A categorical approach to Weyl modules

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Simple complex Lie algebra

- $\mathfrak{g}$ is a simple, complex Lie algebra
- R, R^+ set of roots, Q, Q^+, P, P^+ (positive) root and weight lattice
- $x_\alpha^\pm$ and $h_\alpha = [x_\alpha^+, x_\alpha^-]$, α_i, ω_i
- $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$

Let A be a commutative associative algebra with unit over $\mathbb{C}$. We define a Lie bracket on $\mathfrak{g} \otimes A$ by

$$[x \otimes a, y \otimes b] := [x, y] \otimes ab$$

for $x, y \in \mathfrak{g}$ and $a, b \in A$.

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Weyl modules of current and loop algebras

- Relation to char p
- Chari-Pressley defined global and local Weyl modules for $\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$ (resp. $\mathfrak{g} \otimes \mathbb{C}[t]$) by generators and relations
- Motivated by representations of quantum affine algebras
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Projectives

Let $\mathcal{I}_\mathcal{A}$ be the category of $\mathfrak{g} \otimes A$ modules which are integrable as $\mathfrak{g}$ modules. The morphism are $\mathfrak{g} \otimes A$ module homomorphism. Let V be a left $\mathfrak{g}$–module, define a left $\mathfrak{g} \otimes A$ module

$$P(V) := \mathbf{U}(\mathfrak{g} \otimes A) \otimes_{\mathbf{U}(\mathfrak{g})} V.$$

Proposition

Let V be an integrable $\mathfrak{g}$ module, then $P(V)$ is a projective module in $\mathcal{I}_\mathcal{A}$. If $\lambda \in P^+$, then $P(V(\lambda))$ is generated by $p_\lambda = 1 \otimes v_\lambda$ with relations

$$\mathfrak{n}^+ \otimes 1 = 0 \ , \ (h - \lambda(h)) = 0 \ , \ (x_{\alpha_i}^- \otimes 1)^{\lambda(h_{\alpha_i})+1} = 0.$$

For $\nu \in P^+$ and $V \in \mathrm{Ob}\,\mathcal{I}_{\mathcal{A}}$, let $V^\nu \in \mathrm{Ob}\,\mathcal{I}_{\mathcal{A}}$ be the unique maximal $\mathfrak{g} \otimes A$-quotient of V satisfying

$$\mathrm{wt}(V^\nu) \subset \nu - Q^+.$$

We define $\mathcal{I}_{\mathcal{A}}^\nu$ to be full subcategory of $\mathcal{I}_{\mathcal{A}}$ of objects V s.t.

$$V = V^\nu.$$

We define for $\lambda \in P^+$

$$W_A(\lambda) := P(V(\lambda))^\lambda$$

the "global Weyl module".

Original definition

There is another definition of the global Weyl module by generator and relations, which is the "original" definition by Chari-Pressley in the case $A = \mathbb{C}[t^{\pm 1}]$.

Proposition

For $\lambda \in P^+$, the module $W_A(\lambda)$ is generated by $w_\lambda \neq 0$ with relations:

$$(\mathfrak{n}^+ \otimes A)w_\lambda = 0, \quad hw_\lambda = \lambda(h)w_\lambda, \quad (x_{\alpha_i}^- \otimes 1)^{\lambda(h_{\alpha_i})+1} w_\lambda = 0.$$

Annihilator algebra

Set

$$\mathrm{Ann}_{\mathfrak{h}\otimes A}(w_\lambda) = \{u \in \mathbf{U}(\mathfrak{h} \otimes A) : uw_\lambda = 0\},$$

and define

$$\mathbf{A}_\lambda := U(\mathfrak{h} \otimes A)/\,\mathrm{Ann}_{\mathfrak{h}\otimes A}(w_\lambda).$$

Define a right $\mathfrak{h} \otimes A$-module structure on $W_A(\lambda)$ by

$$zw_\lambda.(h \otimes a) := z(h \otimes a)w_\lambda$$

for $z \in \mathbf{U}(\mathfrak{g} \otimes A)$, $h \otimes a \in \mathfrak{h} \otimes A$.
$W_A(\lambda)$ is a bi-module for $(\mathfrak{g} \otimes A, \mathfrak{h} \otimes A)$, in fact for $(\mathfrak{g} \otimes A, \mathbf{A}_\lambda)$.

Weyl functor

Let mod $\mathbf{A}_\lambda$ be the category of left $\mathbf{A}_\lambda$–modules. Let

$$\mathbf{W}_A^\lambda : \operatorname{mod} \mathbf{A}_\lambda \to \mathcal{I}_\mathcal{A}^\lambda$$

be given by

$$\mathbf{W}_A^\lambda M = W_A(\lambda) \otimes_{\mathbf{A}_\lambda} M, \qquad \mathbf{W}_A^\lambda f = 1 \otimes f,$$

where $M, M' \in \operatorname{mod} \mathbf{A}_\lambda$ and $f \in \operatorname{Hom}_{\mathbf{A}_\lambda}(M, M')$.
We have

- $\mathbf{W}_A^\lambda M \in \operatorname{Ob} \mathcal{I}_\mathcal{A}^\lambda$.
- $\mathbf{W}_A^\lambda$ is right exact.
- $\mathbf{W}_A^\lambda \mathbf{A}_\lambda \cong_{\mathfrak{g} \otimes A} W_A(\lambda)$.

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Restriction functor

- For $\lambda \in P^+$, $V \in \operatorname{Ob} \mathcal{I}_{\mathcal{A}}^{\lambda}$, we have $V_\lambda \in \operatorname{mod} \mathbf{A}_\lambda$
- Define $\mathbf{R}_A^\lambda : \mathcal{I}_A^\lambda \to \operatorname{mod} \mathbf{A}_\lambda$ by $\mathbf{R}_A^\lambda V = V_\lambda$
- $\mathbf{R}_A^\lambda$ is an exact functor
- $\mathrm{id}_{\mathbf{A}_\lambda} \cong \mathbf{R}_A^\lambda \mathbf{W}_A^\lambda$
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We have a categorical definition of $\mathbf{W}_A^\lambda M$, which is maybe the most important improvement in this paper.

Theorem

Let $V \in \mathcal{I}_{\mathcal{A}}^\lambda$. Then $V \cong \mathbf{W}_A^\lambda \mathbf{R}_A^\lambda V$ iff for all $U \in \mathcal{I}_{\mathcal{A}}^\lambda$ with $U_\lambda = 0$, we have

$$\mathrm{Hom}_{\mathcal{I}_{\mathcal{A}}^\lambda}(V, U) = 0, \;\; \mathrm{Ext}^1_{\mathcal{I}_{\mathcal{A}}^\lambda}(V, U) = 0.$$

We can deduce from this

Corollary

The functor $\mathbf{W}_A^\lambda$ is exact iff for all $U \in \mathcal{I}_{\mathcal{A}}^\lambda$ with $U_\lambda = 0$, we have

$$\mathrm{Ext}^2_{\mathcal{I}_{\mathcal{A}}^\lambda}(\mathbf{W}_A^\lambda M, U) = 0, \;\; \forall\, M \in \mathrm{mod}\, \mathbf{A}_\lambda.$$

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Induced maps

For $f : A \to B$, denote also by f the morphism $f : \mathfrak{g} \otimes A \to \mathfrak{g} \otimes B$. For a B-module (resp. $\mathfrak{g} \otimes B$-module) M, denote by f^*M the A (resp. $\mathfrak{g} \otimes A$)-module. For $\lambda \in P^+$ we have

$$f_\lambda : \mathbf{A}_\lambda \to \mathbf{B}_\lambda$$

and bi-module map

$$f_\lambda^* : W_A(\lambda) \to f^*(W_B(\lambda)).$$

For $M \in \operatorname{mod} \mathbf{B}_\lambda$ we have

$$\mathbf{W}_A^\lambda f_\lambda^* M \to f^* \mathbf{W}_B^\lambda M \text{ as } \mathfrak{g} \otimes A - \text{modules}$$

Induced maps

The comultiplication Δ of $\mathbf{U}(\mathfrak{h} \otimes A)$ induces

$$\Delta : \mathbf{A}_{\lambda+\mu} \to \mathbf{A}_\lambda \otimes \mathbf{A}_\mu.$$

The assigment $w_{\lambda+\mu} \mapsto w_\lambda \otimes w_\mu$ induces a bi-module map

$$\tau : W_A(\lambda+\mu) \to W_A(\lambda) \otimes W_A(\mu).$$

For $M \in \operatorname{mod} \mathbf{A}_\lambda$, $N \in \operatorname{mod} \mathbf{A}_\mu$ we have

$$\tau : \mathbf{W}_A^{\lambda+\mu}\Delta^*(M \otimes N) \to \mathbf{W}_A^\lambda M \otimes \mathbf{W}_A^\mu N \text{ as } \mathfrak{g} \otimes A-\text{modules}$$

Recall

$$\mathbf{A}_\lambda = \mathbf{U}(\mathfrak{h} \otimes A) / \operatorname{Ann}_{\mathbf{U}(\mathfrak{h}\otimes A)}(w_\lambda)$$

What is $\mathbf{A}_\lambda$?

$$\lambda = \sum r_i \omega_i \ , \ r_\lambda = \sum r_i \ , \ S_\lambda := S_{r_1} \times \ldots \times S_{r_n} \subset S_{r_\lambda}$$

$$(A^{\otimes r_\lambda})^{S_\lambda} := \bigotimes_i (A^{\otimes r_i})^{S_{r_i}}$$

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Theorem

For $\lambda \in P^+$, we have

$$\mathbf{A}_\lambda \cong (A^{\otimes r_\lambda})^{S_\lambda}$$

as algebras. If A is finitely generated, than $\mathbf{A}_\lambda$ is finitely generated.

From now on, we will suppose that A is finitely generated!

$$\max(\mathbf{A}_\lambda) = \max(\bigotimes_i (A^{\otimes r_i})^{S_{r_i}})$$

which is

$$\max(A)^{\times r_1}/S_{r_1} \times \ldots \times \max(A)^{\times r_n}/S_{r_n}$$

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Irreducibles

Lemma

- *Let $\lambda \in P^+$ and assume that $V \in \mathcal{I}_A^\lambda$ is irreducible. There exists $\mu \in P^+ \cap (\lambda - Q^+)$ such that*

$$\mathit{wt}\, V \subset \mu - Q^+, \quad \dim V_\mu = 1.$$

- *V is the unique irreducible quotient of $\mathbf{W}_A^\mu \mathbf{R}_A^\mu V_\mu$.*
- *If $V' \in \operatorname{Ob} \mathcal{I}_A$, then $V \cong V'$ as $\mathfrak{g} \otimes A$–modules iff $V_\mu \cong V'_\mu$ as $\mathbf{A}_\mu$–modules.*
- *For $M \in \operatorname{irr} \mathbf{A}_\lambda$, we denote the unique irreducible quotient of $\mathbf{W}_A^\lambda M$ by $\mathbf{V}_A^\lambda M$.*

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Finitely supported functions

- Define $\Xi := \{\xi : \max(A) \to P^+ \mid \xi \text{ finitely supported}\}$
- $\operatorname{supp} \xi := \{S \in \max(A) \mid \xi(S) \neq 0\}$
- $\operatorname{wt} \xi := \sum_{S \in \max(A)} \xi(S)$
- $\Xi_\lambda := \{\xi \in \Xi \mid \operatorname{wt} \xi = \lambda\}$
- $M \in \operatorname{mod} \mathbf{A}_\lambda$, finite–dimensional, then $\operatorname{supp} M = \bigcup \operatorname{supp}(\xi_i)$

Ξ_λ parametrizes $\max(\mathbf{A}_\lambda)$, irr $\mathbf{A}_\lambda$, and so irr $\mathbf{U}(\mathfrak{g} \otimes A)$ of highest weight λ.

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Ξ_λ parametrizes $\max(\mathbf{A}_\lambda)$, $\operatorname{irr} \mathbf{A}_\lambda$, and so $\operatorname{irr} \mathbf{U}(\mathfrak{g} \otimes A)$ of highest weight λ.

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- *For $\lambda \in P^+$, $W_A(\lambda)$ is a finitely generated right $\mathbf{A}_\lambda$–module.*
- *If $M \in \operatorname{mod} \mathbf{A}_\lambda$ is finitely generated then $\mathbf{W}_A^\lambda M$ is a finitely generated left $\mathfrak{g} \otimes A$–module. Same for finite–dimensional modules.*
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A generalization of the tensor product phenomenon

Theorem

Suppose that A and B are finite–dimensional commutative, associative algebras and let $\lambda, \mu \in P^+$*. For* $M \in \operatorname{mod} \mathbf{A}_\lambda$, $N \in \operatorname{mod} \mathbf{B}_\mu$*, finite–dimensional, we have,*

$$\mathbf{W}^{\lambda+\mu}_{A \oplus B}(M \otimes N) \cong \mathbf{W}^{\lambda}_{A} M \otimes \mathbf{W}^{\mu}_{B} N,$$

as $\mathfrak{g} \otimes (A \oplus B)$*–modules.*

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Let $\lambda, \mu \in P^+$, *and* $M \in \operatorname{mod} \mathbf{A}_\lambda$, $N \in \operatorname{mod} \mathbf{A}_\mu$, *finite–dimensional with* $\operatorname{supp} M \cap \operatorname{supp} N = \emptyset$. *Then we have*

- $\mathbf{W}_A^{\lambda+\mu}(M \otimes N) \cong_{\mathfrak{g}\otimes A} \mathbf{W}_A^\lambda M \otimes \mathbf{W}_A^\mu N.$
- *If* M, N *are irreducible, then* $\mathbf{V}_A^{\lambda+\mu}(M \otimes N) \cong_{\mathfrak{g}\otimes A} \mathbf{V}_A^\lambda M \otimes \mathbf{V}_A^\mu N.$

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We want to analyze $\mathbf{W}_A^\lambda M_\xi$, $M_\xi \in \operatorname{irr} \operatorname{mod} \mathbf{A}_\lambda$. The tensor product theorem gives

$$\mathbf{W}_A^\lambda M_\xi \cong_{\mathfrak{g}\otimes A} \bigotimes_{S \in \operatorname{supp} \xi} \mathbf{W}_A^{\xi(S)} M_{\xi_S}, \ \operatorname{supp} \xi_S = \{S\}$$

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We define

$$\Xi_\lambda^{\mathrm{ns}} = \{\xi \in \Xi_\lambda : \xi(S) \in \{0, \omega_1, \dots, \omega_n\},\ \forall S \in \max A\}.$$

Then $\Xi_\lambda^{\mathrm{ns}}$ is an open subset and

$\Xi_\lambda^{\mathrm{ns}} \leftrightarrow \{$orbits of non–singular points of the $S_{r_\lambda}-$action on $\max(A^{\otimes r_\lambda})\}$.

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Notations

Let $J_0 \subset I$ be defined as follows:

$$J_0 = \begin{cases} I, & \mathfrak{g} \text{ of type } A_n,\ C_n, \\ \{n\}, & \mathfrak{g} \text{ of type } B_n, \\ \{n-1, n\}, & \mathfrak{g} \text{ of type } D_n. \end{cases}$$

Given $m \in \mathbb{Z}_+$, let $\mathbf{c}(m,k)$ be the dimension of the space of polynomials of degree m in k–variables, i.e

$$\mathbf{c}(m,k) = \#\{\mathbf{s} = (s_1, \cdots, s_k) :\in \mathbb{Z}_+^k : s_1 + \cdots + s_k = m\}.$$

Theorem

Let $\mathfrak{g}$ be of classical type. Let $S \in \max A$ be such that $\dim S/S^2 = k$ and for $i \in I$, let $M_S \in \operatorname{irr} \operatorname{mod} \mathbf{A}_{\omega_i}$.

- *If $i \in J_0$, then $\mathbf{W}_A^{\omega_i} M_S \cong_{\mathfrak{g}} V(\omega_i)$.*
- *If $i \notin J_0$, then*

$$\mathbf{W}_A^{\omega_i} M_S \cong_{\mathfrak{g}} \bigoplus_j V(\omega_{i-2j})^{\oplus \mathbf{c}(j,k)}.$$

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