The Dehn invariants of the Bricard octahedra

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Definiton

A polyhedron (i.e., a polyhedral surface in the Euclidean, Lobachevskij of spherical d-space) is said to be flexible if its spatial shape can be changed continuously due to changes of its dihedral angles only,
i. e., if every face remains congruent to itself during the flex.

What do we know about flexible polyhedra?

- If $d \geqslant 3$, there is no flexible polytope in d-space [A.L. Cauchy, 1813].
- Flexible octahedra (with self-intersections) do exist in 3-space [R. Bricard, 1897].
- Flexible self-intersection free sphere-homeomorphic polyhedra do exist in 3-space [R. Connelly, 1976].
- There exists flexible self-intersection free sphere-homeomorphic polyhedron in 3-space that has only 9 vertices [K. Steffen, ∼1980].
- In 3-space, there exist flexible polyhedra of arbitrary genus, as well as non-orientable ones [Obvious modulo previous results].
- Flexible cross-polytopes (with self-intersections) do exist in 4-space [R. Connelly & A. Walz, ∼1998].

What do we know about flexible polyhedra? (Continued)

- Every flexible polyhdron in $\mathbb{R}^3$ preserves its total mean curvature during the flex, i.e. $\frac{1}{2}\sum\limits_{\ell\in E}|\ell|\big(\pi-\alpha(\ell)\big)=\mathrm{const}$ [R. Alexander, 1985].
- Every flexible polyhdron in $\mathbb{R}^3$ preserves its (oriented) volume during the flex [I.Kh. Sabitov, 1996].
- There exist flexible polyhedra in $\mathbb{S}^3$ that do not preserve neither thier volume nor total mean curvature during the flex [A., 1998].
- Every flexible polyhdron in $\mathbb{R}^4$ preserves its (oriented) volume during the flex [A.A. Gaifullin, 2011].

References

- I.Kh. Sabitov, *"Algebraic methods for solution of polyhedra"*. Russ. Math. Surv., **66**, no. 3 (2011), 445–505. doi: 10.1070/RM2011v066n03ABEH004748
- A.A. Gaifullin, *"Sabitov polynomials for polyhedra in four dimensions"*. arXiv:1108.6014 [math.MG]

Open problems on flexible polyhedra in Euclidean spaces

- Describe all flexible polyhedra in $\mathbb{R}^3$ (i.e., given a boundary-free simplicial complex, describe what edge lengths correspond to flexible polyhedra).
- Decide whether a polyhedron is flexible (i.e., invent an algorithm that receives a polyhedron and returns "yes" if it is flexible and "no" otherwise).
- Do flexible polyhedra exist in $\mathbb{R}^d$ for $d \geqslant 5$?
- For $d \geqslant 5$, do all flexible polyhedra in $\mathbb{R}^d$ necessarily preserve their volume during the flex?

$\Rightarrow$ Everybody can find an open problem on flexible polyhedra according to his / her taste. You are welcome!

The Strong Bellows Conjecture

If an embedded (i. e., self-intersection free) compact boundary-free polyhedron Q is obtained from an embedded polyhedron P by a continuous flex then the domains bounded by Q and P are scissors congruent, i. e., the domain bounded by Q can be partitioned in a finite set of simplices $\{S_j\}_{j=1,\dots,n}$, with the following property: for every $j = 1, \dots, n$ there is an isometry $F_j : \mathbb{R}^3 \to \mathbb{R}^3$ such that the set $\{F_j(S_j)\}_{j=1,\dots,n}$ is a partition of the domain bounded by P.

M. Dehn — J.P. Sydler — B. Jessen's theorem

The following well-known theorem reduces the Strong Bellows Conjecture to the problem whether the Dehn invariants are constant during the flex:

Theorem (M. Dehn — J.P. Sydler — B. Jessen): *Given two embedded compact boundary-free polyhedra P and Q in $\mathbb{R}^3$ the following conditions are equivalent*:

(1) the domains bounded by P *and* Q *are scissors congruent;*

(2) $\mathrm{Vol}\, P = \mathrm{Vol}\, Q$ *and* $\mathrm{D}_f P = \mathrm{D}_f Q$ *for every* $\mathbb{Q}$*-linear function* $f : \mathbb{R} \to \mathbb{R}$ *such that* $f(\pi) = 0$.

Here $\mathrm{Vol}\, P$ stands for the volume of the domain bounded by P;
$\mathrm{D}_f P = \sum_\ell |\ell| f(\alpha_\ell)$ stands for the Dehn invariant of P;
α_ℓ stands for the internal dihedral angle of P at the edge ℓ;
$|\ell|$ stands for the length of ℓ.

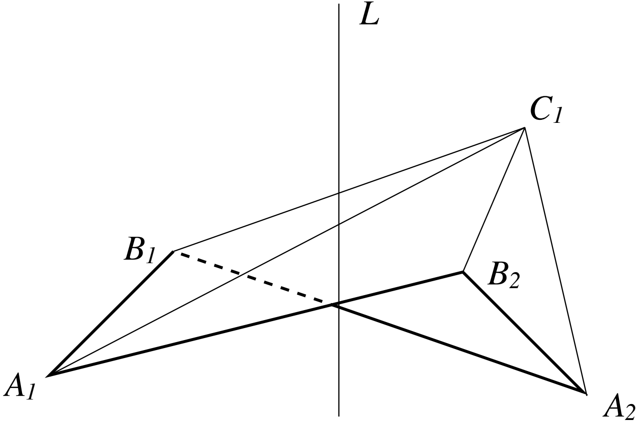
The main result

Theorem (A., *J. Geom.* **99** (2010)): *Any Dehn invariant of any Bricard octahedron remains constant during the flex; moreover, it is equal to zero.*

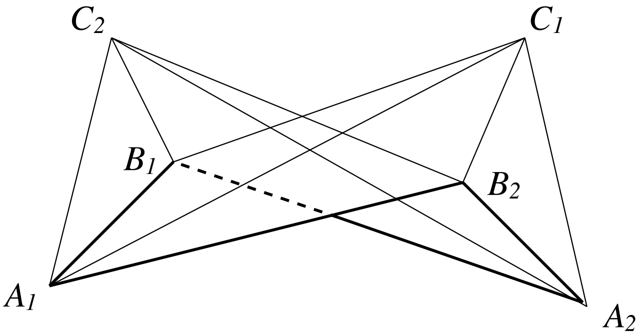
N.B.: Why interested in Bricard octahedra?

- They were the first known flexible polyhedra [R. Bricard, 1897]
- They are the simplest flexible polyhedra
- More complicated flexible polyhedra often incorporate Bricard octahedra as a part [K. Steffen, $\sim$1980]

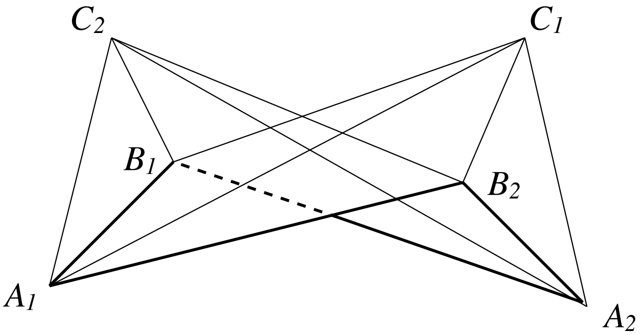
Constructing Bricard octahedron of type 1



Constructing Bricard octahedron of type 1 (Finalized)

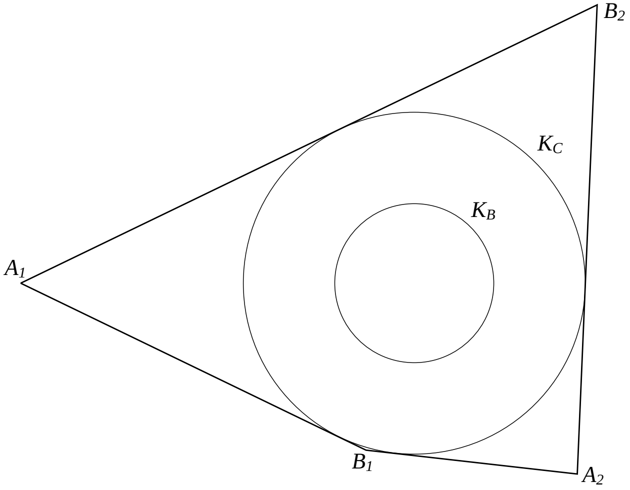


Constructing Bricard octahedron of type 1 (Finalized)

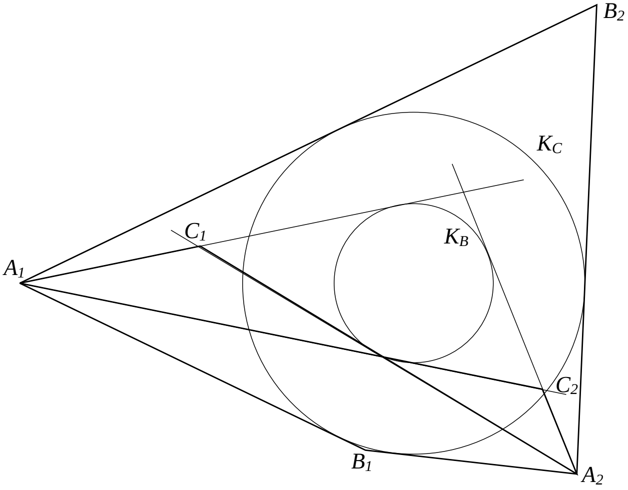


Symmetry $\Rightarrow$ $\mathrm{D}_f P \equiv \sum_\ell |\ell| f(\alpha_\ell) = 0$

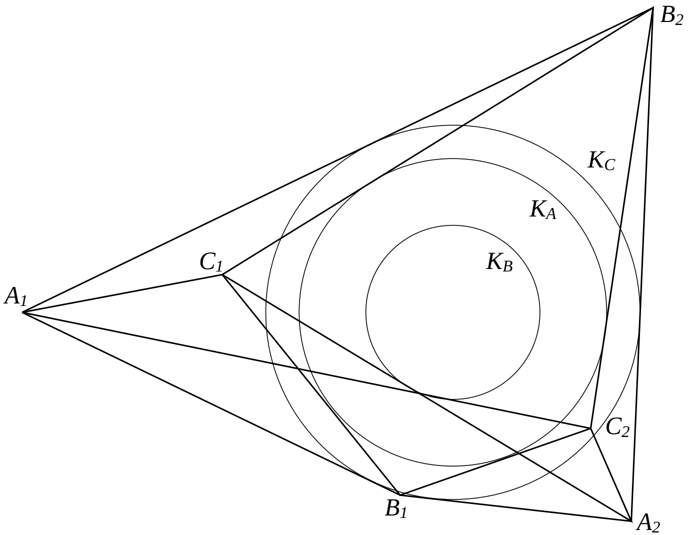
Constructing Bricard octahedron of type 3



Constructing Bricard octahedron of type 3 (Continued)



Constructing Bricard octahedron of type 3 (Finalized)



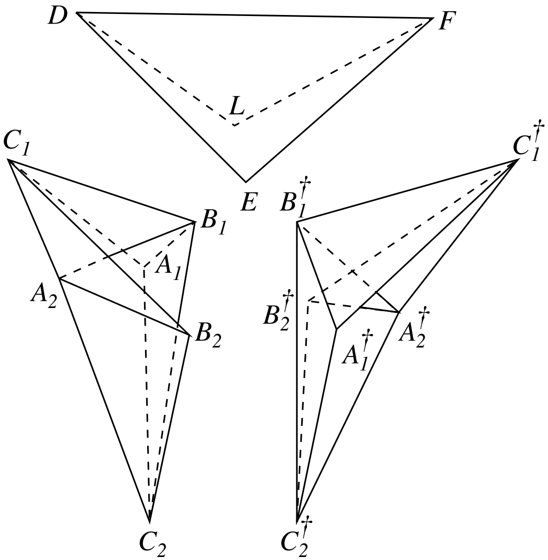
Symmetries

Bricard octaheda of type 3 have unexpectedly many "symmetries":

- There are linear relations between some edge lengths, e.g., $A_1B_2 + A_2B_1 = A_1B_1 + A_2B_2$
- There are linear relations between some planar angles, e.g., $\angle B_2A_1C_1 = \angle B_1A_1C_2$
- There are linear relations between some dihedral angles, e.g., $\alpha_{A_1B_1} + \alpha_{A_2B_1} = 2\pi m$ for some integer m

$\Rightarrow$ Using these symmetries and spherical trigonometry we prove that all Dehn invariants of Bricard octahedra of type 3 identically equal zero

Constructing the Steffen polyhedron



Conclusions

- If P is a Steffen polyhedron and Q is obtained from P by a continuous flex then the domains bounded by P and Q are scissor congruent
- Some flexible suspensions with hexagonal equator do not preserve some Dehn invariants during the flex [A. & Connelly, arXiv:0905.3683 [math.MG]]
- This talk is based on the author's paper *"The Dehn invariants of the Bricard octahedra"*, Journal of Geometry, **99** (2010), 1–13

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`Thank you for your attention!`