《Quantum Groups & Crystal Bases II》 8/16/09 Ottawa

2. Kac-Moody algebras

(Example) Let $L = sl(n+1, \mathbb{F}) = \{x \in gl(n+1, \mathbb{F}) \mid \mathrm{tr}\, x = 0\}$

$\Rightarrow L = \mathrm{Span}\{E_{ij}\ (i \neq j),\ E_{i,i} - E_{i+1,i+1}\ (i = 1, \dots, n)\}$

Set $e_i = E_{i,i+1}$, $f_i = E_{i+1,i}$, $h_i = E_{i,i} - E_{i+1,i+1}$ $(i = 1, \dots, n)$.

$\Rightarrow L$ is generated by e_i, f_i, h_i $(i = 1, \dots, n)$.

Moreover, they satisfy the relations : $[h_i, h_j] = 0$

$$[h_i, e_j] = \begin{cases} 2e_i & j = i \\ -e_j & j = i \pm 1 \\ 0 & \text{others} \end{cases} \qquad [e_i, f_j] = \delta_{ij} h_i$$

$$[h_i, f_j] = \begin{cases} -2f_i & j = i \\ f_j & j = i \pm 1 \\ 0 & \text{others} \end{cases}$$

$$\left(\begin{array}{l} [e_i, [e_i, e_j]] = 0 = [f_i, [f_i, f_j]] \quad \text{if } j = i \pm 1 \quad (j \neq i) \\ [e_i, e_j] = [f_i, f_j] = 0 \quad \text{others} \end{array} \right.$$

Set $A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}_{i,j=1,\dots,n}$: Cartan matrix of L

Then $[h_i, h_j] = 0$, $[e_i, f_j] = \delta_{ij} h_i$,

$[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$,

$(\mathrm{ad}\, e_i)^{1-a_{ij}}(e_j) = (\mathrm{ad}\, f_i)^{1-a_{ij}}(f_j) = 0 \quad (i \neq j)$,

where $\mathrm{ad}\,x(y) = [x,y]$ $(x, y \in L)$.

Motivated by this, we will consider the class of Lie algebras defined by ~~a matrix A & the relations determined by A~~. generators & relations which are determined by a matrix A. More precisely,

$\boxed{\text{Def}}$ GCM. $A = (a_{ij})_{i,j \in I}$

i) $a_{ii} = 2$ $\forall i \in I$ ii) $a_{ij} \in \mathbb{Z}_{\le 0}$ $i \neq j$,

iii) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

We assume A is <u>symmetrizable</u>; i.e., $\exists$ a diagonal matrix $D = \mathrm{diag}(s_i \in \mathbb{Z}_{>0} \mid i \in I)$ s.t. DA is symmetric.

Set $P^\vee = \left(\bigoplus_{i \in I} \mathbb{Z} h_i\right) \oplus \left(\bigoplus_{j=1}^{\operatorname{corank} A} \mathbb{Z} d_j\right)$, $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$,

$P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z}\}$, $\Pi^\vee = \{h_i \mid i \in I\}$,

$\Pi = \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$, linearly indep. s.t.

$\langle h_i, \alpha_j \rangle = a_{ij}$

$\boxed{\text{Def}}$ $(A, P^\vee, \Pi^\vee, P, \Pi)$: <u>Cartan datum</u>

$Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ (<u>root lattice</u>), $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$, $Q_- = -Q_+$

$P^+ = \{\lambda \in P \mid \lambda(h_i) \geq 0 \ \forall i \in I\}$: <u>dominant integral weights</u>

$\boxed{\text{Def}}$ $\mathfrak{g}$ = Kac-Moody alg. assoc. with $(A, \cdots, \Pi)$
= Lie alg / $\mathbb{C}$ generated by e_i, f_i $(i \in I)$, $h \in P^\vee$
with defining relations:

$[h, h'] = 0 \quad \forall h, h' \in P^\vee$, $\quad [e_i, f_j] = \delta_{ij} h_i$,

$[h, e_j] = \alpha_j(h) e_j$, $\quad [h, f_j] = -\alpha_j(h) f_j$,

$(\mathrm{ad}\, e_i)^{1-a_{ij}}(e_j) = 0$, $(\mathrm{ad}\, f_i)^{1-a_{ij}}(f_j) = 0 \quad \forall i \neq j$.

(Example) ① $\mathfrak{g}$: f.d. simple Lie alg (finite type)

② $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$;
affine Kac-Moody alg (affine type)

③ $\mathfrak{g}$: ~~indef~~ Kac-Moody alg of indef type

Def $\mathcal{O}_{\mathrm{int}} \overset{\mathrm{def}}{=}$ category of $\mathfrak{g}$-modules M s.t.

i) $M = \bigoplus_{\lambda \in P} M_\lambda$, where $M_\lambda = \{m \in M \mid h \cdot m = \lambda(h) m \ \forall h \in \mathfrak{h}\}$

(ii) & $\dim M_\lambda < \infty \quad \forall \lambda$,

ii) $\exists \lambda_1, \cdots, \lambda_s \in P$ s.t. $\mathrm{wt}(M) \subset \bigcup_{j=1}^{s} (\lambda_j - Q_+)$,

iii) e_i, f_i are loc nilp on M.

Def $M \in \mathcal{O}_{\mathrm{int}}$; $M = \bigoplus_{\lambda \in P} M_\lambda$

$\mathrm{ch}\, M$ = <u>character</u> of M

$$= \sum_{\lambda \in P} (\dim M_\lambda) e^\lambda$$

Rmk ① 1st clue to understanding M

② algebraic invariant

③ important & interesting mathematical objects: symm fns, modular fns, L-pt fns, etc.

$\boxed{\text{Def}}$ V: $\mathfrak{g}$-module

V is a <u>h.w. module</u> <u>with</u> <u>h.w. $\lambda \in P$</u> if

$\exists v_\lambda \neq 0$ in V s.t. i) $h v_\lambda = \lambda(h) v_\lambda$ $\forall h \in \mathfrak{h}$,

ii) $e_i v_\lambda = 0$ $\forall i \in I$, iii) $V = U(\mathfrak{g}) v_\lambda$.

$\boxed{\text{Note}}$ $U^+ = \langle e_i \mid i \in I \rangle = U(\mathfrak{g}_+)$

$U^- = \langle f_i \mid i \in I \rangle = U(\mathfrak{g}_-)$

$U^0 = U(\mathfrak{h})$

$\Longrightarrow V = U(\mathfrak{g}) v_\lambda = U^- v_\lambda = \bigoplus_{\mu \leq \lambda} V_\mu$.

(~~Example~~) $J(\lambda)$ = left ideal of $U(\mathfrak{g})$ generated by

e_i $(i \in I)$, $h - \lambda(h)\mathbb{1}$ $(h \in \mathfrak{h})$.

Set $M(\lambda) \overset{\text{def}}{=} U(\mathfrak{g}) \big/ J(\lambda)$: <u>Verma module</u>

Prop ① $M(\lambda)$ is a h.w. module with h.w. λ

② $M(\lambda)$ is a free U^--module of rank 1

③ Every h.w. $\mathfrak{g}$-module with h.w. λ is a homomorphic image of $M(\lambda)$

④ $\exists !$ maximal submodule $R(\lambda)$ of $M(\lambda)$.

Def $V(\lambda) \overset{\text{def}}{=} M(\lambda)/R(\lambda)$: irred h.w. module

Prop ① $V \in \mathcal{O}_{int}$, irred $\Rightarrow V \cong V(\lambda)$, $\lambda \in P^+$

② $\mathcal{O}_{int}$ is semisimple

③ $M \in \mathcal{O}_{int}$, $\forall i \in I$

$\Rightarrow M \cong \bigoplus$ (f.d. irred U_i-submodules),

where $U_i = \langle e_i, f_i, h_i \rangle \cong U(\mathfrak{sl}_2)$.

Rmk Key input : Weyl-Kac char formula for $V(\lambda)$, $\lambda \in P^+$

3. Quantum Groups

q: indeterminate, $q_i = q^{s_i}$, $[n]_i = \dfrac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$

Def $U_q(\mathfrak{g})$ = quantum group
= $\mathbb{C}(q)$-alg with 1 generated by e_i, f_i $(i \in I)$
& q^h $(h \in P^\vee)$ with defing relations

$q^h q^{h'} = q^{h+h'}$,
$q^h e_j q^{-h} = q^{\alpha_j(h)} e_j$, $q^h f_j q^{-h} = q^{-\alpha_j(h)} f_j$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad K_i = q^{s_i h_i}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i e_i^{1-a_{ij}-k} e_j e_i^k = 0$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i f_i^{1-a_{ij}-k} f_j f_i^k = 0 \quad (i \neq j)$$

$\Delta(q^h) = q^h \otimes q^h$

$\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i$, $\Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i$

$S(e_i) = -e_i K_i$, $S(f_i) = -K_i^{-1} f_i$, $S(q^h) = q^{-h}$

$\boxed{\text{Def}}$ $\mathcal{O}^q_{int}$ = category of $U_q(\mathfrak{g})$-modules M^q s.t.

i) $M^q = \bigoplus_{\lambda \in P} M^q_\lambda$, where $M^q_\lambda = \{ m \in M^q \mid q^h m = q^{\lambda(h)} m \ \forall h \in P^\vee \}$,
$\dim_{\mathbb{Q}(q)} M^q_\lambda < \infty$

ii) $\exists \lambda_1, \dots, \lambda_s \in P$ s.t. $\mathrm{wt}(M^q) \subset \bigcup_{j=1}^{s} (\lambda_j - Q_+)$,

iii) e_i, f_i $(i \in I)$ are loc. n.lp on M^q.

Define $\mathrm{ch}\, M^q = \sum_{\lambda \in P} (\dim_{\mathbb{Q}(q)} M^q_\lambda) e^\lambda$

= <u>character</u> of M^q.

$\boxed{\text{Def}}$ V^q: $U_q(\mathfrak{g})$-module

V^q is a <u>h.w. module</u> with h.w. $\lambda \in P$ if
$\exists v_\lambda \neq 0$ in V^q s.t. i) $q^h v_\lambda = q^{\lambda(h)} v_\lambda$ $\forall h \in P^\vee$,
ii) $e_i v_\lambda = 0$ $\forall i \in I$, iii) $V^q = U_q(\mathfrak{g}) v_\lambda$.

<u>Note</u>: $U_q^+ = \langle e_i \mid i \in I\rangle$, $U_q^- = \langle f_i \mid i \in I \rangle$,
$U_0 = \langle q^h \mid h \in P^\vee \rangle$

$$\Rightarrow V^q = U_q(g)\, v_\lambda = U_q^-\, v_\lambda = \bigoplus_{\mu \leq \lambda} V^q_\mu .$$

Defn: $J^q(\lambda)$ = left ideal of $U_q(g)$ generated by
e_i ($i \in I$), $q^h - q^{\lambda(h)} \mathbb{1}$ ($h \in P^\vee$)

Set $M^q(\lambda) = U_q(g) / J^q(\lambda)$: <u>Verma module</u>

$\boxed{\text{Prop}}$ (Exercise: State similar props.)

① $M^q(\lambda)$: h.w. module

② : free U_q^--module

③ V^q: h.w. module $\Rightarrow$ $M^q(\lambda) \twoheadrightarrow V^q$

④ $\exists!$ maximal submodule $R^q(\lambda)$.

$$V^q(\lambda) = M^q(\lambda) / R^q(\lambda) \ : \ \underline{\text{irred}}\ \underline{\text{h.w.}}\ \underline{\text{module}}.$$

Set $A_1 = \{ f/g \in \mathbb{C}(q) \mid f, g \in \mathbb{C}[q],\ g(1) \neq 0 \}$.

$U_{A_1} \overset{\text{def}}{=} A_1$-subalg of $U_q(\mathfrak{g})$ gen by

$e_i, f_i\ (i \in I),\ q^h\ (h \in P^\vee),\ \dfrac{q^h - 1}{q - 1}\ (h \in P^\vee)$

$V^q_{A_1} \overset{\text{def}}{=} U_{A_1} \cdot v_\lambda \left(= U^-_{A_1} v_\lambda \right)$

$J_1 =$ ideal of A_1 gen by $q - 1$

$\Rightarrow A_1 / J_1 \xrightarrow{\sim} \mathbb{C}, \quad f + J_1 \longmapsto f(1)$

$\hphantom{\Rightarrow A_1 / J_1 \xrightarrow{\sim} \mathbb{C}, \quad f + } q \longmapsto 1$

Set $U^1 \overset{\text{def}}{=} \mathbb{C} \otimes_{A_1} U_{A_1} \simeq U_{A_1} / J_1 U_{A_1}$

$V^1 \overset{\text{def}}{=} \mathbb{C} \otimes_{A_1} V^q_{A_1} \simeq V^q_{A_1} / J_1 V^q_{A_1}$

Thm (classical limit)

① $\bar{U}_1 \cong U(\mathfrak{g})$

② $\bar{V}^q = \bar{V}^q(\lambda) \Rightarrow \bar{V}^1 \cong V(\lambda)$

③ $\dim_{\mathbb{C}(q)} \bar{V}^q_\mu = \mathrm{rank}_{A_1}(\bar{V}^q_{A_1})_\mu = \dim_{\mathbb{C}} \bar{V}^1_\mu$.

In particular, $\mathrm{ch}\, \bar{V}^q = \mathrm{ch}\, \bar{V}^1$ $\forall q$: indeterminate.

Prop ① $\bar{V}^q \in \mathcal{O}^q_{int}$: irred $\Rightarrow \bar{V}^q \cong \bar{V}^q(\lambda)$ $\lambda \in P^+$

② $\mathcal{O}^q_{int}$ is semisimple

③ $M^q \in \mathcal{O}^q_{int}$. $\forall i \in I$,

$\Rightarrow M^q = \bigoplus$ (f.d. irred U_i-submodules),

where $U_i = \langle e_i, f_i, K_i^{\pm 1} \rangle \cong U_{q_i}(\mathfrak{sl}_2)$.