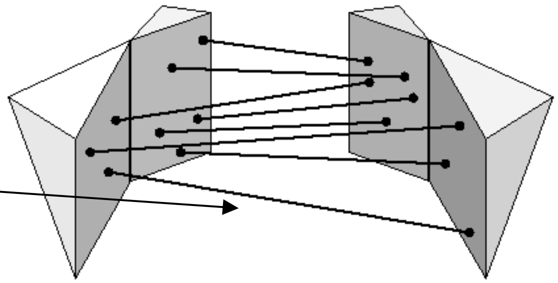
SolidWorks

- Example assembly



Known rigidity: body-bar-hinge model

- Building blocks are *rigid bodies*
- Constraints are
  - *fixed-length bars* rigidly attached to bodies by *universal joints*
  - *hinges*



3D Body-and-bar

(Generically) minimally rigid body-and-bar frameworks characterized by Tay's **counting condition**:

A body-and-bar framework is generically minimally rigid

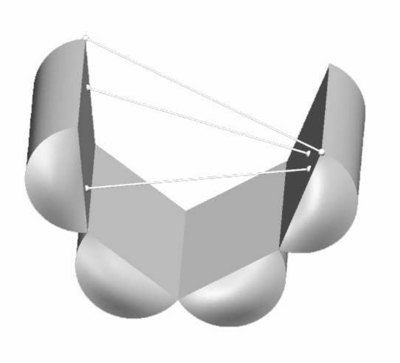
□

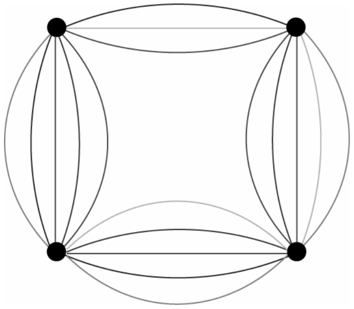
for associated graph on *n* vertices and *m* edges:

any *n'* vertices span at most *6n'-6* edges

and

m = 6n-6





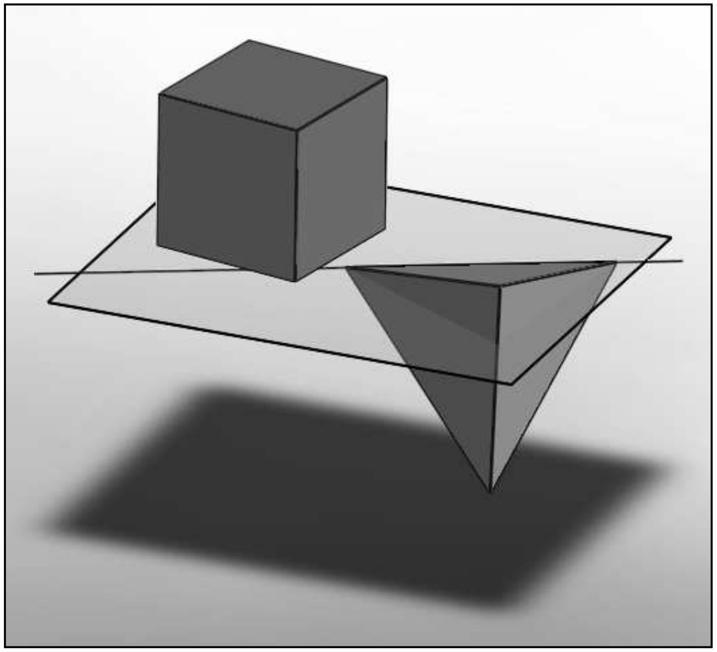
A *recipe* for rigidity theory

- Formulate algebraic rigidity concept
  - Often quadratic systems
  - Even defining rigidity is challenging and subtle!
    - Differential geometry, topology, algebraic geometry
- Study associated infinitesimal rigidity
  - Linear algebra
- Try to find combinatorial characterizations
  - Graph theory
- … that hopefully lead to efficient algorithms!
  - (Pebble games)
  - Matroid theory

Body-and-cad

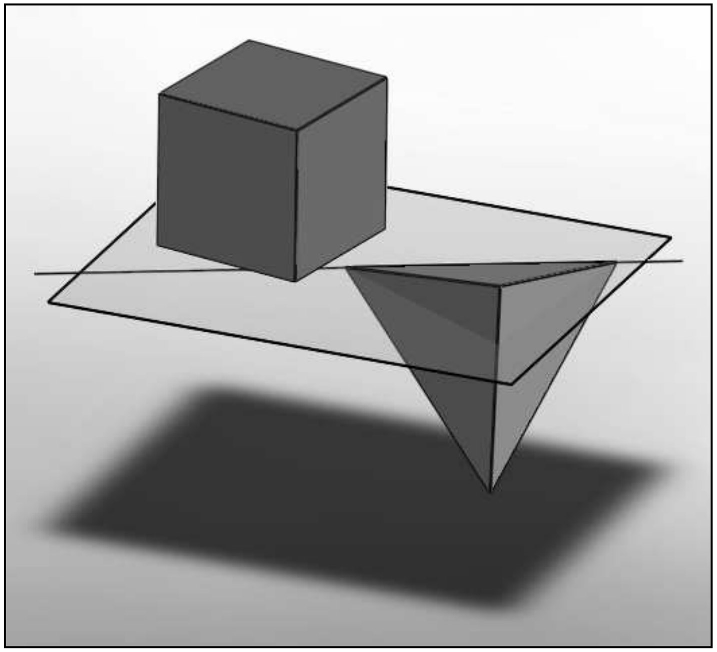
Body-and-cad frameworks

- Rigid bodies
  - Geometric elements
    - lines, planes, points
- Pairwise constraints



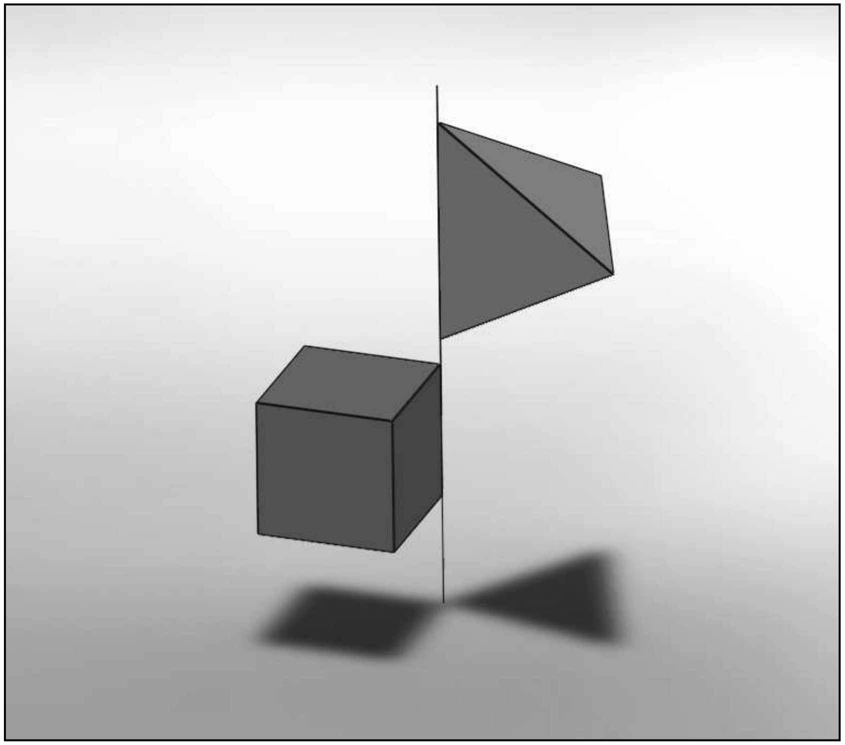
Body-and-cad frameworks

- Rigid bodies
  - Geometric elements
    - lines, planes, points
- Pairwise constraints
  - Coincidence



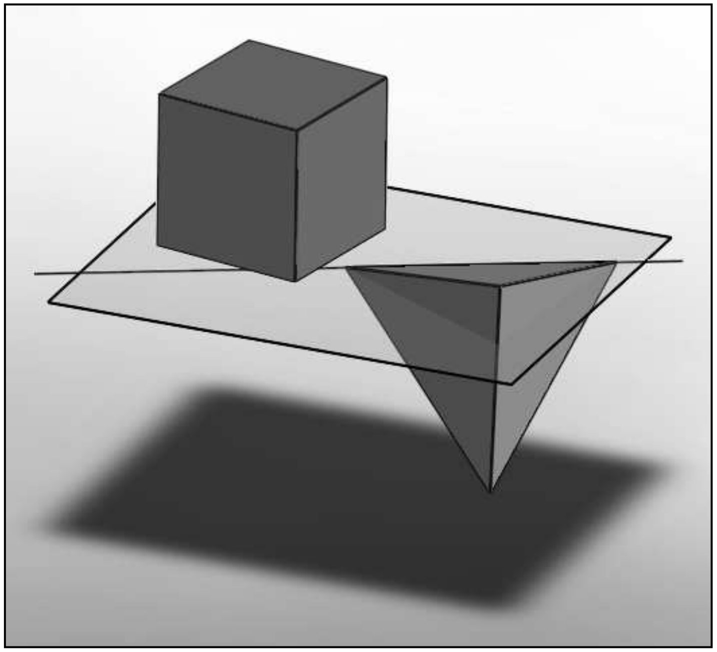
Example coincidence constraint

- Line-line coincidence



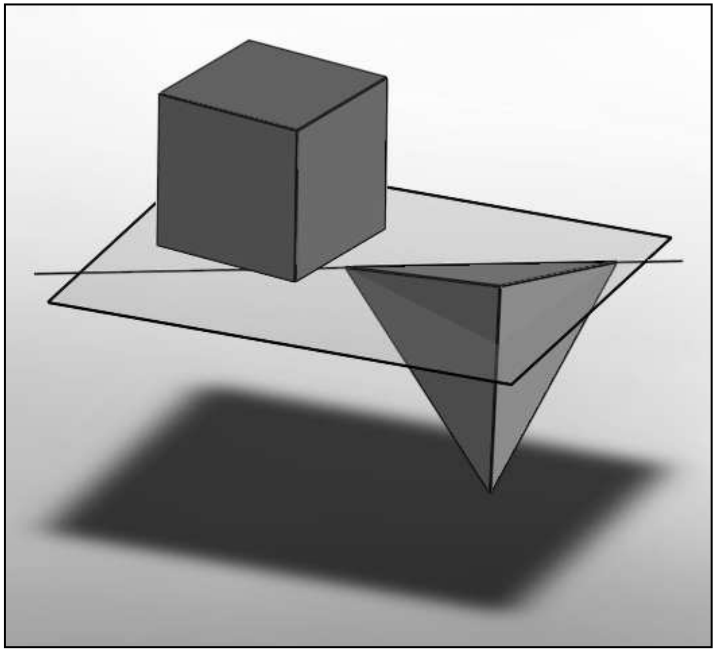
Body-and-cad frameworks

- Rigid bodies
  - Geometric elements
    - lines, planes, points
- Pairwise constraints
  - Coincidence
  - Angle



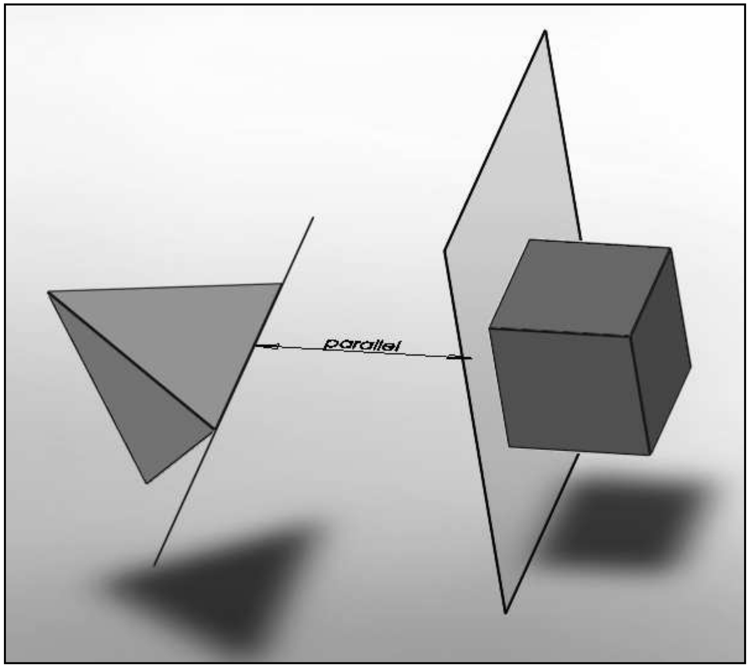
Body-and-cad frameworks

- Rigid bodies
  - Geometric elements
    - lines, planes, points
- Pairwise constraints
  - Coincidence
  - Angle
    - Parallel
    - Perpendicular
    - Arbitrary fixed angle

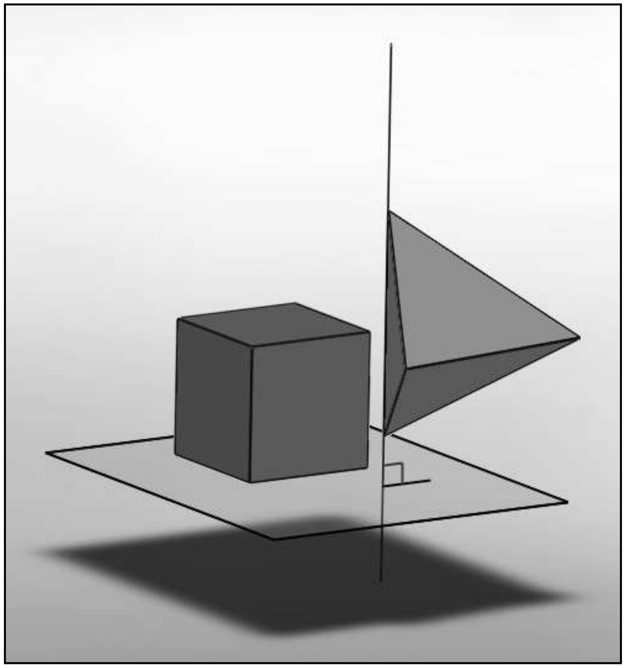


Example angle constraints

- Plane-line parallel

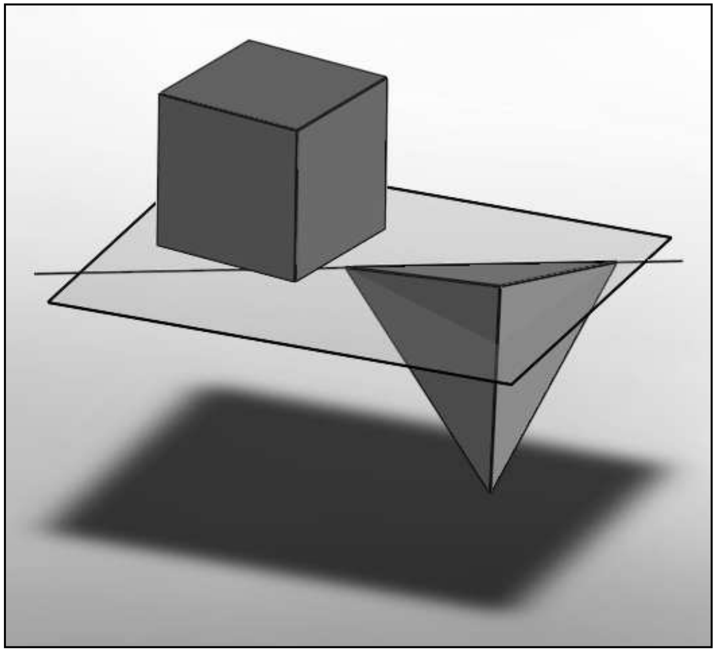


- Plane-line perpendicular



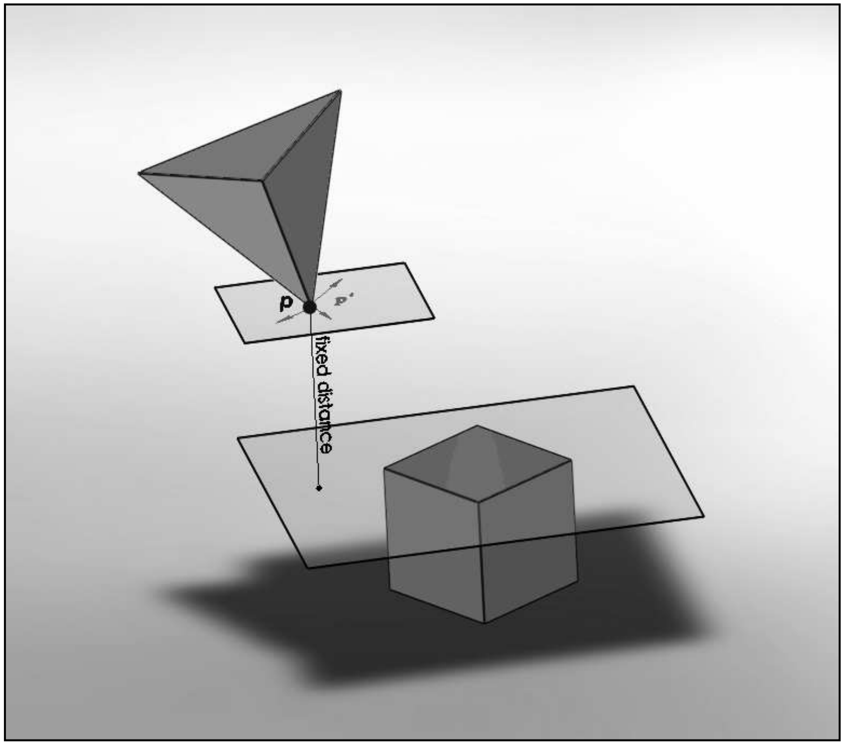
Body-and-cad frameworks

- Rigid bodies
  - Geometric elements
    - lines, planes, points
- Pairwise constraints
  - Coincidence
  - Angle
    - Parallel
    - Perpendicular
    - Arbitrary fixed angle
  - Distance



Example distance constraint

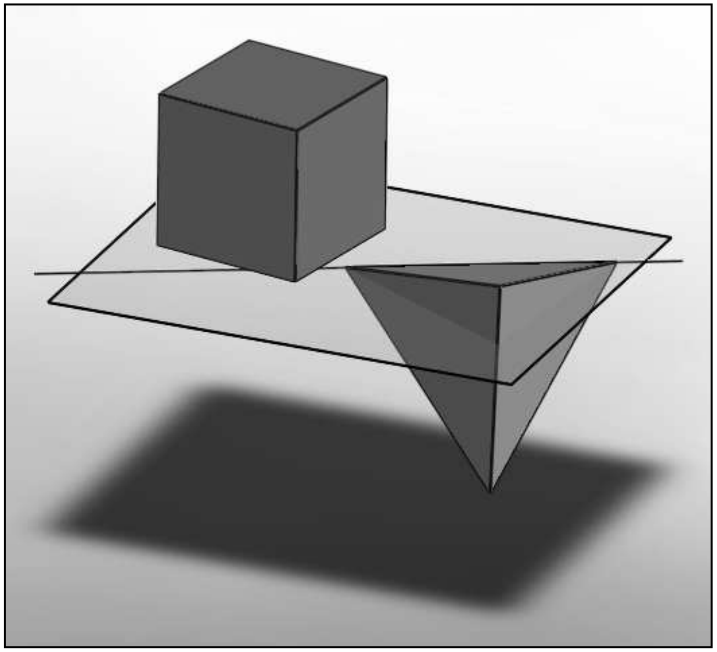
- Plane-point distance



Body-and-**cad** frameworks

- Rigid bodies
  - Geometric elements
    - lines, planes, points
- Pairwise constraints
  - **C**oincidence
  - **A**ngle
  - **D**istance

21

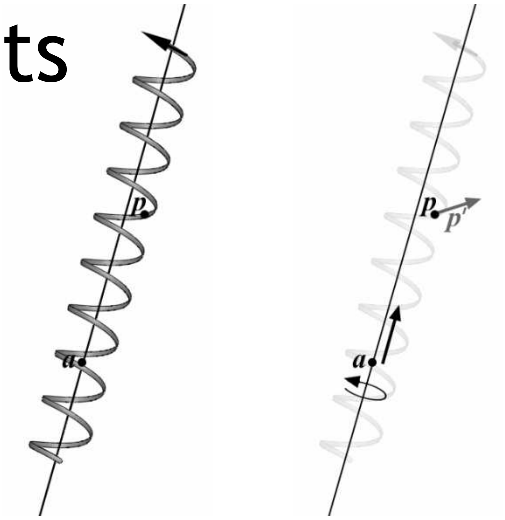


A *recipe* for rigidity theory

- Formulate algebraic rigidity concept
  - Often quadratic systems
  - Even defining rigidity is challenging and subtle!
    - Differential geometry, topology, algebraic geometry
- Study associated infinitesimal rigidity
  - Linear algebra
- Try to find combinatorial characterizations
  - Graph theory
- … that hopefully lead to efficient algorithms!
  - (Pebble games)
  - Matroid theory

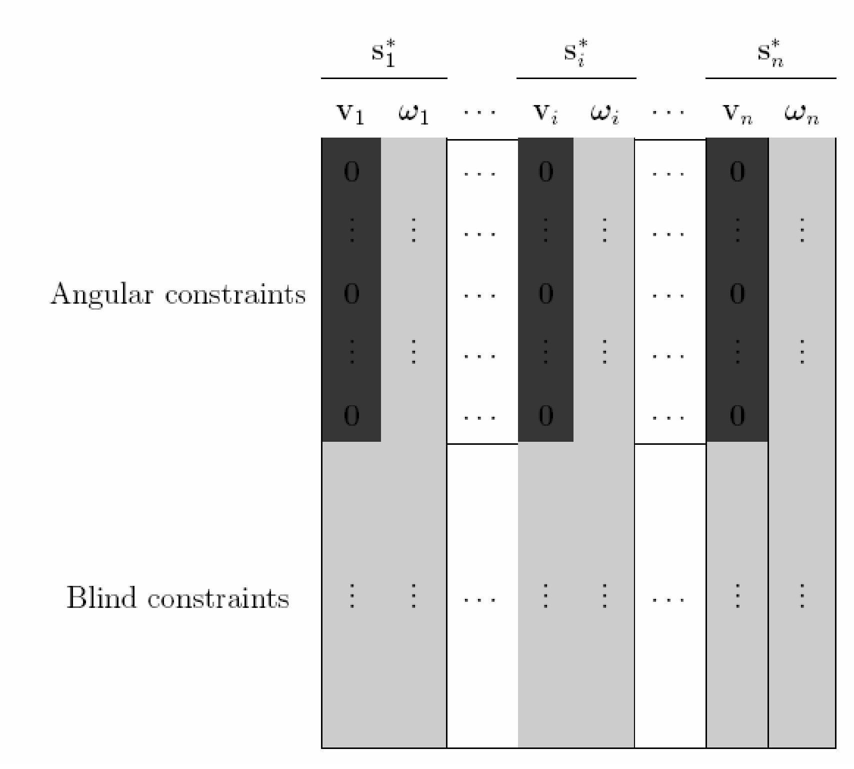
Tools

- Grassmann-Cayley algebra
  - extensors and tensors
- Plücker coordinates
- Theory of screws/twists



Goal: rigidity matrix

- Angular and blind constraints



Constraint types

degree of freedom

- Observe:

  Some constraints may affect > 1 dof
  - Plane-plane parallel
  - Line-line parallel
- Introduce primitive constraints
  - May affect ≤ 1 dof
  - Line-line perpendicular

Primitive constraint types

- Angular
  - May affect only rotational dofs
  - Line-line perpendicular
- Blind
  - May affect either rotational or translational dofs
  - Point-point distance (bar)

Basic building blocks

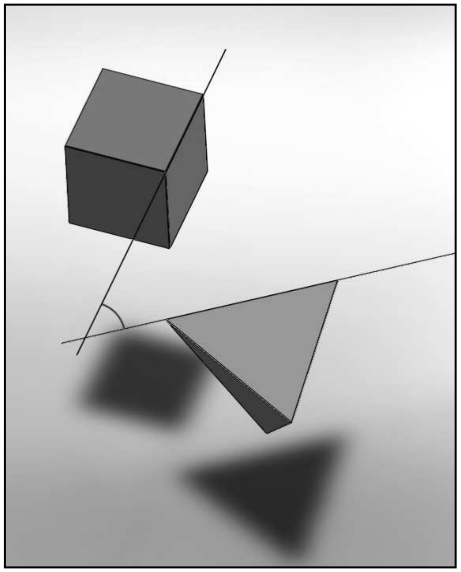
- 20 of 21 body-and-cad constraints decompose into 2 basic building blocks:
  - 1 angular
  - 1 blind

- Point-point coincidence requires separate treatment

Basic building blocks

Basic angular:

line-line non-parallel

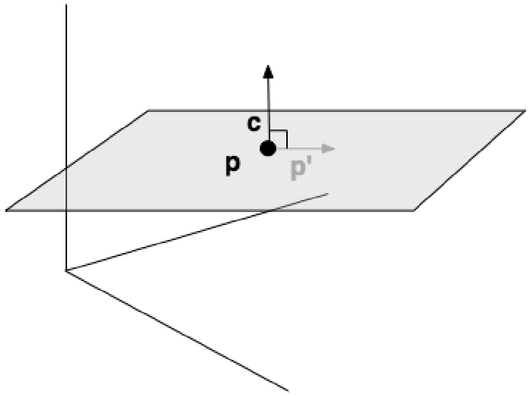


$\cdots$	$\mathbf{v}_i$	$-\boldsymbol{\omega}_i$	$\cdots$	$\mathbf{v}_j$	$-\boldsymbol{\omega}_j$	$\cdots$
$\cdots\vec{0}\cdots$	$\vec{0}$	$\mathbf{b}\times\mathbf{a}$	$\cdots\vec{0}\cdots$	$\vec{0}$	$\mathbf{a}\times\mathbf{b}$	$\cdots\vec{0}\cdots$

Basic blind:

For point ***p*** and direction ***c***,

p' is constrained to be orthogonal to ***c***

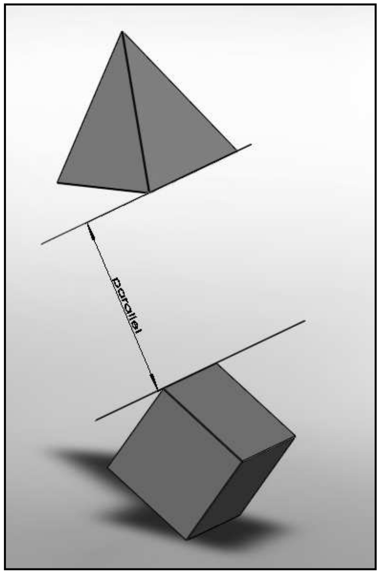


$\cdots$	$\mathbf{v}_i$	$-\boldsymbol{\omega}_i$	$\cdots$	$\mathbf{v}_j$	$-\boldsymbol{\omega}_j$	$\cdots$
$\cdots\vec{0}\cdots$	$(\mathbf{p}:1)\vee(\mathbf{c}:0)$		$\cdots\vec{0}\cdots$	$-(\mathbf{p}:1)\vee(\mathbf{c}:0)$		$\cdots\vec{0}\cdots$

Building blocks

Angular:

line-line **parallel**

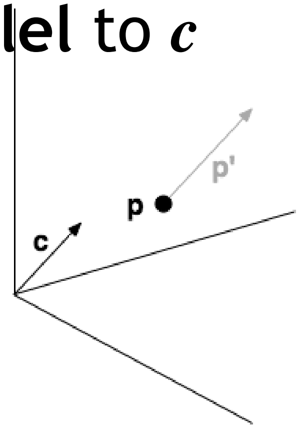


$\cdots$	$\mathbf{v}_i$	$-\boldsymbol{\omega}_i$	$\cdots$	$\mathbf{v}_j$	$-\boldsymbol{\omega}_j$	$\cdots$
$\cdots\vec{0}\cdots$	$\vec{0}$	$\mathbf{c}\times\mathbf{a}$	$\cdots\vec{0}\cdots$	$\vec{0}$	$\mathbf{a}\times\mathbf{c}$	$\cdots\vec{0}\cdots$
$\cdots\vec{0}\cdots$	$\vec{0}$	$\mathbf{c}\times\mathbf{b}$	$\cdots\vec{0}\cdots$	$\vec{0}$	$\mathbf{b}\times\mathbf{c}$	$\cdots\vec{0}\cdots$

Blind:

For point $\boldsymbol{p}$ and direction $\boldsymbol{c}$,

$\boldsymbol{p'}$ is constrained to be **parallel** to $\boldsymbol{c}$

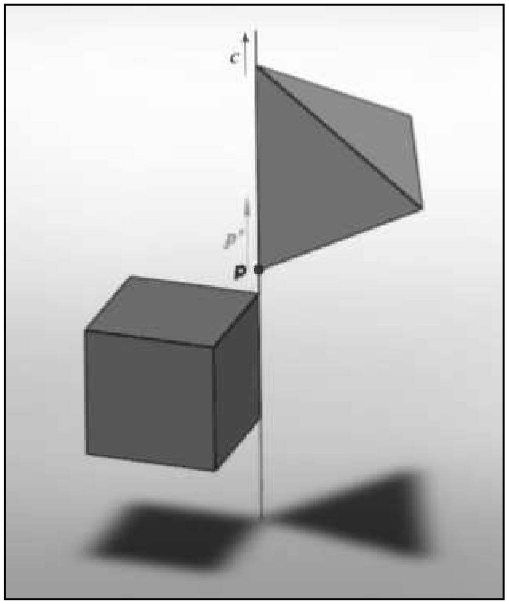


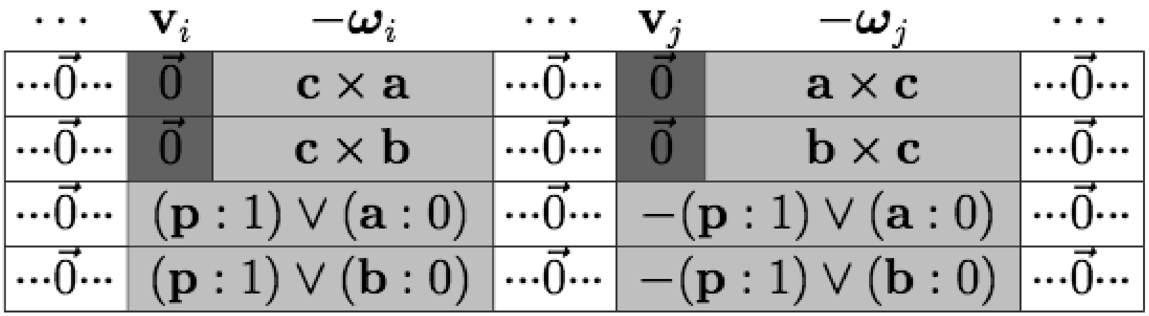
$\cdots$	$\mathbf{v}_i$	$-\boldsymbol{\omega}_i$	$\cdots$	$\mathbf{v}_j$	$-\boldsymbol{\omega}_j$	$\cdots$
$\cdots\vec{0}\cdots$	$(\mathbf{p}:1)\vee(\mathbf{a}:0)$		$\cdots\vec{0}\cdots$	$-(\mathbf{p}:1)\vee(\mathbf{a}:0)$		$\cdots\vec{0}\cdots$
$\cdots\vec{0}\cdots$	$(\mathbf{p}:1)\vee(\mathbf{b}:0)$		$\cdots\vec{0}\cdots$	$-(\mathbf{p}:1)\vee(\mathbf{b}:0)$		$\cdots\vec{0}\cdots$

Example: line-line coincidence

(i) line-line parallel

(iii) ***p'*** same direction as ***c***

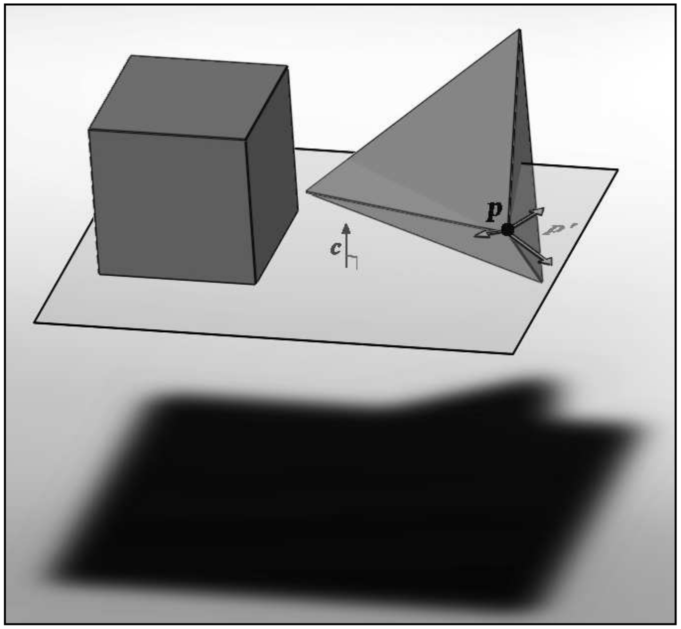




$\cdots$	$\mathbf{v}_i$	$-\boldsymbol{\omega}_i$	$\cdots$	$\mathbf{v}_j$	$-\boldsymbol{\omega}_j$	$\cdots$
$\cdots\vec{0}\cdots$	$\vec{0}$	$\mathbf{c} \times \mathbf{a}$	$\cdots\vec{0}\cdots$	$\vec{0}$	$\mathbf{a} \times \mathbf{c}$	$\cdots\vec{0}\cdots$
$\cdots\vec{0}\cdots$	$\vec{0}$	$\mathbf{c} \times \mathbf{b}$	$\cdots\vec{0}\cdots$	$\vec{0}$	$\mathbf{b} \times \mathbf{c}$	$\cdots\vec{0}\cdots$
$\cdots\vec{0}\cdots$	$(\mathbf{p}:1) \vee (\mathbf{a}:0)$		$\cdots\vec{0}\cdots$	$-(\mathbf{p}:1) \vee (\mathbf{a}:0)$		$\cdots\vec{0}\cdots$
$\cdots\vec{0}\cdots$	$(\mathbf{p}:1) \vee (\mathbf{b}:0)$		$\cdots\vec{0}\cdots$	$-(\mathbf{p}:1) \vee (\mathbf{b}:0)$		$\cdots\vec{0}\cdots$

Example: plane-point coincidence

(iv) ***p'*** is constrained to be orthogonal to ***c***



$$\begin{array}{ccccccc} \cdots & \mathbf{v}_i & -\boldsymbol{\omega}_i & \cdots & \mathbf{v}_j & -\boldsymbol{\omega}_j & \cdots \\ \hline \cdots\mathbf{0}\cdots & (\mathbf{p}:1)\vee(\mathbf{c}:0) & & \cdots\mathbf{0}\cdots & -(\mathbf{p}:1)\vee(\mathbf{c}:0) & & \cdots\mathbf{0}\cdots \end{array}$$

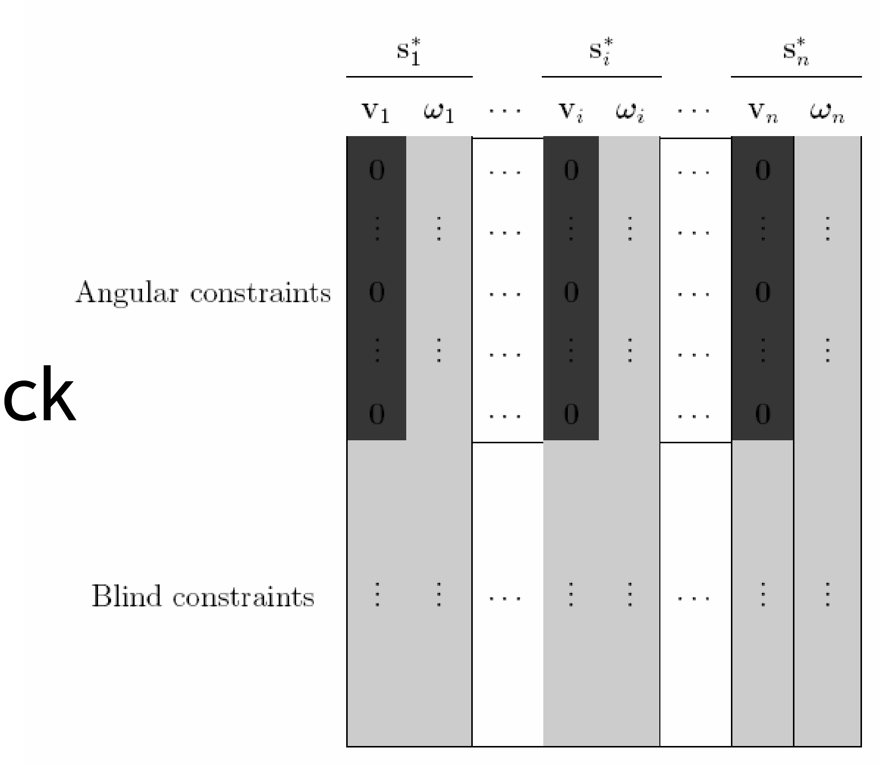
Rigidity matrix

21 body-and-cad constraints

- Each associated to

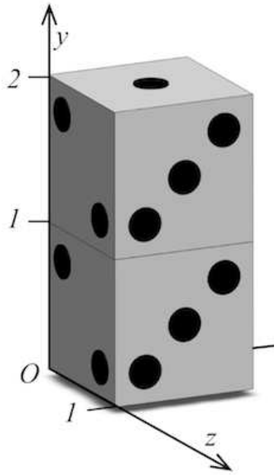
  ≤ 1 angular building block

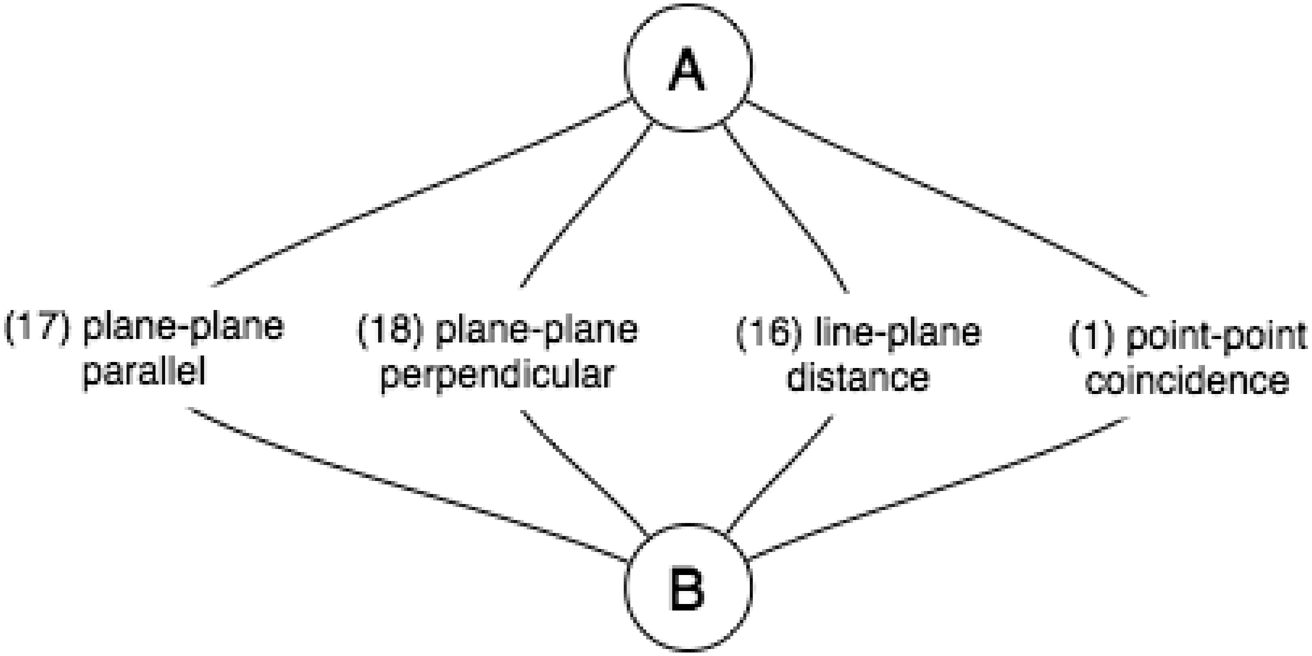
  ≤ 1 blind building block



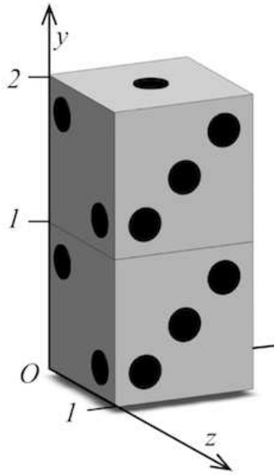
Combinatorics

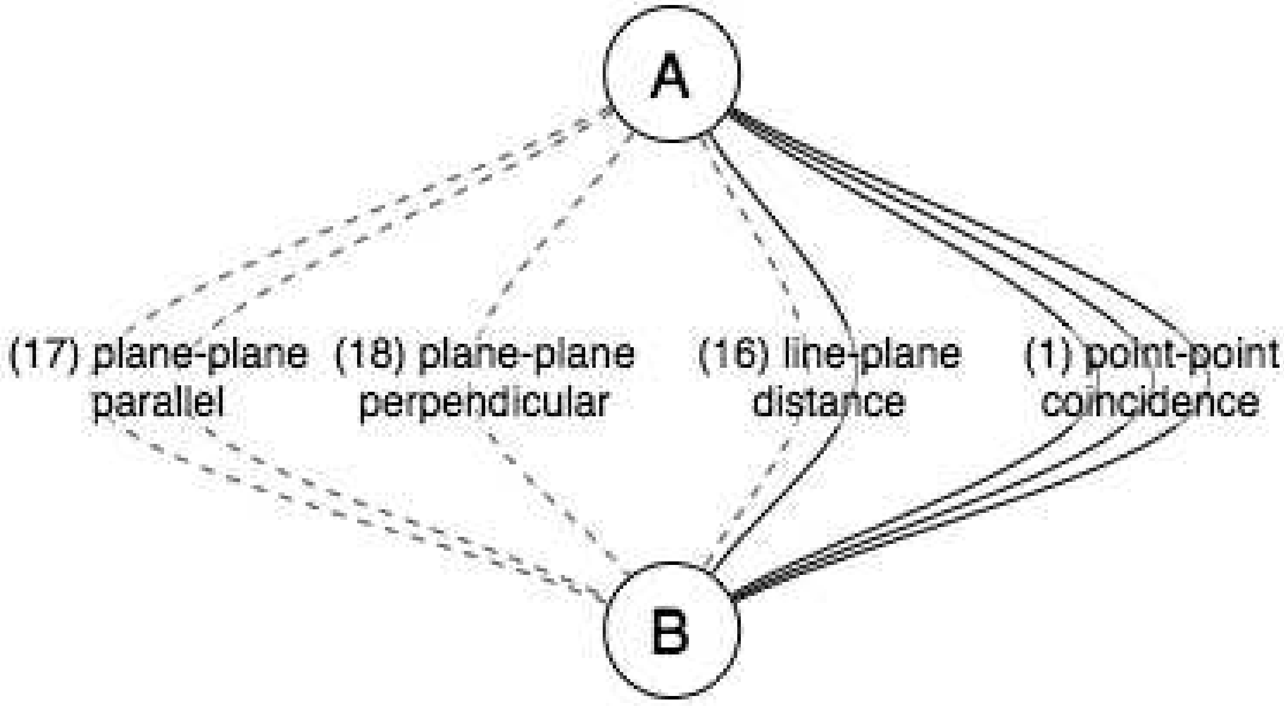
Cad graph


y
2
1
O
1
z


A
(17) plane-plane parallel
(18) plane-plane perpendicular
(16) line-plane distance
(1) point-point coincidence
B

Primitive cad graph




A
(17) plane-plane parallel
(18) plane-plane perpendicular
(16) line-plane distance
(1) point-point coincidence
B

A *recipe* for rigidity theory

- Formulate algebraic rigidity concept
  - Often quadratic systems
  - Even defining rigidity is challenging and subtle!
    - Differential geometry, topology, algebraic geometry
- Study associated infinitesimal rigidity
  - Linear algebra
- Try to find combinatorial characterizations
  - Graph theory
- … that hopefully lead to efficient algorithms!
  - (Pebble games)
  - Matroid theory

Nested sparsity

Matroid intersection algorithm

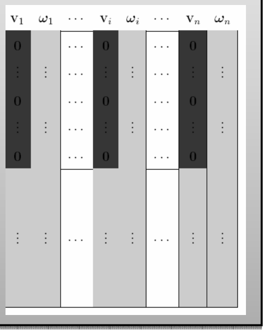
(k_1, a_1, k_2, a_2)-nested sparse

On a graph with edges colored black and red

1. All edges (black and red): (k_1, a_1)-sparse
2. Red edges: (k_2, a_2)-sparse

Theorem: (6,6,3,3)-nested sparsity is a necessary condition

- –All constraints: (6,6)-sparse
- –Angular constraints: (3,3)-sparse

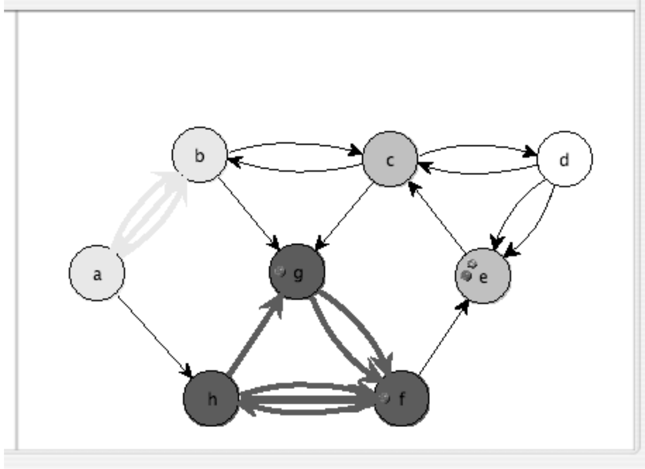


$$\begin{array}{cccccccc}
\mathbf{v}_1 & \omega_1 & \cdots & \mathbf{v}_i & \omega_i & \cdots & \mathbf{v}_n & \omega_n \\
0 & & \cdots & 0 & & \cdots & 0 & \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & & \cdots & 0 & & \cdots & 0 & \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & & \cdots & 0 & & \cdots & 0 & \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots
\end{array}$$

Sparsity

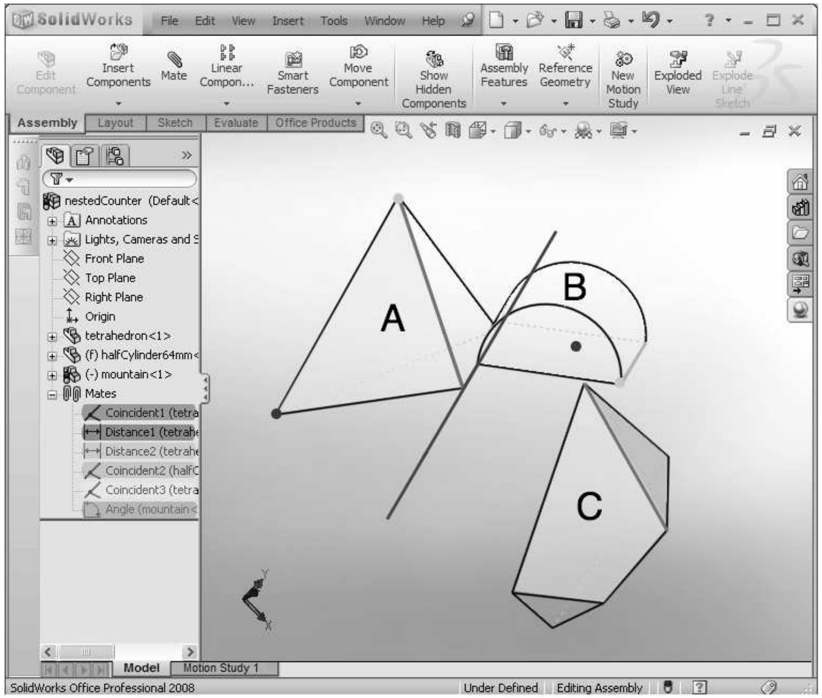
- A graph G on n vertices is (k,a)-sparse if every set of n' vertices spans at most $kn'-a$ edges

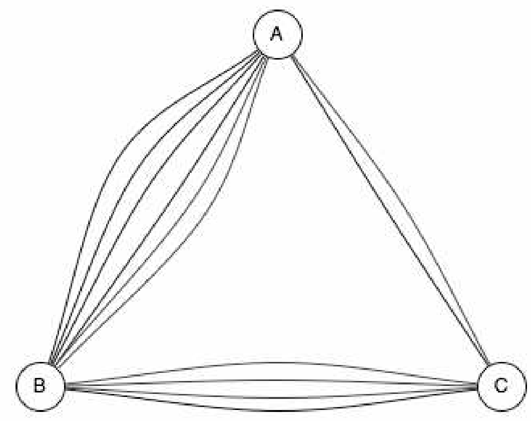
- Pebble game algorithms for sparsity (Lee, Streinu '05)
  - Matroidal properties of sparse graphs

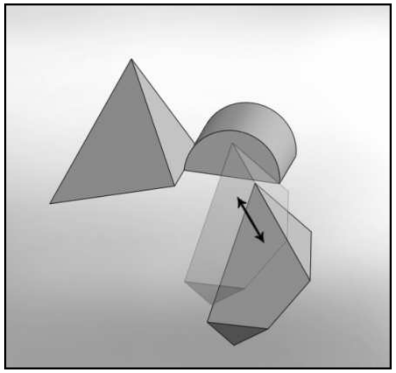


Nested sparsity for body-and-cad

- Not sufficient





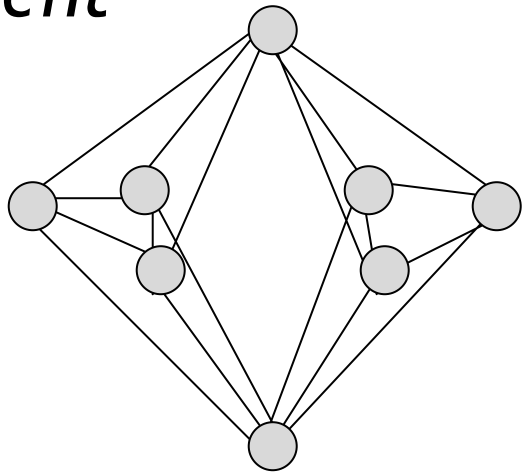


This sounds familiar…

Bar-and-joint: 3D

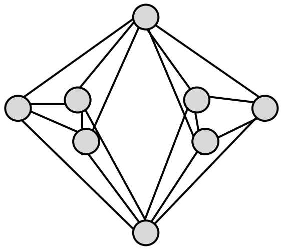
- Combinatorial characterization is open!!
  - Generalized Laman count (3,6) is *necessary* but not *sufficient*
    - Double banana example

Even this is not well-understood!

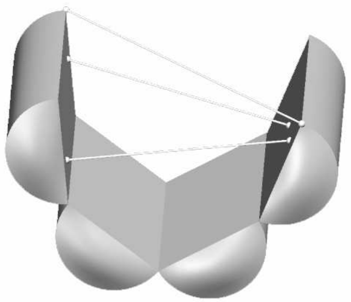


Body-and-cad: 3D

- Can we come up with a combinatorial characterization for body-and-cad rigidity?

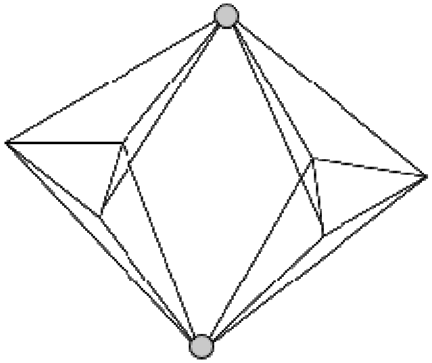


- *Is this as hard as 3D bar-and-joint?*
- *Where does it lie between body-and-bar and bar-and-joint?*

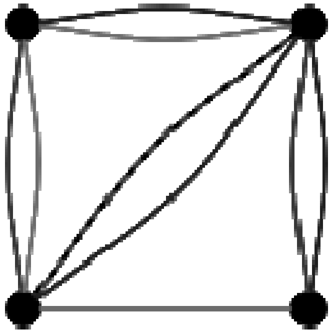


Audrey Lee-St.John
Mount Holyoke College

Without point-point coincidences…



… we have a combinatorial characterization!



Main theorem

A body-and-cad framework

(with no point-point coincidences)

is generically minimally rigid

□

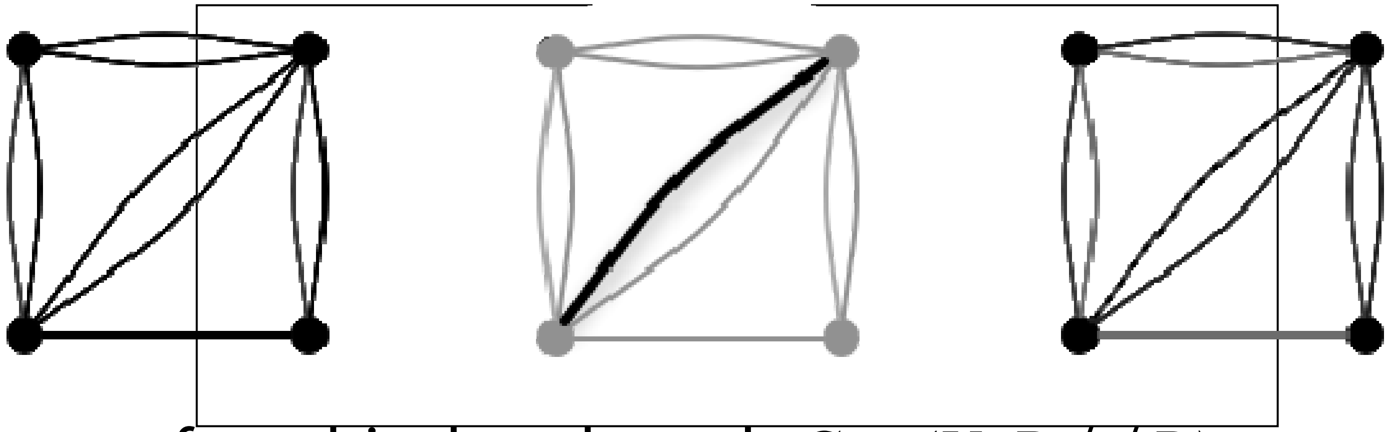
for its associated primitive cad graph $G = (V, B \sqcup / R)$,

exists $B' □ B$ s.t.

(1) $B \setminus B'$ is edge-disjoint union of 3 spanning trees, and

(2) $R □ B'$ is edge-disjoint union of 3 spanning trees.

Combinatorics



for a bicolored graph $G = (V, B \square R)$,

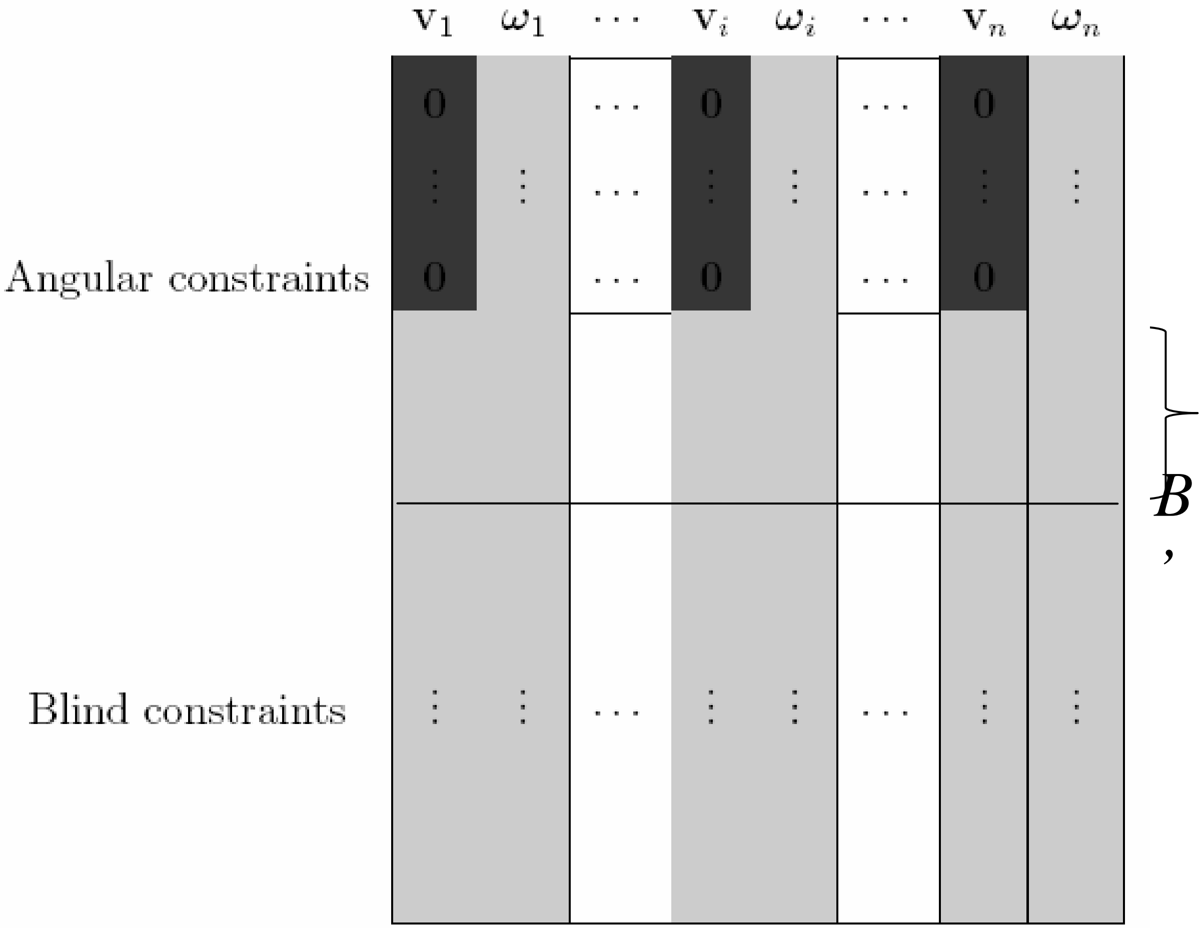
exists $B' \square B$ s.t.

(1) $B \setminus B'$ is edge-disjoint union of $\overset{2}{\cancel{3}}$ spanning trees,

and

(2) $R \square B'$ is edge-disjoint union of $\overset{1}{\cancel{3}}$ spanning trees.

Intuitively…


$\mathbf{v}_1$ ω_1 $\cdots$ $\mathbf{v}_i$ ω_i $\cdots$ $\mathbf{v}_n$ ω_n
Angular constraints
Blind constraints
0
0
B'

Generalize White and Whiteley

- A bi-colored graph G is (k,g)-counted if
  - $m = kn - k$ (count on total edges), and
  - $m_R \leq gn - g$ (count on red edges)
- A (k,g)-frame is a pair (G,x),
  - $x{:}E \rightarrow R^k$, where $x(r)_j=0$ for $j=1,\ldots,k-g$ and r□R

Body-and-cad (without point-point coincidence)
are (6,3)-frames

Main theorem

A (k,g)-frame,

with $G = (V, B \sqcup R)$ a bicolored graph,

is generically minimally rigid

□

exists B' □ B s.t.

(1) $B \setminus B'$ is edge-disjoint union of k-g spanning trees, and

(2) R □ B' is edge-disjoint union of g spanning trees.

Pure condition

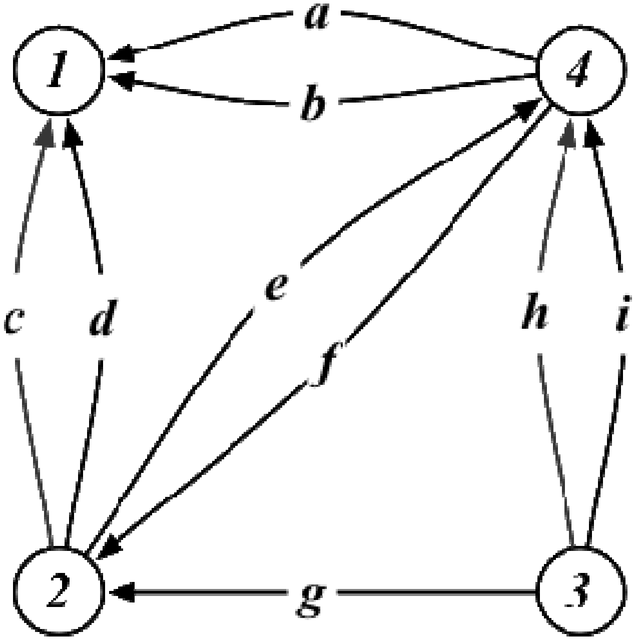
Tied-down rigidity matrix

- The pure condition for G is a polynomial in indeterminate edge vectors: $C(G) = det(M_T(G))$

- A (k,g)-fan φ is a partitioning of E into $n - 1$ ordered sets $(\varphi_2,\ldots,\varphi_n)$ such that each $\varphi_i=(\varphi_{i,1},\ldots,\varphi_{i,k})$ contains
  - exactly k edges incident to vertex i,
  - $\leq g$ (red) edges from R

Fans and the pure condition

Proposition. $C(G) = \Sigma\varphi \pm [\varphi_2] \cdots [\varphi_n]$,
for all distinct (k,g)-fans φ of G.



- (3,1)-fan diagram corresponds to one *distinct* fan

- more than one (3,1)-fan, including:
  $((c,d,e),(a,b,f),(g,h,i))$
  $((e,c,d),(f,a,b),(i,g,h))$
  $((c,e,d),(a,f,b),(g,i,h))$

Proof sketch

- => If $C(G) \neq 0$, then can pick out k trees
  - Red edges can only appear in g of the trees due to 0 entries
  - B' is composed of black edges in those g trees
- <= If have tree decomposition, assign

$$x'(e) = \begin{cases} (a_{j,1}, a_{j,2}, \ldots, a_{j,k}) & \text{if } e \in B \setminus B' \\ (0, 0, \ldots, 0, a_{j,k-g+1}, \ldots, a_{j,k}) & \text{if } e \in R \cup B' \end{cases}$$

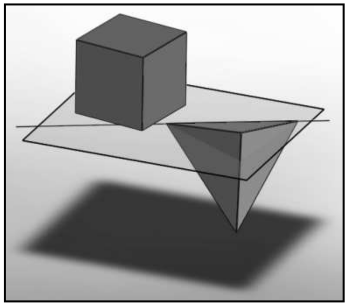
Then only one distinct (k,g)-fan (root trees at vertex 1 and direct edges to root) contributes a non-zero term to $C(G)$ and $C(G) \neq 0$.

Conclusions

Summary of contributions

- New rigidity model
  - Body-and-*cad*
    - 21 pairwise constraints
    - **C**oincidence
    - **A**ngle
    - **D**istance
- Infinitesimal rigidity theory foundation
  - Rigidity matrix

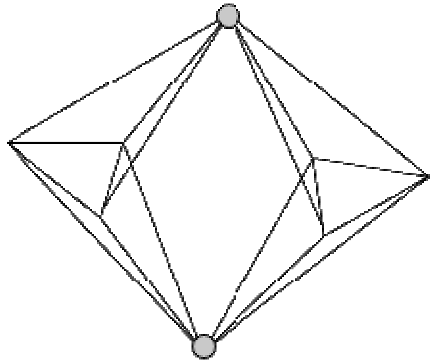
- Combinatorics
  - Nested sparsity
    - Necessary, not sufficient
  - Trees
    - Characterization for 20 of 21 constraints



Open questions

- Other constraints for CAD
  - Symmetry, equality

- Combinatorial characterization for point-point coincidence



Thank you!

- Infinitesimal rigidity theory based on dissertation work at UMass Amherst
  - Advisor: Ileana Streinu
  - Other collaborators: Kirk Haller, Meera Sitharam, Neil White
- Combinatorial characterization
  - Joint work with Jessica Sidman
- Questions?

Symmetric Mate

- **Allows two similar entities to be symmetric about a face or plane**
- **Features allowed:**
  - Points
  - Planes or planar faces
  - Spheres of equal radius
  - Cylinders of equal radius

http://www.neswuc.com/Presentation/Advanced_Mates.pdf

SolidWorks

Alpha helices

- Line-line angle

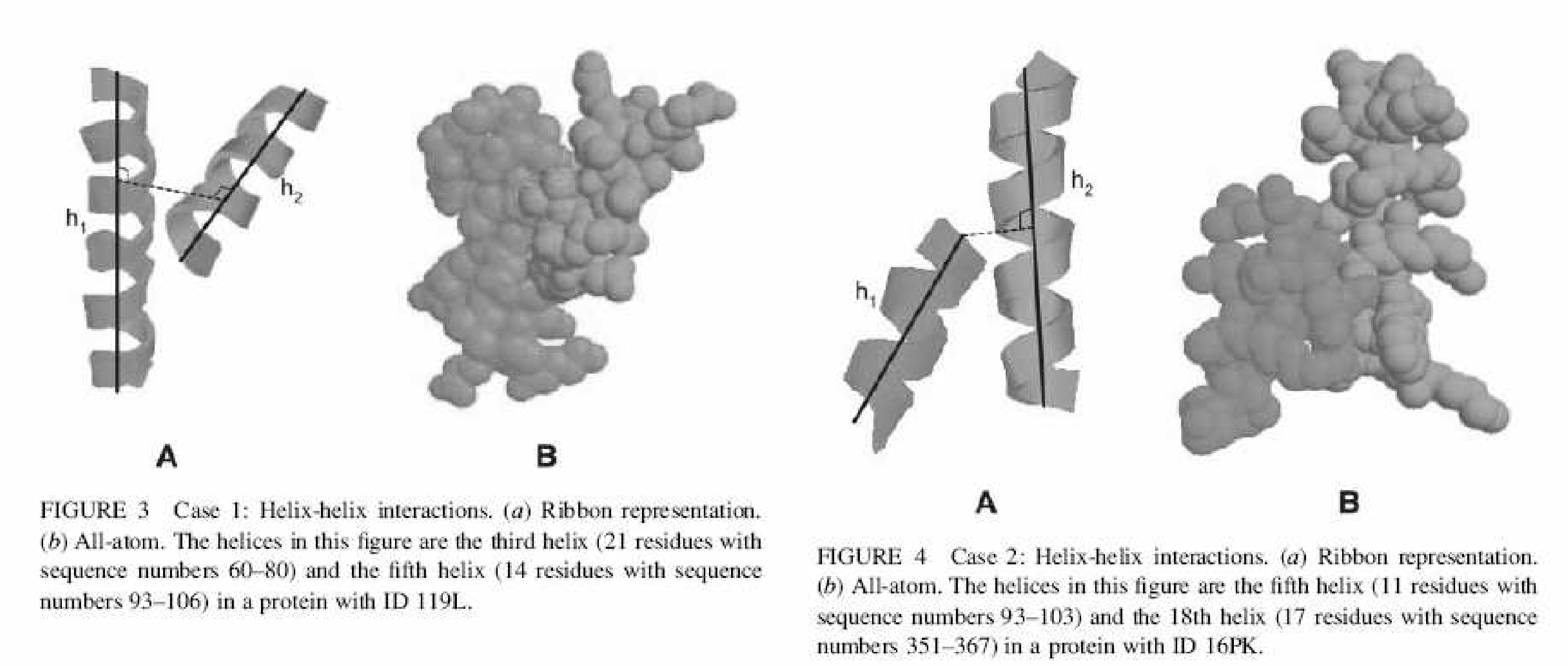


FIGURE 3 Case 1: Helix-helix interactions. (*a*) Ribbon representation. (*b*) All-atom. The helices in this figure are the third helix (21 residues with sequence numbers 60–80) and the fifth helix (14 residues with sequence numbers 93–106) in a protein with ID 119L.

FIGURE 4 Case 2: Helix-helix interactions. (*a*) Ribbon representation. (*b*) All-atom. The helices in this figure are the fifth helix (11 residues with sequence numbers 93–103) and the 18th helix (17 residues with sequence numbers 351–367) in a protein with ID 16PK.

Lee, Sangyoon, and Chirikjian, Gregory. Interhelical angle and distance prefer-ences in globular proteins. Biophysical Journal 86 (2004), 1105–1117.

Beta sheets

- Plane-plane ||, coincident

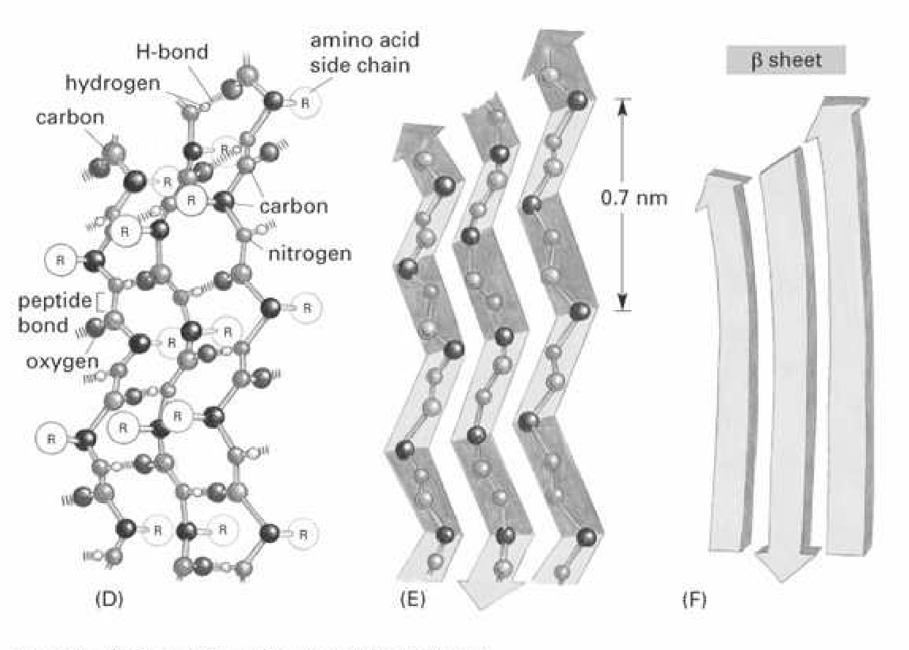
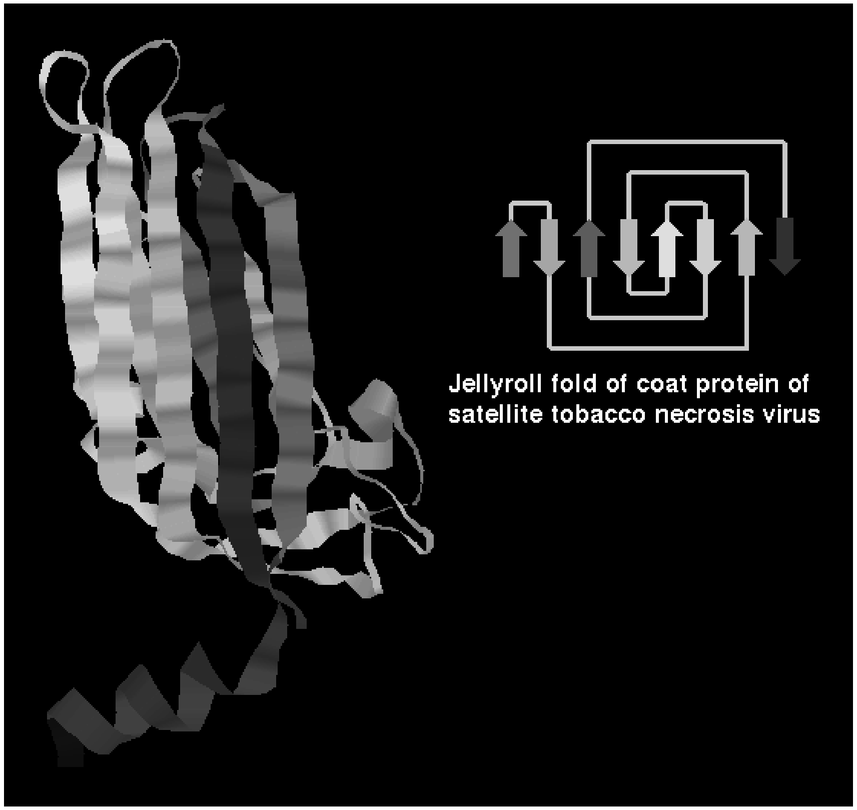


Figure 4-10 part 2 of 2 Essential Cell Biology, 2/e. (© 2004 Garland Science)

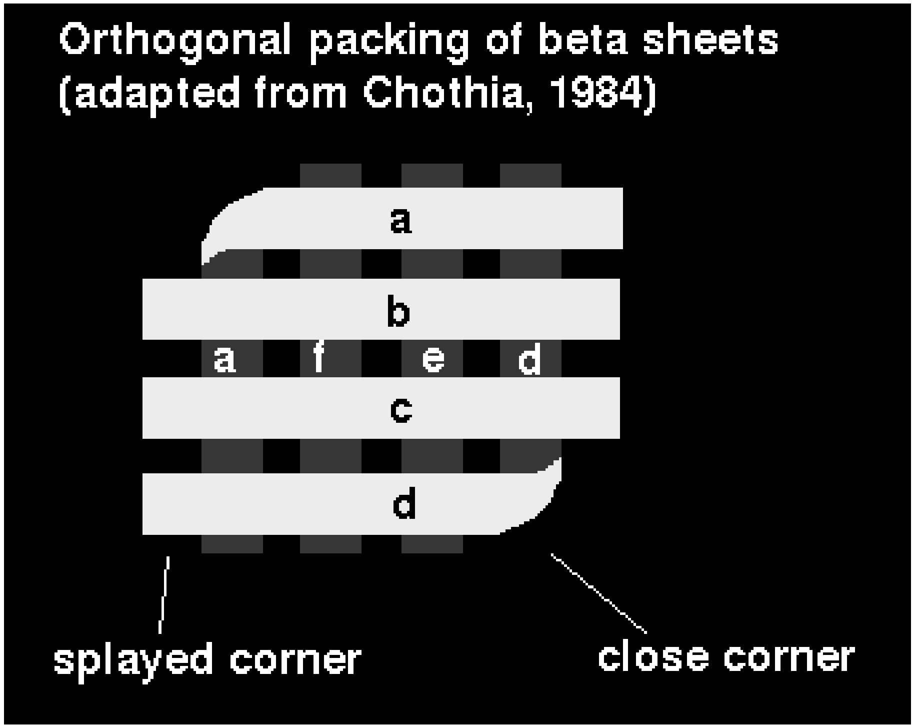
http://fig.cox.miami.edu/~cmallery/255/255prot/ecb4x10b.jpg

Beta sheets


Jellyroll fold of coat protein of
satellite tobacco necrosis virus

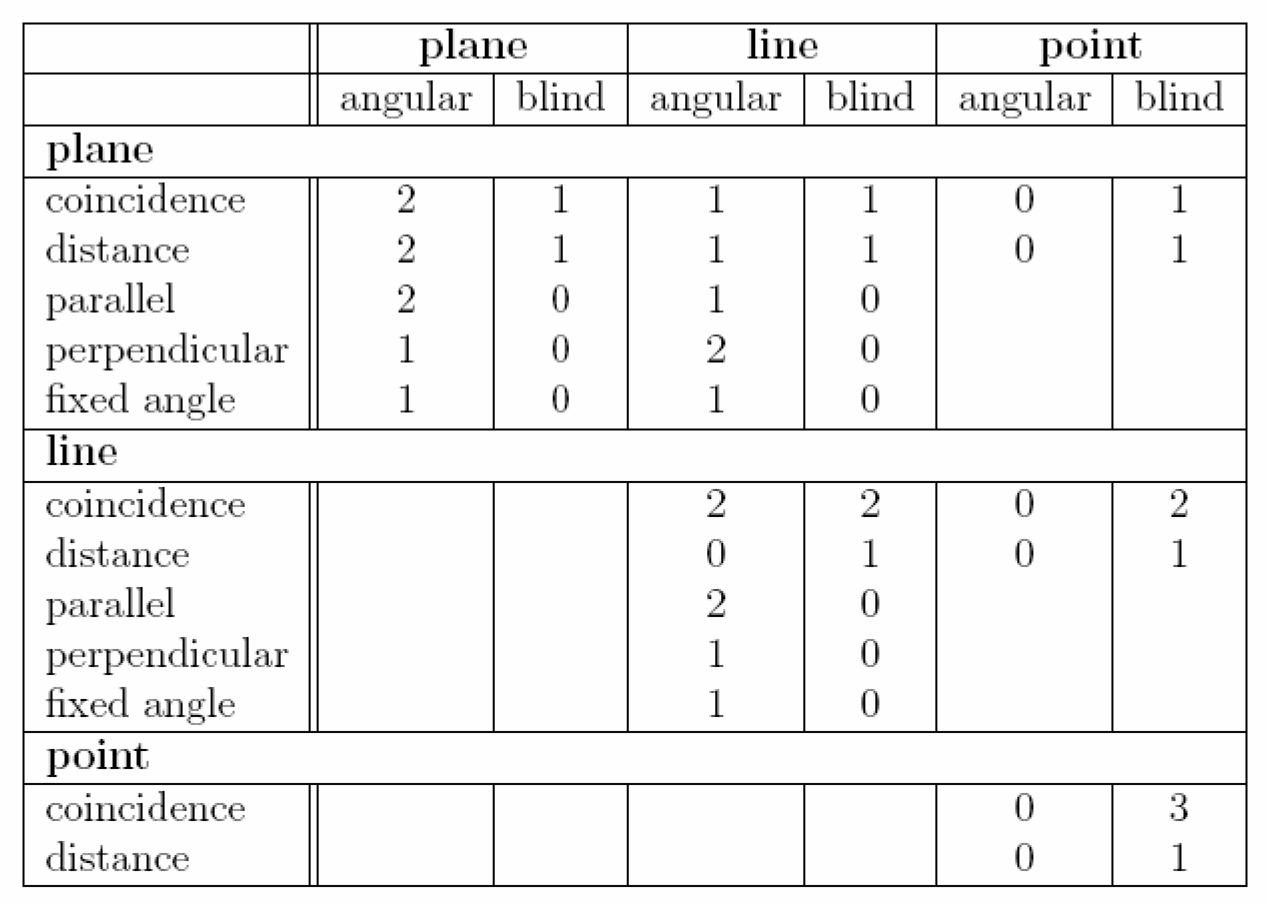
http://www.cryst.bbk.ac.uk/PPS95/course/8_folds/8_stv.gif

Orthogonal beta sheets


Orthogonal packing of beta sheets
(adapted from Chothia, 1984)
a
b
a
f
e
d
c
d
splayed corner
close corner

http://www.cryst.bbk.ac.uk/PPS2/course/section9/9_sheshe.html

Body-and-cad



	plane		line		point	
	angular	blind	angular	blind	angular	blind
plane						
coincidence	2	1	1	1	0	1
distance	2	1	1	1	0	1
parallel	2	0	1	0		
perpendicular	1	0	2	0		
fixed angle	1	0	1	0		
line						
coincidence			2	2	0	2
distance			0	1	0	1
parallel			2	0		
perpendicular			1	0		
fixed angle			1	0		
point						
coincidence					0	3
distance					0	1