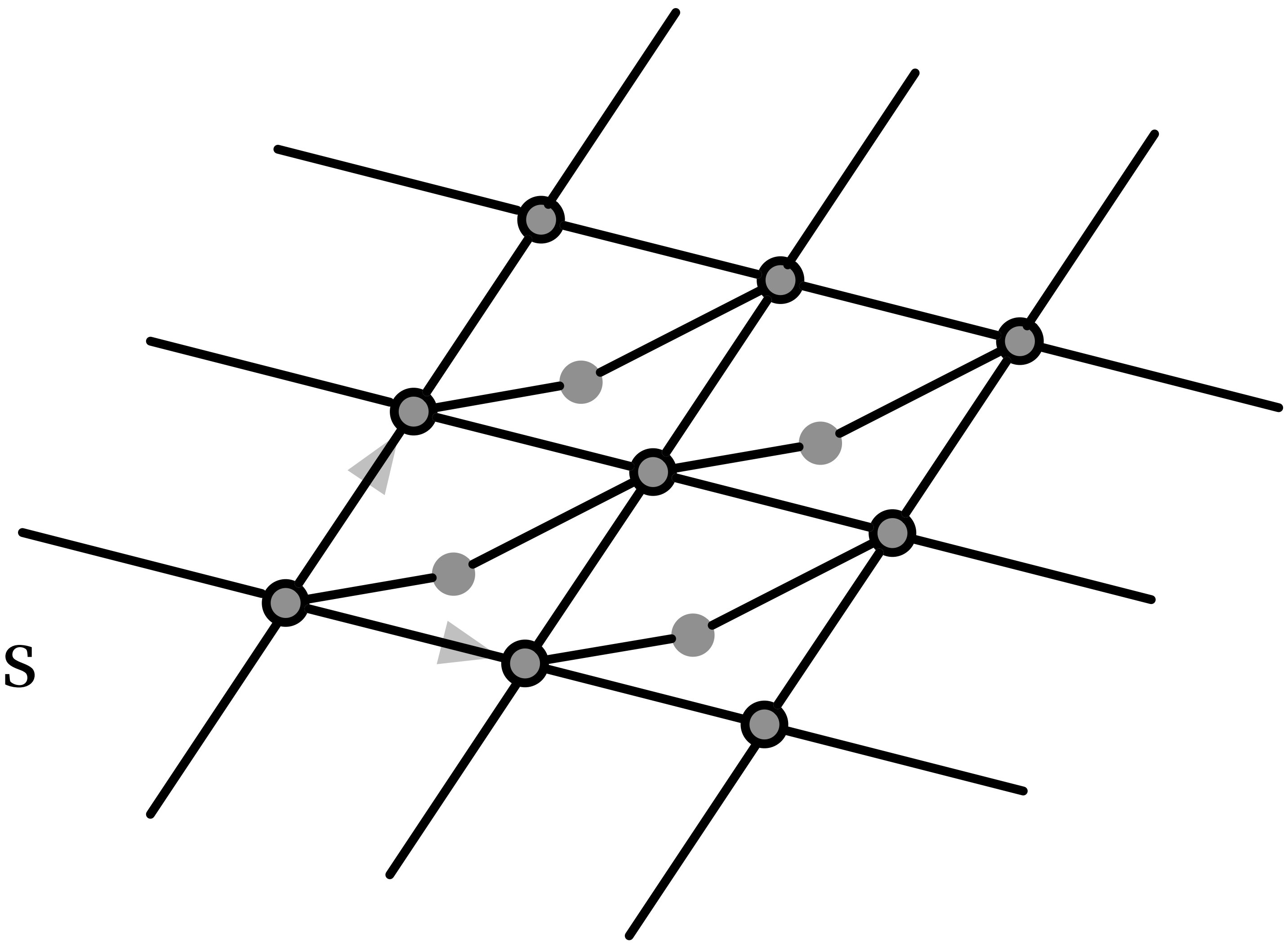
Generic crystallographic rigidity

Louis Theran
(joint work with Justin Malestein)

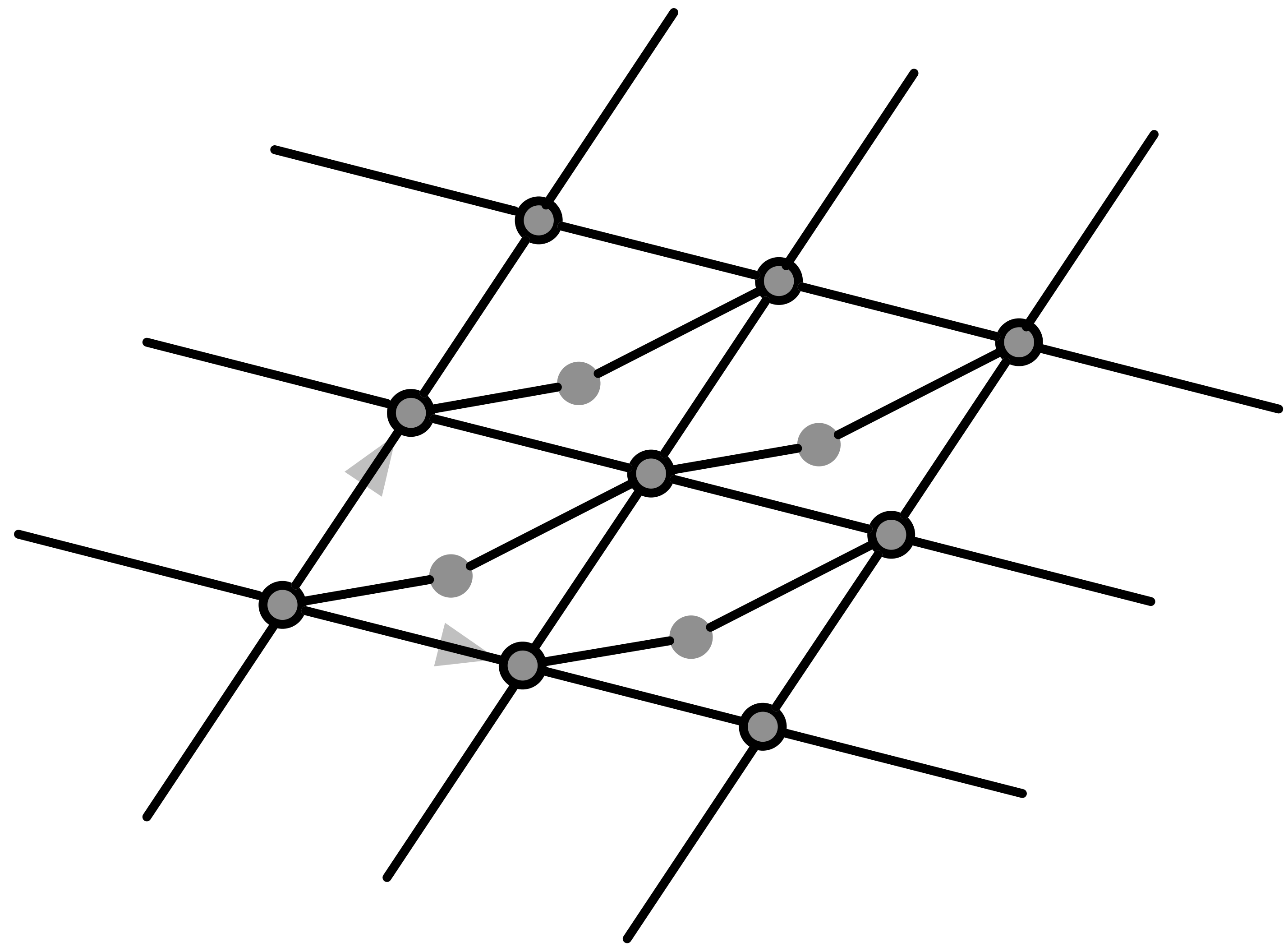
Periodic frameworks

- Infinite planar structure
- Periodic with respect to a lattice
- Finite quotient
- Fixed-length bars and universal joints



Periodic motions

- Allowed motions preserve...
- *Lengths* of the bars
- Connectivity
- *Periodicity*
- but *not necessarily* the specific *lattice*
- Def. from [Borcea-Streinu '10]
- *Rigid* if all motions isometries



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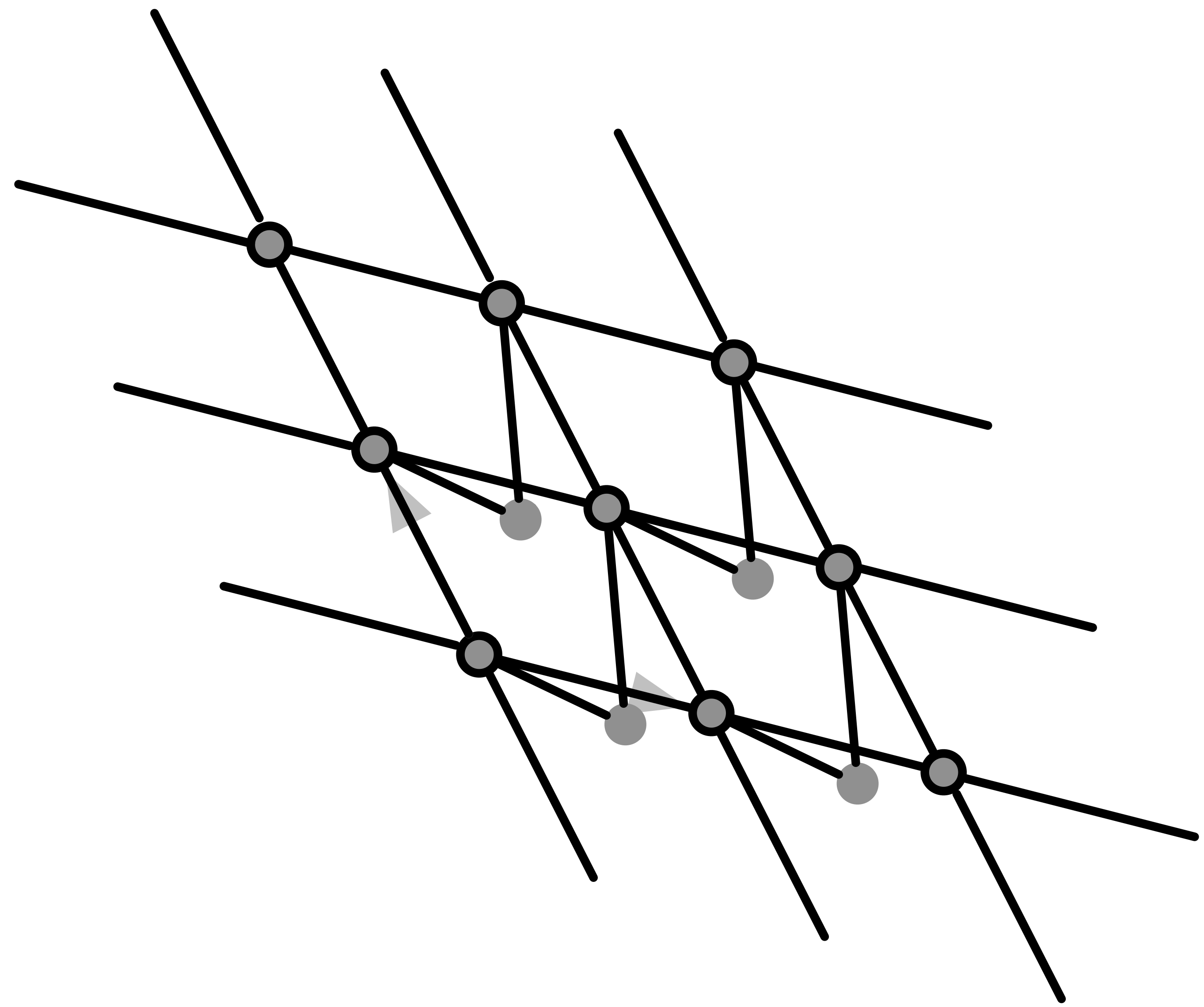
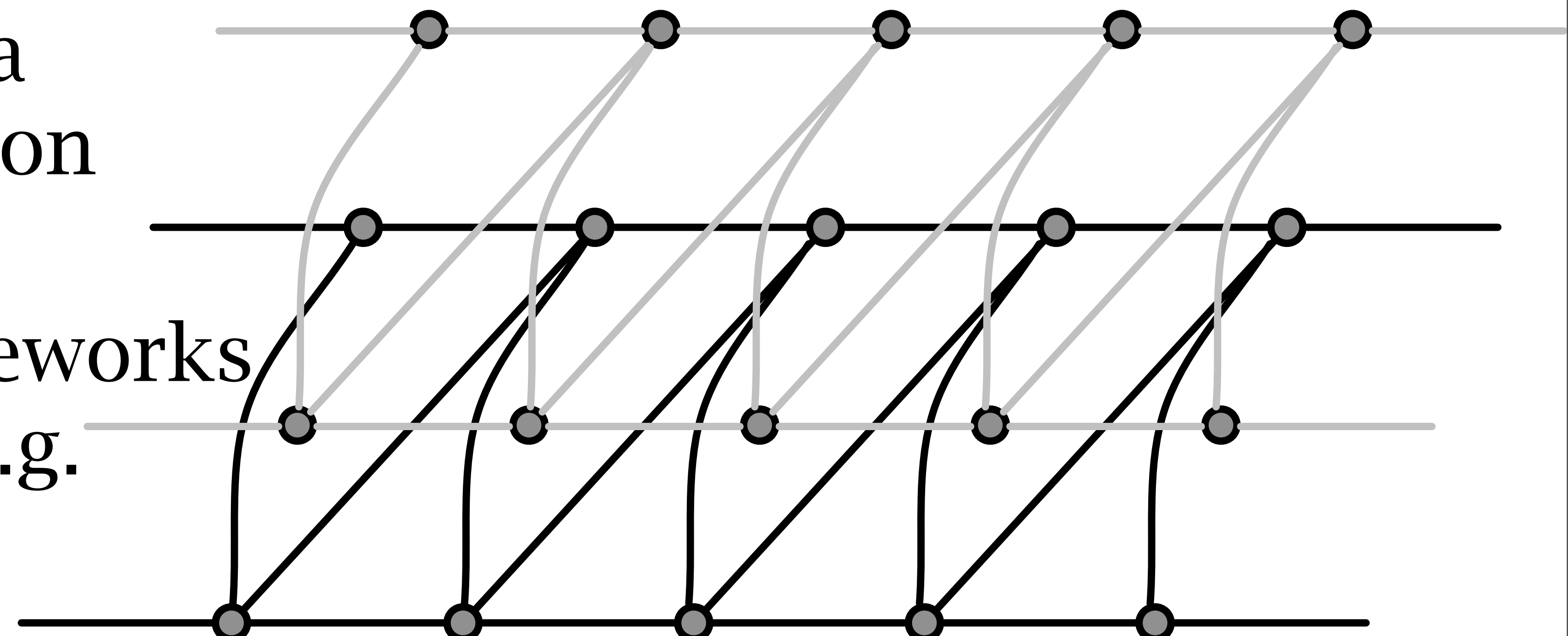


Forced periodicity

- Even with a “flexible lattice” forced periodicity is a strong condition
- Disconnected periodic frameworks can be rigid, e.g.



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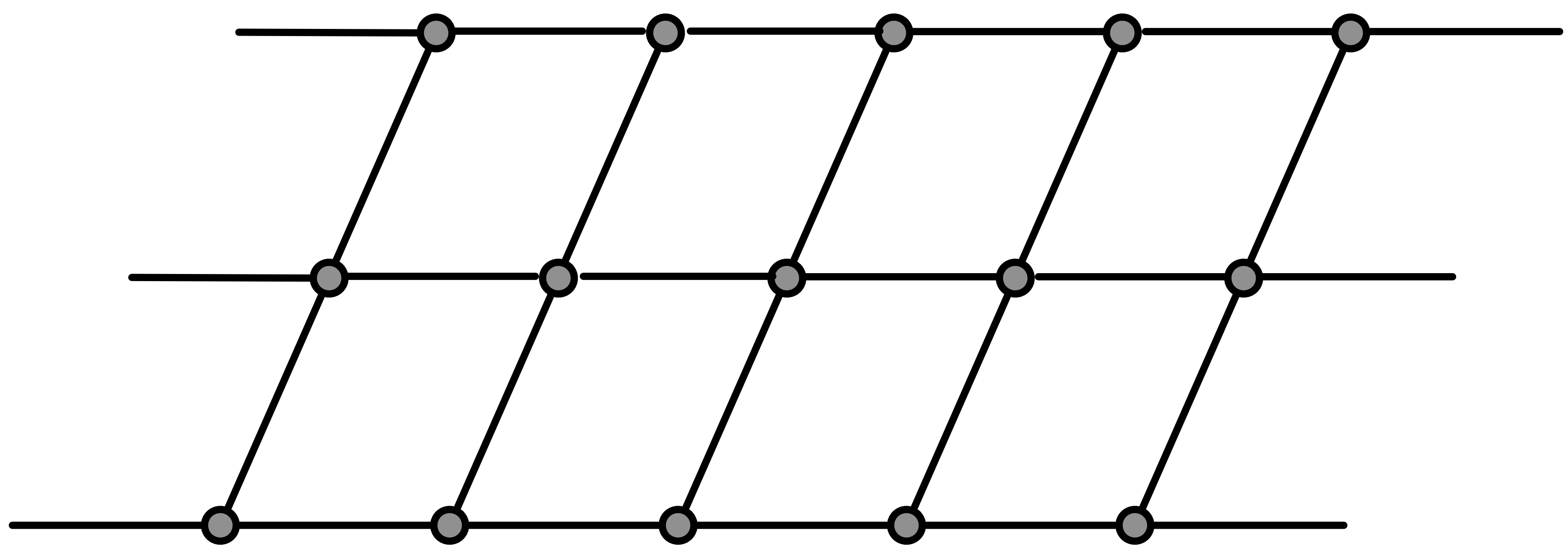

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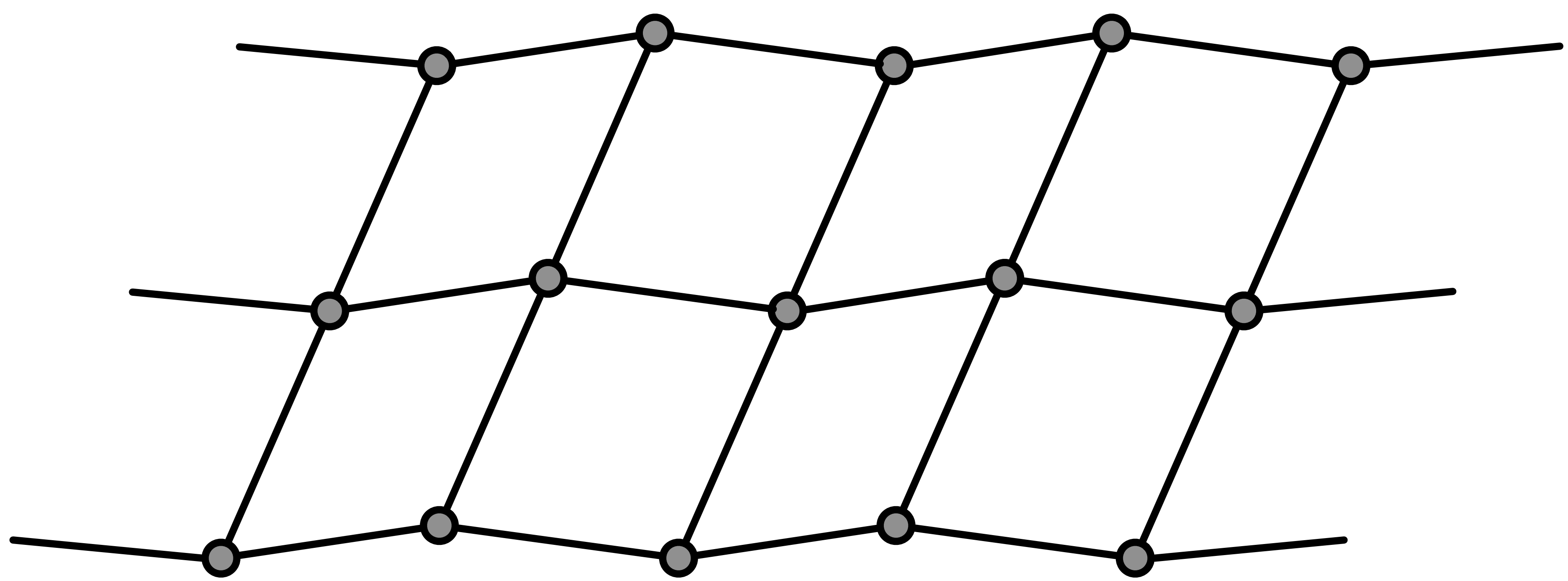
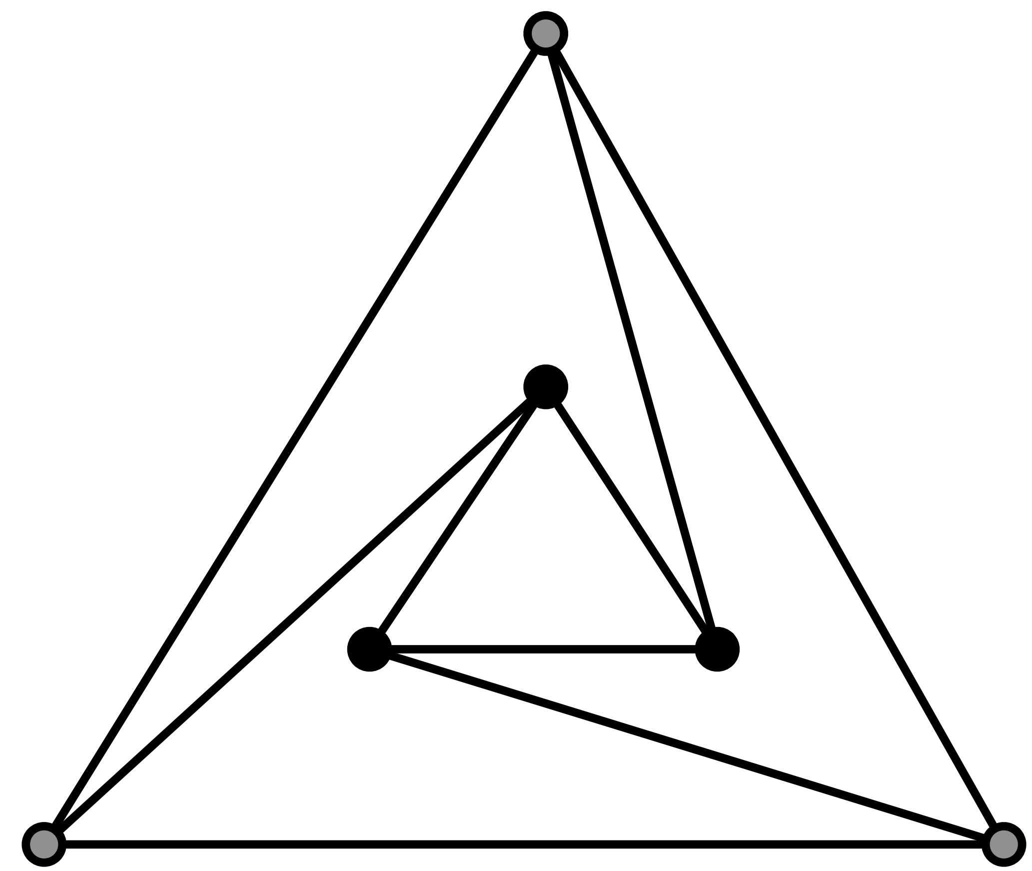
Not Allowed!



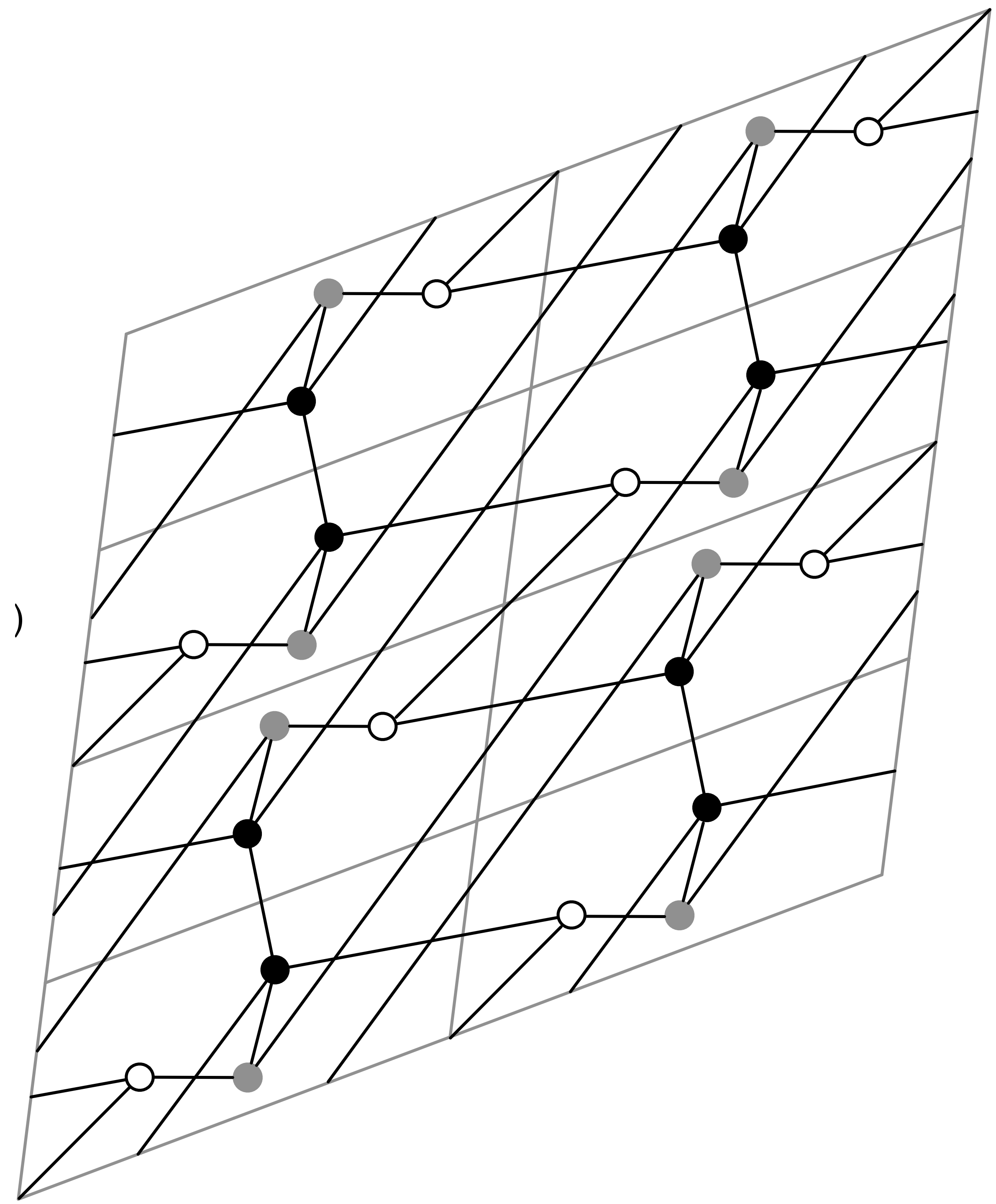
Cone frameworks

- Frameworks that are symmetric with respect to a finite order rotation
- Allowed motions preserve bars and rotational symmetry



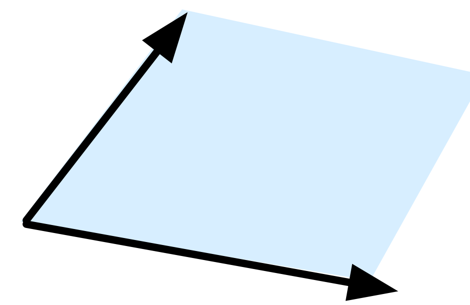
Crystal frameworks

- More generally...
- Let Γ_k be a group generated by
  - translations
  - finite order rotation
- with cocompact action
- A Γ_k-framework moves with Γ_k-symmetry

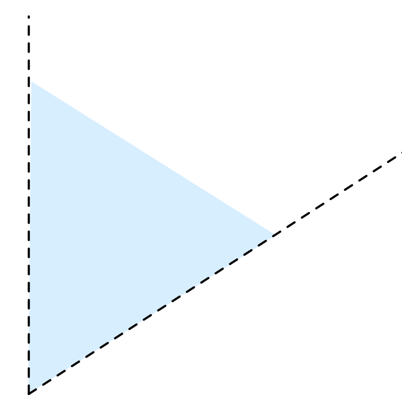


Symmetry groups

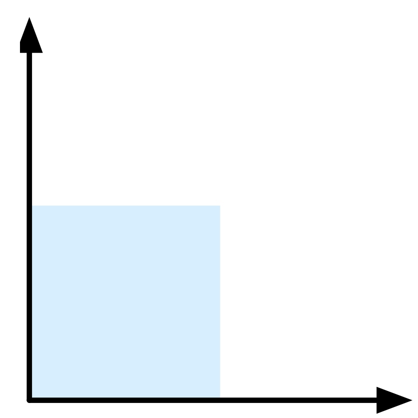
- $\mathbf{Z}^2$
  - translations



- $\mathbf{Z}/k\mathbf{Z}$
  - finite-order rotation



- Γ_k
  - translations
  - order k=2,3,4,6 rot.
- $\mathbf{Z}/2\mathbf{Z}$
  - reflection



Generic rigidity

- *Which frameworks are rigid?*
  - seems intractible
- Which...
  - *combinatorial types* of frameworks
  - are *generically* rigid*?*
- Tractable and useful
  - small perturbations of any frameworks are generic
  - efficiently checkable

Results

- *Combinatorially* characterize *generic* 2d rigidity for:
- *periodic* frameworks
- *cone* frameworks
- Γ_k crystal frameworks
- reflection frameworks

Some corollaries

- Via combinatorial equivalences:
- Fixed-lattice [Ross ’09]
- One period “flat cylinder”
- Partially-fixed lattice
- 2d periodic body-bar frameworks

Lots of related work

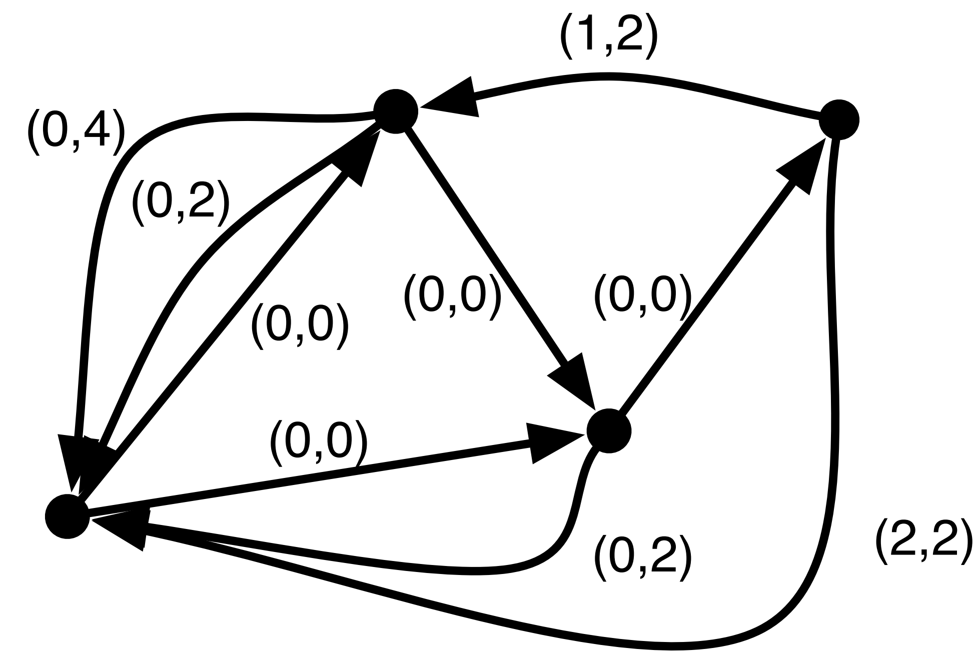
- [Borcea-Streinu ’10, ’11]
- [Guest et al.]
- [Owen & Power]
- [Ross ’09-’11]
- [Whiteley ’88]

Rest of the talk

- Combinatorial models
  - colored and periodic graphs
  - from subgraphs to subgroups
- How to count the degrees of freedom
  - the "Maxwell direction"
- Detailed statements of results
- Proof strategy
  - cf. Justin's talk later today

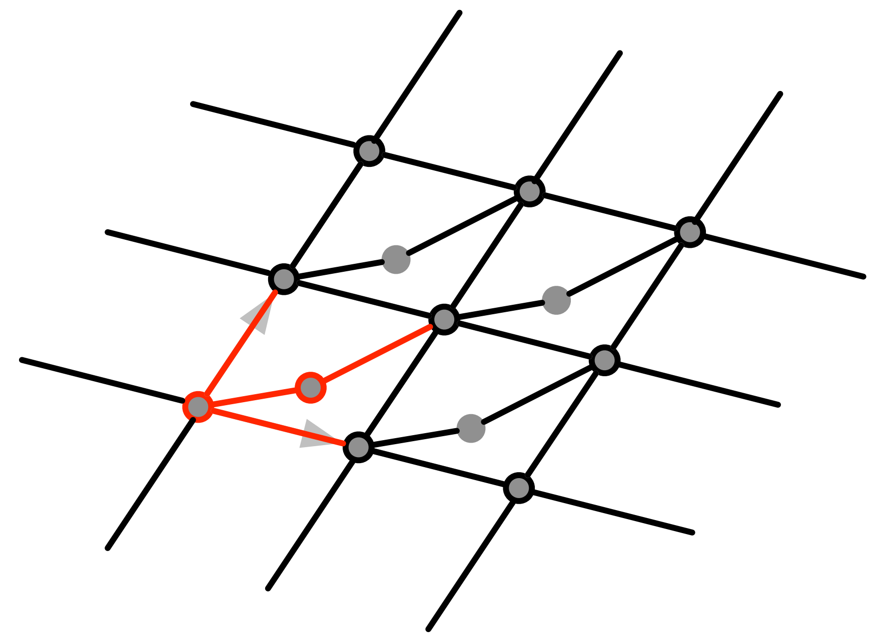
Colored graphs

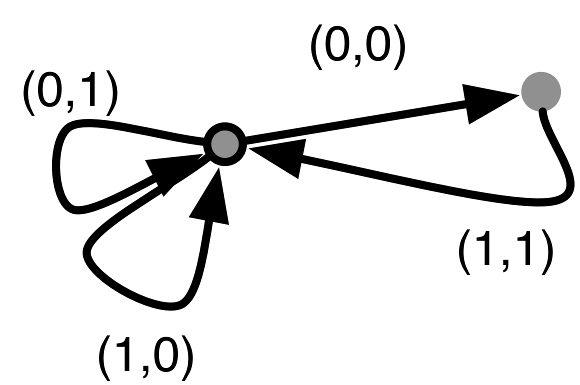
- A Γ-colored graph (G,γ) is a:
- Finite directed graph G
- Assignment of a group element γ_{ij} of Γ to each edge



Colored quotient

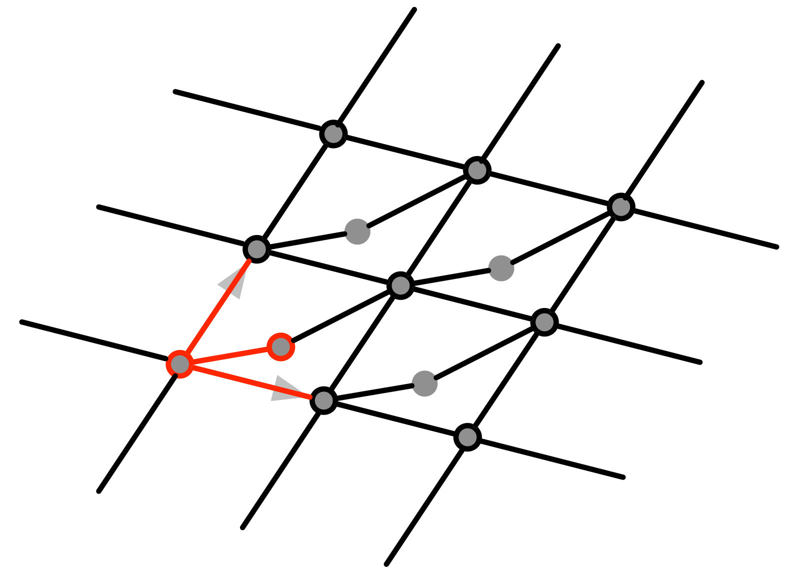
- Let G have a free Γ action
- Representative from each vertex orbit
- Representative from each edge orbit
- Determines colors and orientations in the quotient G/Γ

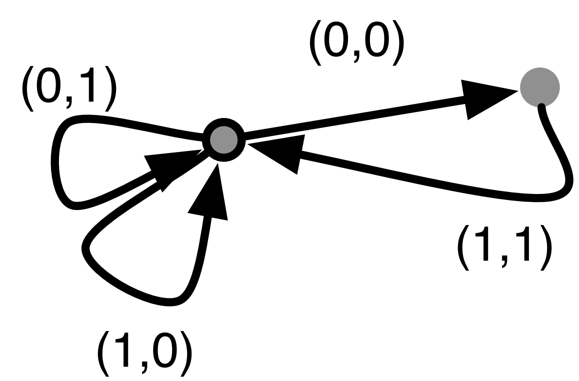




Lifting colored graphs

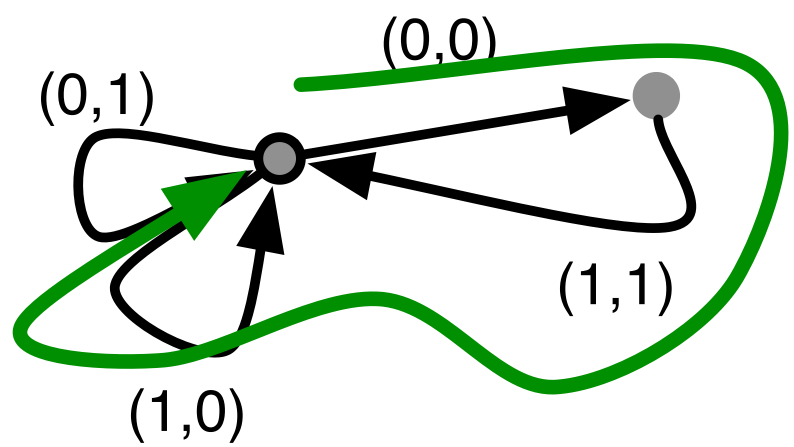
- Can go in the other direction too
- Fiber over every vertex *i* is
  - $\{\gamma \cdot i : \gamma \text{ in } \Gamma\}$
- Fiber over every edge *ij* is
  - $\{(\gamma \cdot i, (\gamma_{ij} \cdot \gamma) \cdot j)\}$





The map ρ

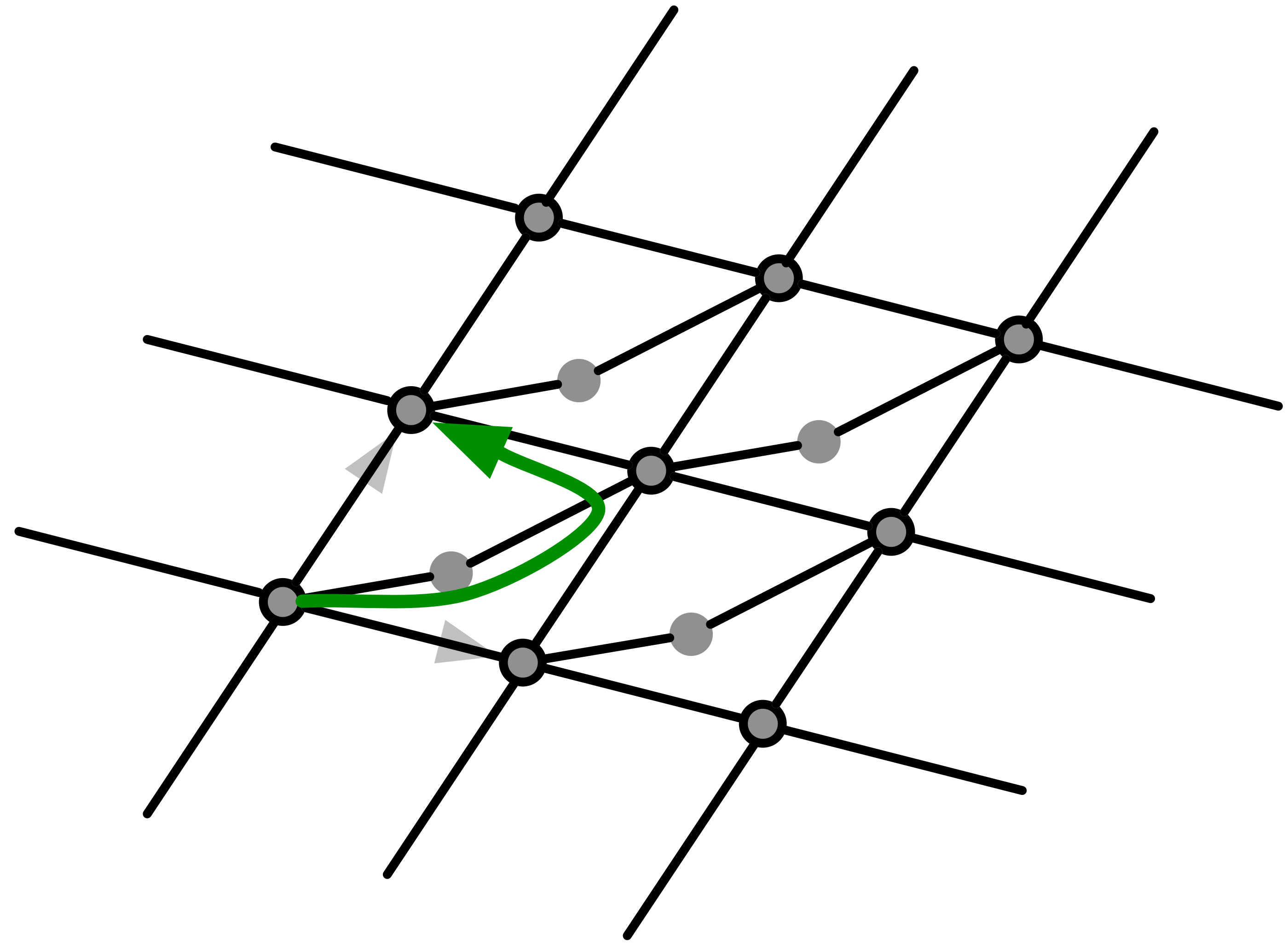
- Pick a base vertex *b* in (G,γ)
- For a closed path *P* starting and ending at *b*...
- $\rho(P) = \Pi\ \varepsilon_{ij}$,
  - *ij* on P in order
  - $\varepsilon_{ij} = \gamma_{ij}$ if *ij* traversed forwards
  - $\varepsilon_{ij} = (\gamma_{ij})^{-1}$ otherwise

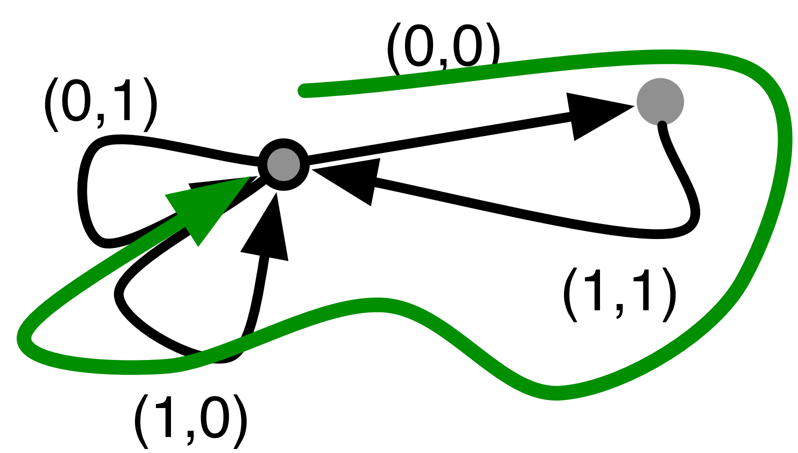


(0,0) + (1,1) - (1,0)
=
(0,1)

The map rho

- Easier interpretation in the lift
- Path with trivial rho-image lifts to a closed walk
- Path with non-trivial ρ-image ends at a different copy of the start
    - “sees” the group action

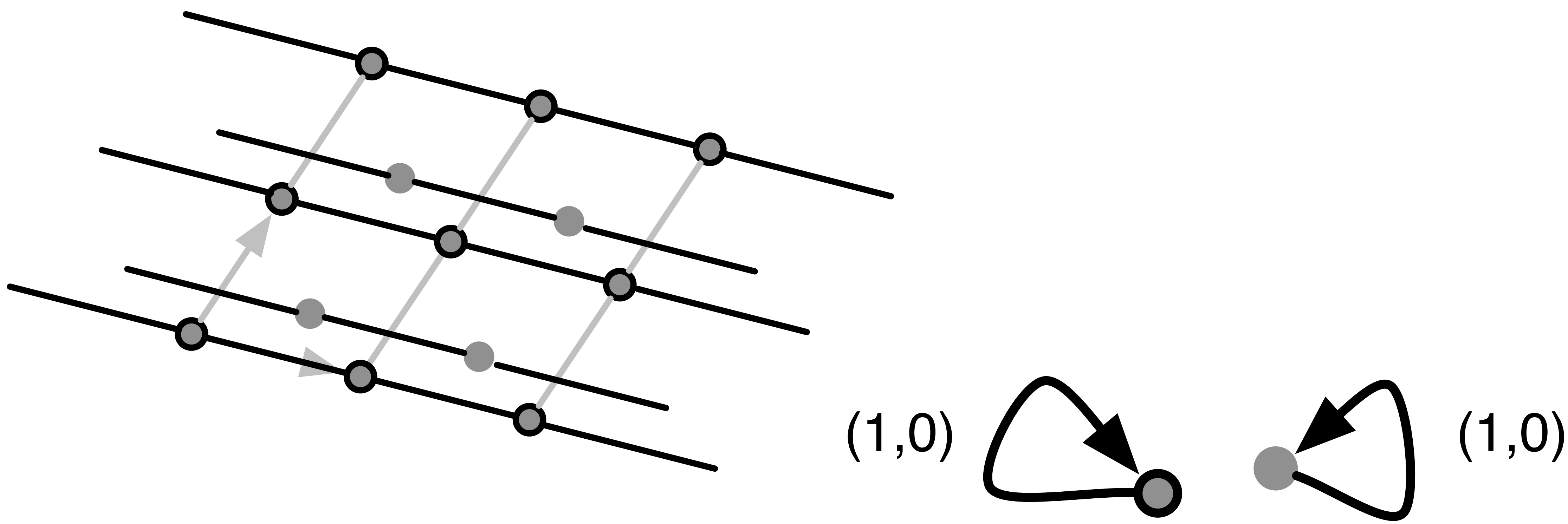




Subgraph's subgroup

- The map ρ induces a homomorphism
  - $\rho(G,b)$ from $\pi_1(G,b)$ to Γ
- If G has more than one connected component:
  - pick a base vertex for each component
  - defines $\rho(G_i,b_i)$

Example

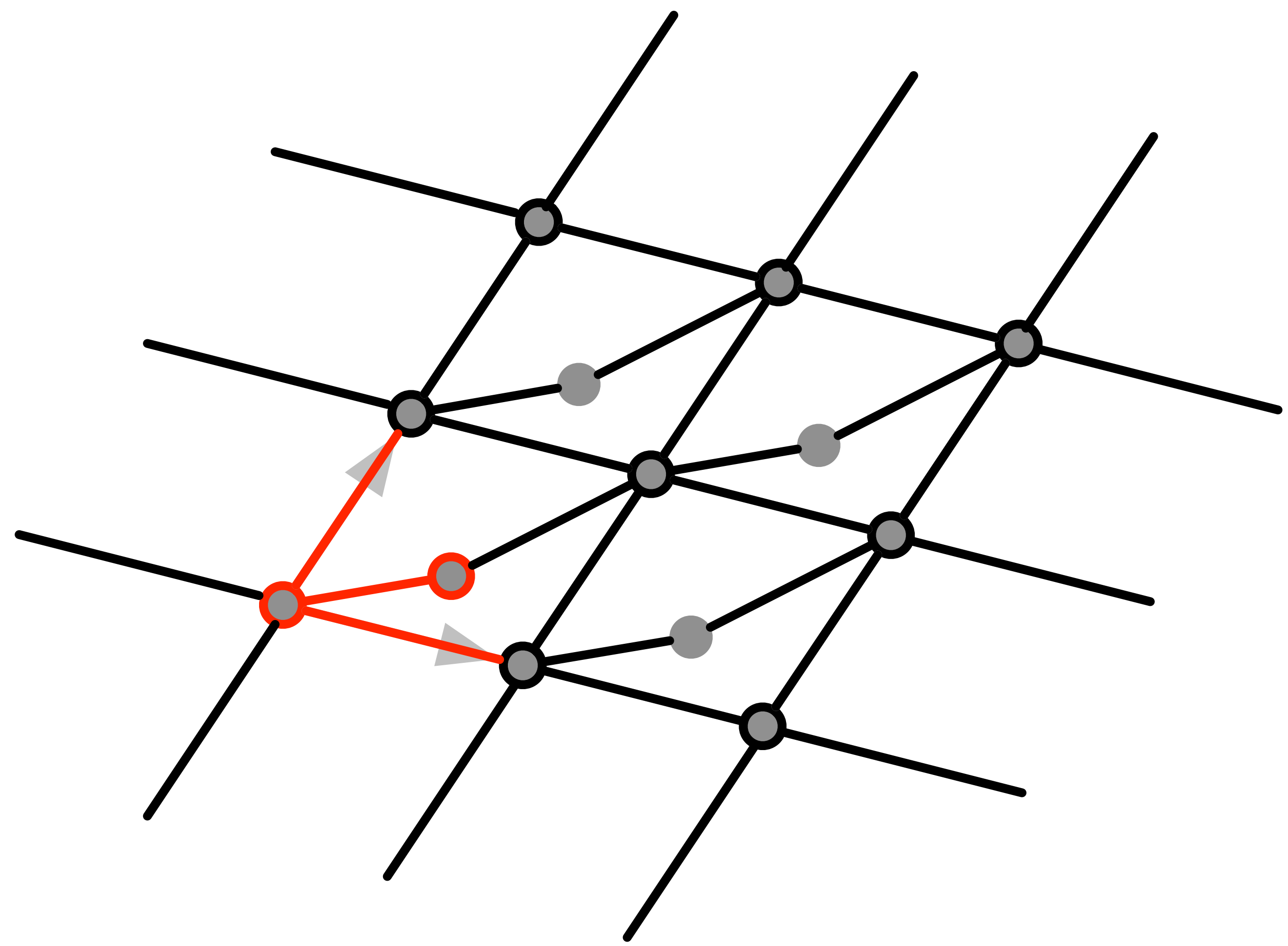


Translation subgroup

- If G is connected define $\Lambda(G,b)$ as translation subgroup of $\rho(G,b)$
- If multiple conn. components
  - $\Lambda(G,b)$ is generated by elements of all generated by $\Lambda(G_i,b_i)$
- Intuition: sees all translations generated by a walk in *some* component

Periodic realizations

- To specify a *realization* $G(\mathbf{p},\mathbf{L})$of a periodic framework need
  - coordinates $\mathbf{p}$ for each vertex of the colored quotient
  - a vector for each generator of the lattice, given by a matrix $\mathbf{L}$

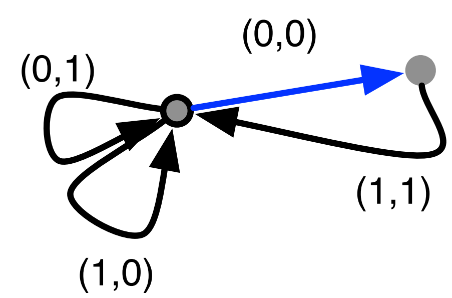


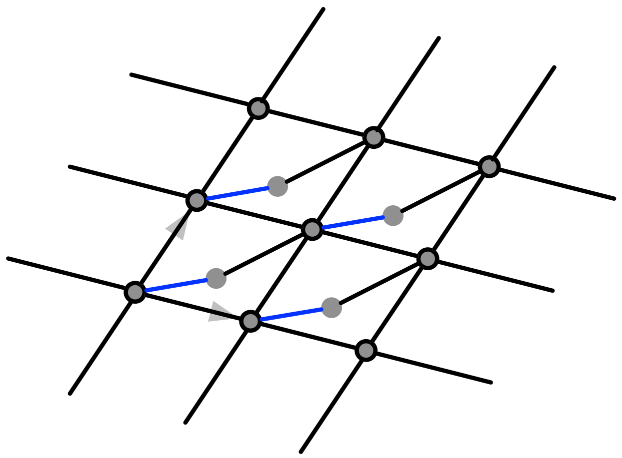
Counting d.o.f.s

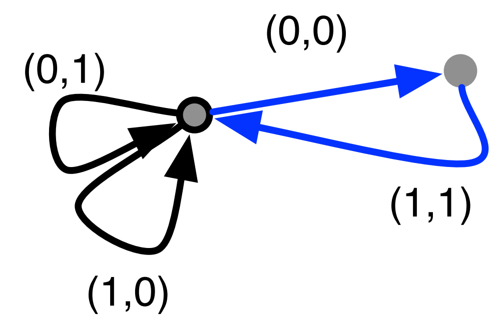
- Maxwell heuristic looks like
  - #eqns ≤ #vars - #(triv. motions)
- For finite frameworks in the plane this is
  - $m' \leq 2n' - 3$
- Since...
  - #edges = #equations
  - #vertices = #variables
  - #(triv. motions) = 3

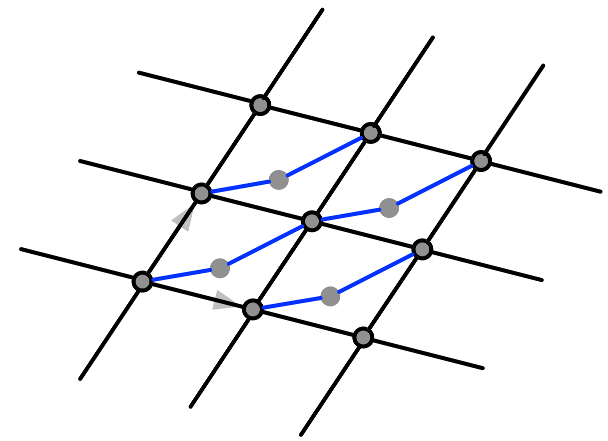
Periodic d.o.f.s

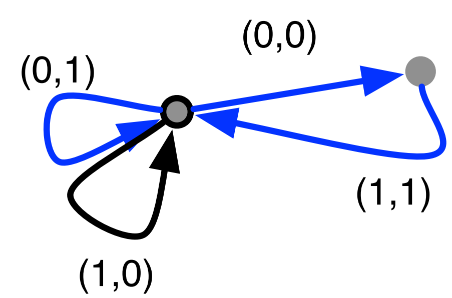
- The number of variables a subgraph influences depends on its ρ-image
- For *n*' vertices
  - 2*n*' if trivial rho-image
  - 2*n*' + 2 if one indep. translation
  - 2*n*' + 4 if two indep. translations

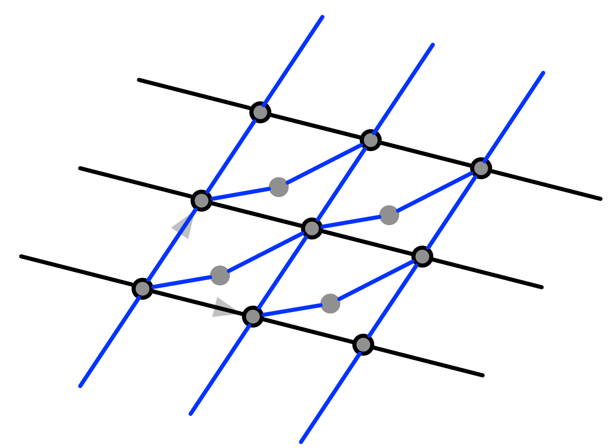






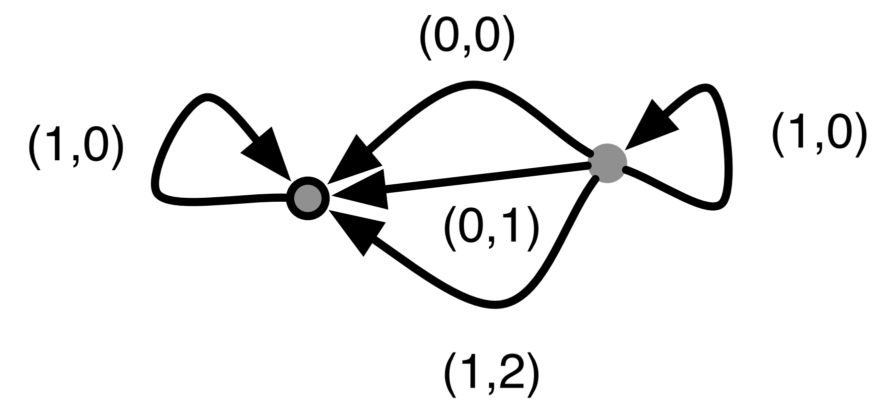


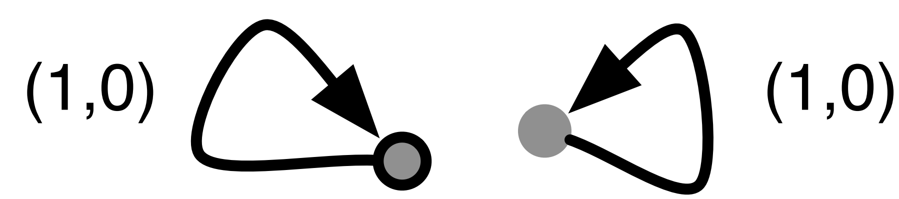


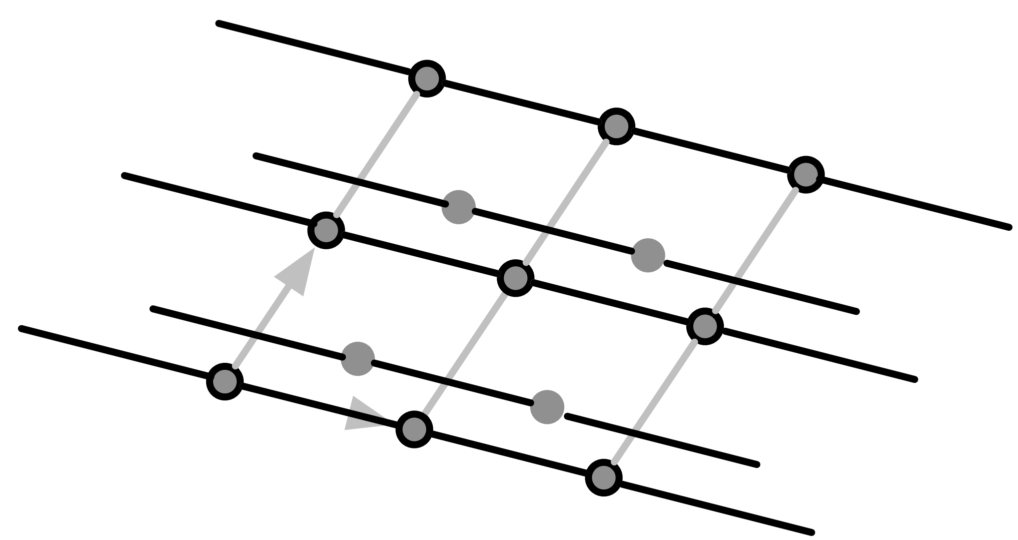


Periodic d.o.f.s

- Now count trivial motions
- Pin down one connected component:
  - 3 triv. d.o.f.
- Every other c.c. translates freely
  - 2(c - 1) triv. d.o.f.
- *Necessary* to look at connected components

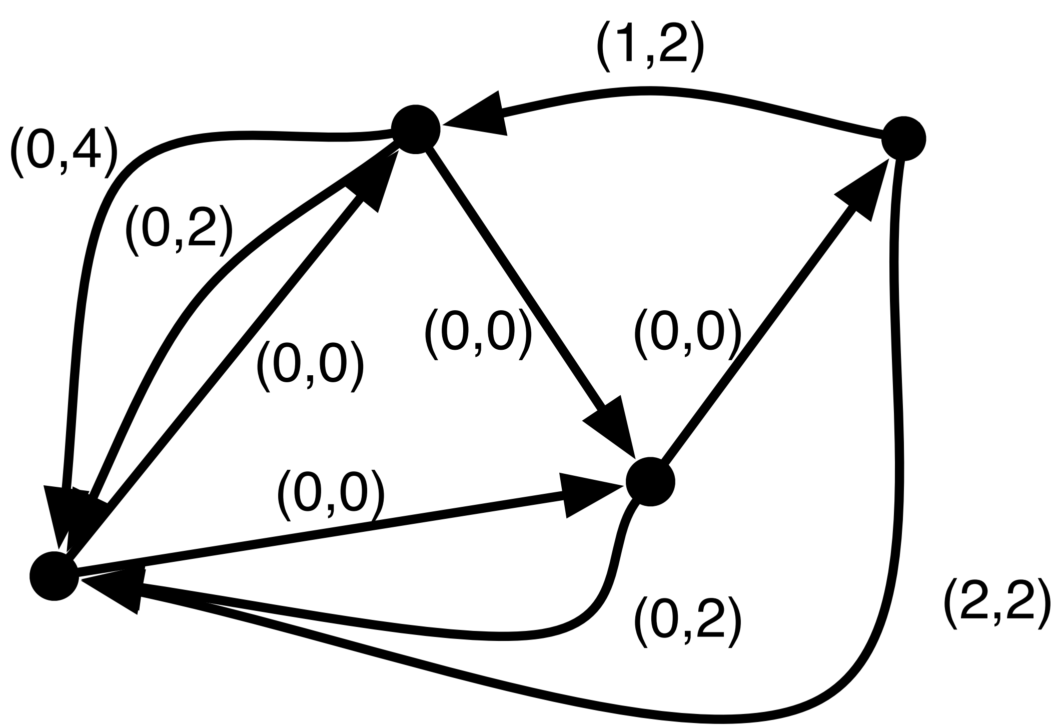


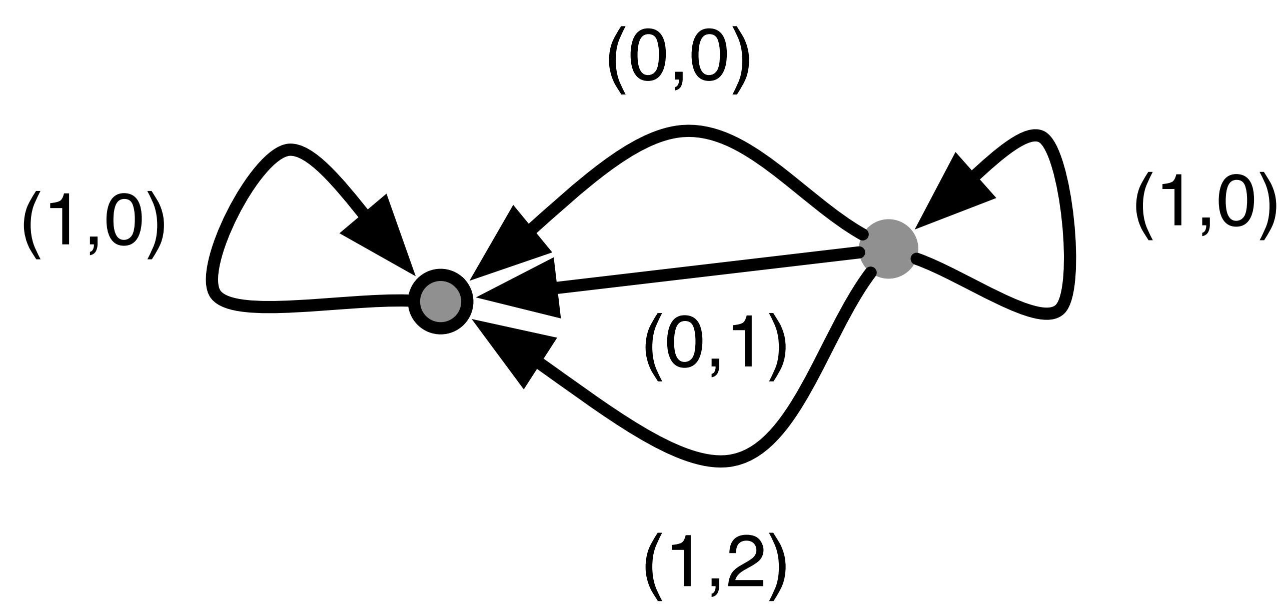




colored-Laman graphs

- A colored graph is colored-Laman if:
  - $m = 2n + 1$
  - For *all* subgraphs
    - $m' \leq 2(n' + r') - 3 - 2(c' - 1)$





Theorem

A generic periodic framework is minimally rigid

if and only if

The associated colored graph is colored-Laman.

Remarks

- Almost all realizations are generic
- If a realization is not generic, a small perturbation of the *points only* is generic
  - colored graph is the same for the perturbation

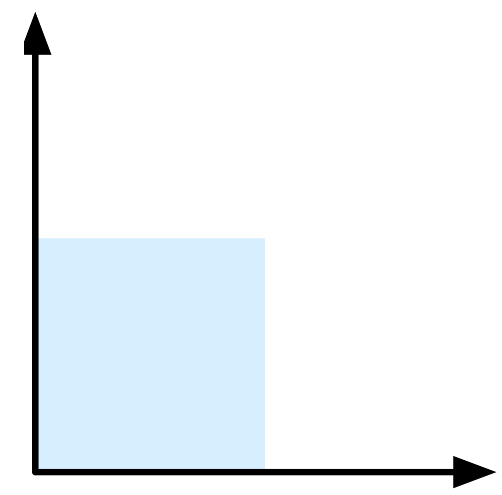
Other groups

- For other groups we can follow a general "recipe" to count d.o.f.s
- We define several spaces for subgroups
  - Representation space
  - Teichmüller space
  - Centralizer
- The dimensions of these will play the role of the "$\# \cdot$"

Spaces for subgroups

- *Representation space*: reps. extending to a rep of Γ
- *Teichmuller space*: Rep(Γ')/Euc(2)
- *Centralizer*: isometries commuting with a representation

- To give coordinates, can just specify translation vectors
- (assume the origin is a rotation center)



Dimensions

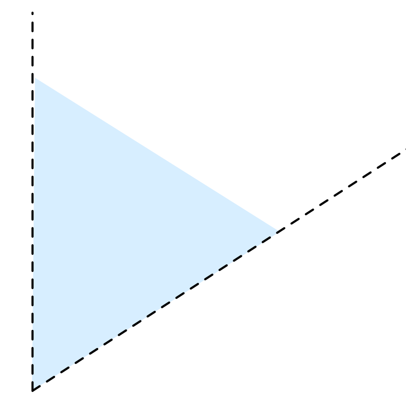
- For a colored graph (G,γ)
  - with connected components G_i
- teich(G) = dim(Teich(Λ(G)))
- cent(G_i) = dim(Cent($\rho(G_i,b_i)$))
- All these quantities are:
  - well-defined
  - independent of representations and base vertices

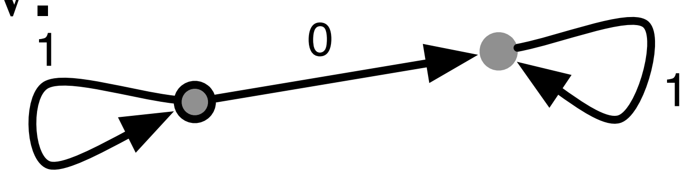
Γ-Laman graphs

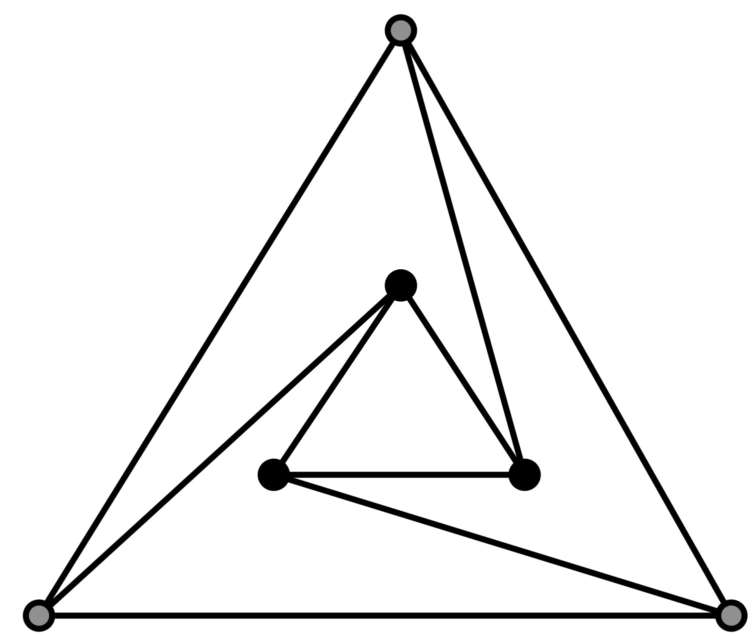
- A colored graph G with colors from a group Γ is Γ-Laman if
  - $m = 2n + \dim(\mathrm{Teich}(\Lambda(\Gamma)))$
  - For *all* subgraphs,
    - $m' \leq 2n' + \mathrm{teich}(G) - \Sigma\, \mathrm{cent}(G_i)$
- Slight refinement of the periodic colored Laman counts

Example: cone

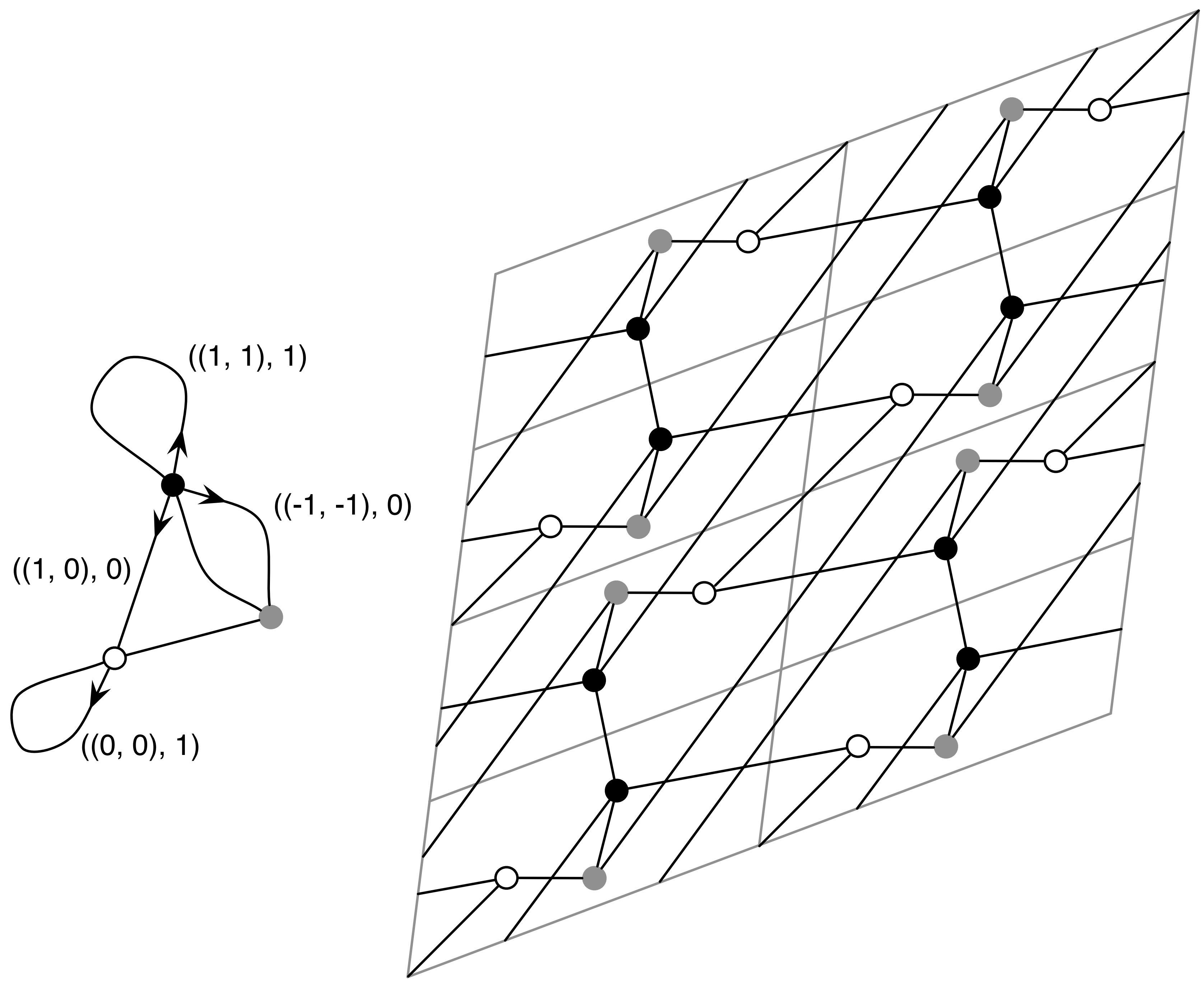
- Reps. are all defined by the rotation center
  - all eqv. by Euc(2)
- teich(Γ’) = 0
- cent(Γ’) =
  - 3 if trivial
  - 1 o.w.







Example: Γ_2

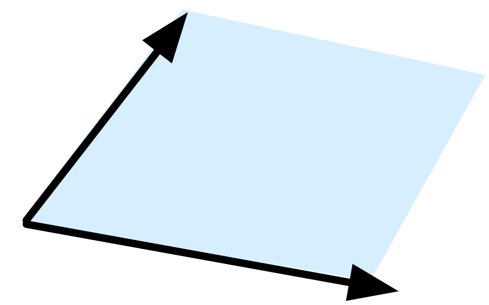


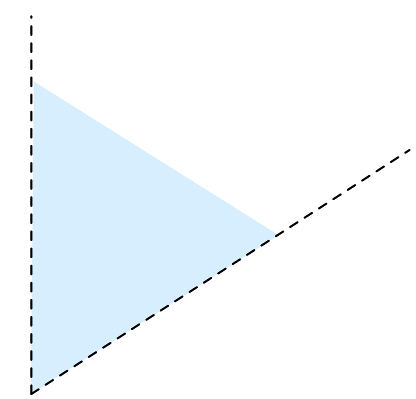
Example: Γ_2

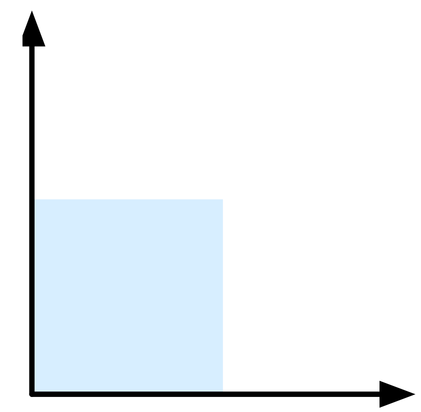
- Two indep translations:
  - teich(G') = {0,1,3}
- Centralizer has more possibilities:
  - 3 if trivial image
  - 2 if only translations
  - 1 if only rotations
  - 0 if translations and rotations
- So for *minimal rigidty* with n vertices need
  - $2n + 3$ edges

Symmetry groups

- $\mathbf{Z}^2$
  - translations
- $\mathbf{Z}/k\mathbf{Z}$
  - finite-order rotation
- Γ_k
  - translations
  - order k=2,3,4,6 rotation
- $\mathbf{Z}/2\mathbf{Z}$
  - reflection







Theorem

- For groups Γ from the prev. slide,
  A generic Γ-framework is minimally rigid

  if and only if

  The associated colored graph is Γ-Laman

Proof overview

- We use a "direction network method" for the difficult direction
- Assign directions instead of lengths to the edges
- Characterize when these directions are realizable by non-zero distinct points
  - iff the graph Γ-Laman
  - corresponds to infinitesimal rigidity

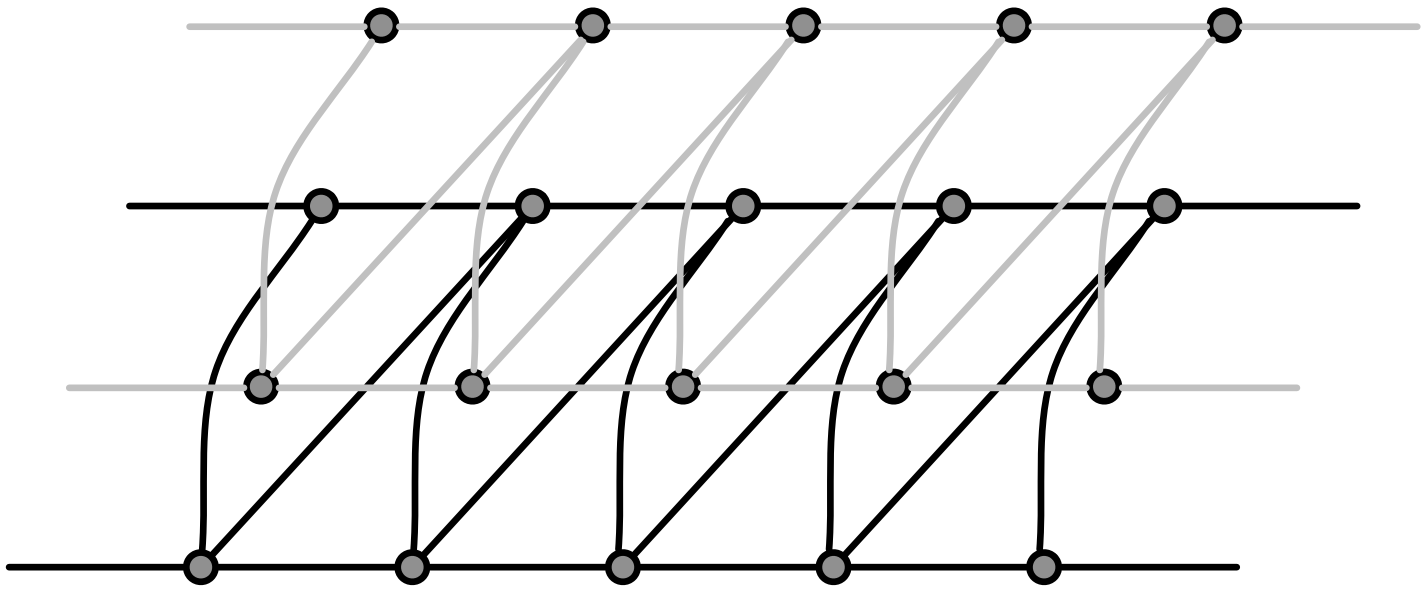
Summary

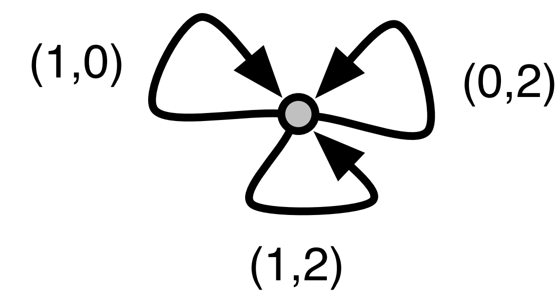
- Studied generic rigidity with forced symmetry in 2d
  - “Flexible” representation space
- Combinatorial (Laman-type) theorems for a number of groups
- New matroidal families of colored graphs
- Direction network theorems

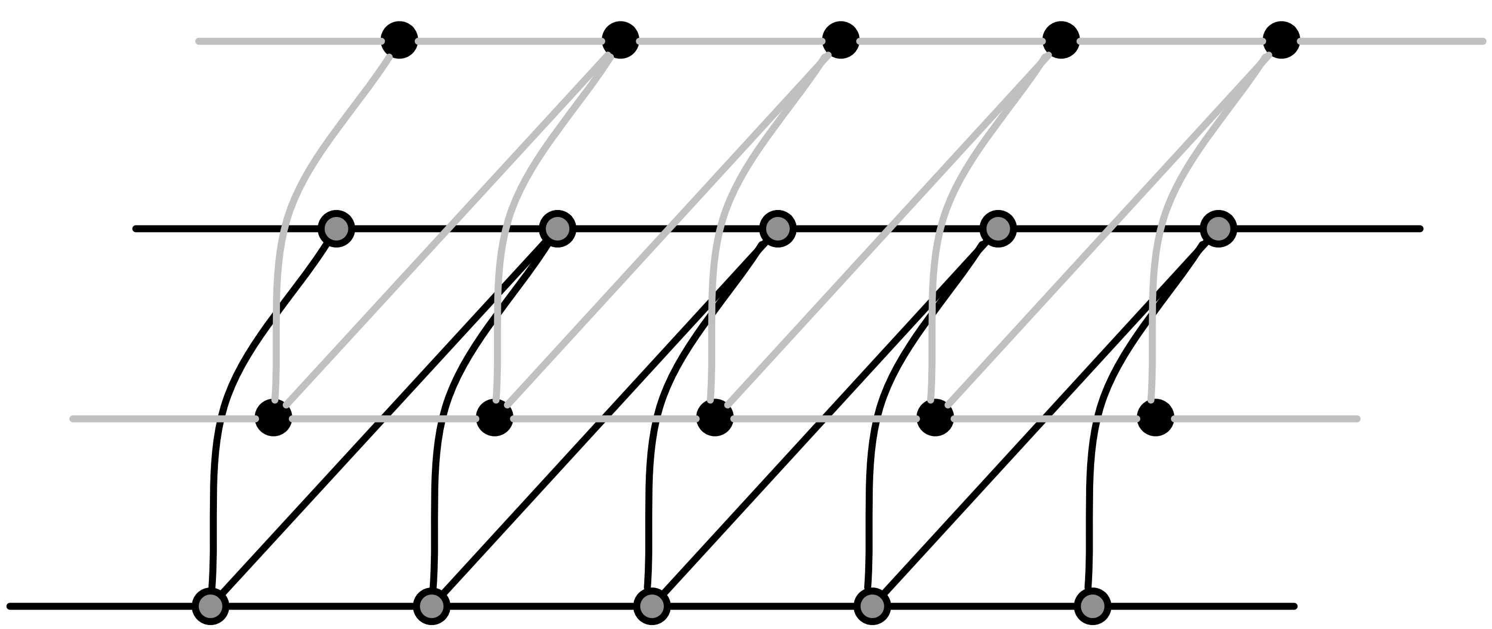
Sublattice question

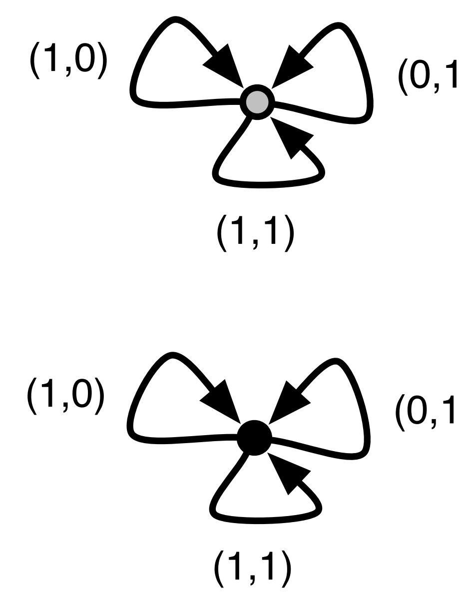
- Which ($\mathbf{Z}^2$) colored-Laman graphs (G,γ) have the property that
  - For every sub-lattice, the finite cover of (G,γ) gives a colored-Laman spanning graph
  - Are the induced frameworks always rigid if we start with a generic framework?

Example









Questions/Extensions?

- Similar result for more groups?
  - All crystallographic groups
  - $PSL(2,\mathbf{R})$ (i.e., hyperbolic surfaces)
- Can we extend more of 2d rigidity to the symmetric setting?
- Body-bar in higher dimensions?

Questions/Extensions?

- Can we extend more (k, ℓ)-sparse graph theory to colored graphs?
  - more sparsity parameters?
    - "matroidal range" can be pretty large
  - inductive constructions?
  - faster (than linear algebra) algorithms