≪ Quantum Groups & Crystal Bases IV ≫ 6/18/09

Ottawa

6. Perfect Crystals

$A = (a_{ij})_{i,j\in I}$: Cartan matrix of affine type

$I = \{0, 1, \cdots, n\}$, $(A_n^{(1)}, B_n^{(1)}, \cdots, G_2^{(1)}, D_4^{(3)})$

$P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d$, $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$

$\Pi^\vee = \{h_0, h_1, \cdots, h_n\}$

$\Pi = \{\alpha_0, \alpha_1, \cdots, \alpha_n\}$, where

$$\alpha_j(h_i) = a_{ij}, \quad \alpha_j(d) = \begin{cases} 1 & j = 0 \\ 0 & j \neq 0 \end{cases}$$

$P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{Z}\left(\frac{1}{d_0}\delta\right)$, where

$$\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d) = 0,$$

$$\delta = d_0\alpha_0 + d_1\alpha_1 + \cdots + d_n\alpha_n, \quad A\begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_n \end{pmatrix} = 0.$$

$\rightsquigarrow (A, P^\vee, \Pi^\vee, P, \Pi)$: affine Cartan datum

P^+ : affine dominant integral wt.

$c \overset{\text{def}}{=} c_0 h_0 + c_1 h_1 + \cdots + c_n h_n$ s.t. $(c_0, \cdots, c_n) A = 0$

canonical central elt

$U_q(\mathfrak{g})$ = quantum affine alg

$\cup$

$U_q'(\mathfrak{g}) = \langle e_i, f_i, K_i^{\pm 1} \mid i \in I \rangle$

= (derived) quantum affine alg

$\overline{P}^\vee = \mathbb{Z} h_0 \oplus \mathbb{Z} h_1 \oplus \cdots \oplus \mathbb{Z} h_n \supset \Pi^\vee$

$\overline{P} = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n \supset \Pi$

$\leadsto$ classical Cartan datum

$\overline{P}^+$: classical dominant integral wt

$$P \underset{\text{aff}}{\overset{\text{cl}}{\rightleftarrows}} \overline{P}$$

$\lambda = a_0\Lambda_0 + \cdots + a_n\Lambda_n + k\delta \longmapsto \overline{\lambda} = a_0\Lambda_0 + \cdots + a_n\Lambda_n$

$a_0\Lambda_0 + \cdots + a_n\Lambda_n \longleftarrow a_0\Lambda_0 + \cdots + a_n\Lambda_n$

$-2\Lambda_0 + \Lambda_1 + \Lambda_n + \delta \longleftarrow \alpha_0 = -2\Lambda_0 + \Lambda_1 + \Lambda_n$

$\boxed{\text{Def}}$ $\mathbb{B}$: finite $U_q'(\mathfrak{g})$-crystal

$b \in \mathbb{B}$; $\varepsilon(b) \overset{\text{def}}{=} \sum_{i\in I} \varepsilon_i(b)\Lambda_i$

$\varphi(b) \overset{\text{def}}{=} \sum_{i\in I} \varphi_i(b)\Lambda_i$.

$\mathbb{B}$ is a <u>perfect</u> <u>crystal of level ℓ</u> if

i) $\exists$ a f.d. $U_q'(\mathfrak{g})$-module V whose crystal is $\mathbb{B}$

ii) $\mathbb{B}\otimes\mathbb{B}$ is connected,

iii) $\exists\, \lambda_0 \in \overline{P}$ s.t. $\mathrm{wt}(\mathbb{B}) \subset \lambda_0 + \frac{1}{d_0}\sum_{i\neq 0}\mathbb{Z}_{\le 0}\alpha_i$, $\#\mathbb{B}_{\lambda_0}=1$

iv) $\forall b\in\mathbb{B}$, $\langle c, \varepsilon(b)\rangle \ge \ell$

v) $\forall \lambda \in \overline{P}^+$ with $\lambda(c)=\ell$, $\exists!\, b^\lambda$ & $\exists!\, b_\lambda$ in $\mathbb{B}$ s.t. $\varepsilon(b^\lambda)=\lambda$, $\varphi(b_\lambda)=\lambda$.

(Example) ① $\mathfrak{g}=A_2^{(1)}$; $\mathbb{B}$:

$$\begin{tikzcd} \boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \boxed{3} \arrow[ll, bend left, "0"] \end{tikzcd}$$

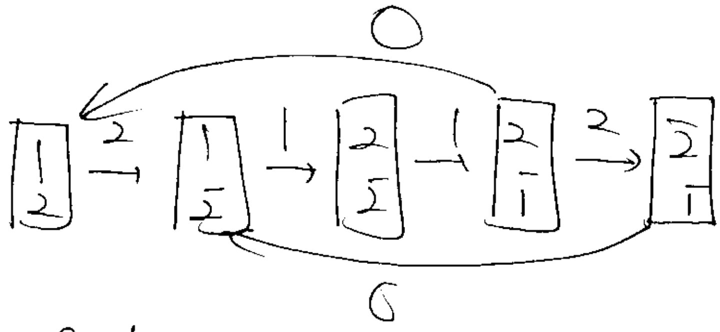
perfect of level 1

② $\mathfrak{g}=C_2^{(1)}$, $\mathbb{B}$:

$$\begin{tikzcd} \boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \boxed{\bar{2}} \arrow[r, "1"] & \boxed{\bar{1}} \arrow[lll, bend left, "0"] \end{tikzcd}$$

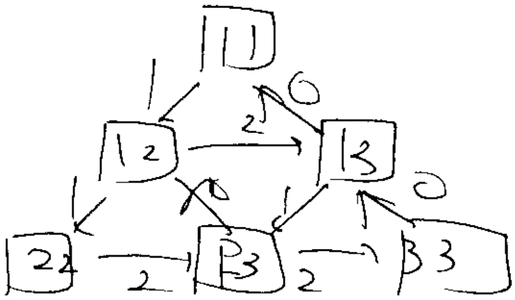
<u>not</u> perfect

③ $\mathfrak{g} = C_2^{(1)}$, $\mathbb{B}$:



perfect of level 1

④ $\mathfrak{g} = A_2^{(1)}$, $\mathbb{B}$:



perfect of level 2

$\boxed{\text{Thm}}$ $(\text{KMN})^2$

$\mathbb{B}$: perfect of level $\ell > 0$, $\lambda \in \bar{P}^+$, $\lambda(c) = \ell$

$$\Longrightarrow B(\lambda) \xrightarrow{\sim} B(\varepsilon(b_\lambda)) \otimes \mathbb{B} \qquad \varphi(b_\lambda) = \lambda.$$

$$u_\lambda \longmapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda,$$

(idea) $u_{\varepsilon(b_\lambda)} \otimes b_\lambda$ is the unique maximal vector in $B(\varepsilon(b_\lambda)) \otimes \mathbb{B}$; wt $= \lambda$.

$$\begin{array}{ccccc} & & B(\lambda_1)\otimes \mathbb{B} & & B(\lambda_2)\otimes\mathbb{B}\otimes\mathbb{B} \\ & & \| & & \| \\ B(\lambda) & \xrightarrow{\simeq} & B(\varepsilon(b_\lambda))\otimes\mathbb{B} & \xrightarrow{\simeq} & B(\varepsilon(b_{\lambda_1}))\otimes\mathbb{B}\otimes\mathbb{B} \\ u_\lambda & \longmapsto & u_{\varepsilon(b_\lambda)}\otimes b_\lambda & \longmapsto & u_{\varepsilon(b_{\lambda_1})}\otimes b_{\lambda_1}\otimes b_\lambda \\ & & \| & & \| \\ & & u_{\lambda_1}\otimes b_0 & & u_{\lambda_2}\otimes b_1\otimes b_0 \end{array}$$

$\lambda_0=\lambda,\ b_0=b_\lambda$

$\xrightarrow{\simeq}\ \cdots$

$\lambda_0=\lambda,\ b_0=b_\lambda\ ;\ \lambda_{k+1}=\varepsilon(b_k),\ b_{k+1}=b_{\lambda_{k+1}}$

$$\Longrightarrow\ B(\lambda)\xrightarrow{\simeq}B(\lambda_1)\otimes\mathbb{B}\xrightarrow{\simeq}\cdots\xrightarrow{\simeq}B(\lambda_{k+1})\otimes\mathbb{B}\otimes\cdots\otimes\mathbb{B}$$

$$u_\lambda\longmapsto u_{\lambda_1}\otimes b_0\longmapsto\cdots\longmapsto u_{\lambda_{k+1}}\otimes b_k\otimes\cdots\otimes b_0$$

Def

① $\mathbb{b}=(b_k)_{k\geq 0}$: ground-state path of λ

② $\mathbb{p}=(p_k)_{k\geq 0}$: $\underline{\lambda\text{-path}}$ in $\mathbb{B}$ if $p_k\in\mathbb{B}\ \forall k$ & $p_k=b_k\ \forall k\gg 0$.

$\mathcal{P}(\lambda)\overset{\text{def}}{=}\{\lambda\text{-paths in }\mathbb{B}\}$

$=\{\mathbb{p}=(p_k)_{k\geq 0}=\cdots\otimes p_{k+1}\otimes p_k\otimes\cdots\otimes p_0\}$

For $\mathbb{p} = (p_k)_{k\geq 0} \in \mathcal{P}(\lambda)$, let $N > 0$ be s.t. $p_k = b_k \ \forall k \geq N$, and we define

$$\mathrm{wt}\,\mathbb{p} = \lambda_N + \sum_{k=0}^{N-1} \mathrm{wt}(p_k)$$

$$\hat{e}_i \mathbb{p} = \cdots \otimes p_{N+1} \otimes \hat{e}_i(p_N \otimes \cdots \otimes p_0)$$

$$\hat{f}_i \mathbb{p} = \cdots \otimes p_{N+1} \otimes \hat{f}_i(p_N \otimes \cdots \otimes p_0)$$

$$\varepsilon_i(\mathbb{p}) = \max(\varepsilon_i(\mathbb{p}') - \varphi_i(b_N), 0)$$

$$\varphi_i(\mathbb{p}) = \varphi_i(\mathbb{p}') + \max(\varphi_i(b_N) - \varepsilon_i(\mathbb{p}'), 0),$$

where $\mathbb{p}' = p_{N-1} \otimes \cdots \otimes p_0$.

(KMN²)

Thm ① $\mathcal{P}(\lambda)$ is a $U_q'(\mathfrak{g})$-crystal

② $\mathcal{P}(\lambda) \simeq B(\lambda)$ as a $U_q'(\mathfrak{g})$-crystal

Q: What is the character of $V(\lambda)$?

Example) ① $\mathfrak{g} = A_2^{(1)}$, ② $\mathfrak{g} = C_2^{(1)}$

$\boxed{\text{Def}}$ B : $U_q'(\mathfrak{g})$-crystal

$H : B \otimes B \to \mathbb{Z}$ is an energy fn if

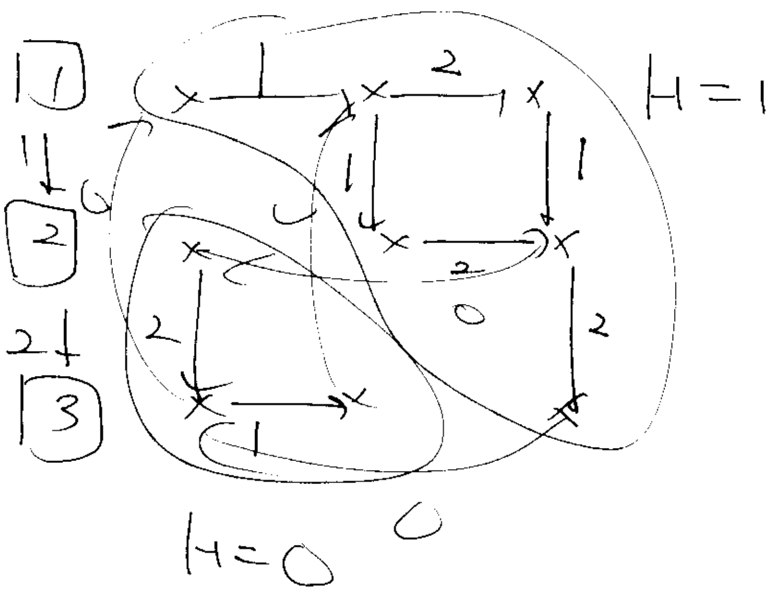
$$H\left(\tilde{e}_i(b_1 \otimes b_2)\right) = \begin{cases} H(b_1 \otimes b_2) & \text{if } i \neq 0 \\ H(b_1 \otimes b_2) + 1 & \text{if } i = 0,\ \varphi_0(b_1) \geq \varepsilon_0(b_2) \\ H(b_1 \otimes b_2) - 1 & \text{if } i = 0,\ \varphi_0(b_1) < \varepsilon_0(b_2) \end{cases}$$

(Example) B : $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$, $\boxed{3} \xrightarrow{0} \boxed{1}$

$$H(\boxed{i} \otimes \boxed{j}) = \begin{cases} 1 & i \geq j \\ 0 & i < j \end{cases}$$

$\cancel{\boxed{1} \otimes \boxed{1} \xrightarrow{2} \boxed{2} \otimes \boxed{1} \longrightarrow}$

$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$



Thm $\mathbb{B}$: perfect crystal of level l. $\lambda \in \bar{P}^+$, $\lambda(c) = l$

$\mathcal{P}(\lambda) = \{\lambda\text{-paths in } \mathbb{B}\}$, $\mathbb{p} \in \mathcal{P}(\lambda)$

$$\Longrightarrow \operatorname{wt} \mathbb{p} = \lambda + \sum_{k=0}^{\infty}\left(\operatorname{wt}(p_k) - \operatorname{wt}(b_k)\right) - \left(\sum_{k=0}^{\infty}(k+1)\left(H(p_{k+1}\otimes p_k) - H(b_{k+1}\otimes b_k)\right)\right)\delta.$$

Cor $$\operatorname{ch} V(\lambda) = \sum_{\mathbb{p}\in\mathcal{P}(\lambda)} e^{\operatorname{wt}\mathbb{p}}.$$

Rmk ① Vertex model theory can be explained in the language of perfect crystals

② It is not so easy to compute ~~the multiplicity~~ ~~of~~ wt multiplicities.

(Example) figures in [HK2002]

$\boxed{\text{Problem}}$ How to construct & classify perfect crystals?

Known examples

More details (Examples)

1) $\mathfrak{g} = A_n^{(1)}$; $\mathbb{B} = B(l\omega_k)$ (KMN)^2

2) $\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}$; $\mathbb{B} = B(l\varpi_1)$ (KMN)^2

3) $\mathfrak{g} = C_n^{(1)}, D_n^{(1)}$; $\mathbb{B} = B(l\varpi_n)$ (KMN)^2

4) $\mathfrak{g} = A_{2n}^{(2)}, D_{n+1}^{(2)}$; $\mathbb{B} = B(0) \oplus B(\omega_1) \oplus \cdots \oplus B(l\omega_1)$ (KMN)^2

5) $\mathfrak{g} = C_n^{(1)}$: $\mathbb{B} = B(0) \oplus B(2\omega_1) \oplus \cdots \oplus B(2l\omega_1)$ (KKM)

6) $\mathfrak{g} = G_2^{(1)}$: $\mathbb{B} = B(l\omega_1)$ (Yamada, Hirose)?

7) $\mathfrak{g} = D_4^{(3)}$: $\mathbb{B} = B(0) \oplus B(\omega_1) \oplus \cdots \oplus B(l\omega_1)$ [KMO] ?

8) $\mathfrak{g}$: all affine; $\mathbb{B} = B(0) \oplus B(\theta)$, level 1 (BFKL)

More details

9) $\mathfrak{g}$: $D_n^{(1)}, B_n^{(1)}, \cancel{A_{2n-1}^{(2)}}, \cancel{D_{n+1}^{(2)}}$: all classical affine

$\mathbb{B} = B(0) \oplus B(\theta) \oplus \cdots \oplus B(l\theta)$ Schilling, Sternberg

$\boxed{\text{Conjecture}}$ Cartan Kirillov–Reshetikhin crystals are perfect

$\boxed{\text{Thm}}$ (Fourier, Okado, Schilling)

KR crystals are perfect for classical affine types.
Cartan

(Example) path realization of $B(\Lambda_0)$ for $\mathfrak{g} = D_4^{(3)}$
using $B = B^{ad}$