Categorification of quantum groups

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Available at http://www.math.columbia.edu/~lauda/talks/

The goal: categorify $U_q^+(\mathfrak{g})$

The quantum enveloping algebra $U_q(\mathfrak{g})$ of a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ has a decomposition

$$U_q(\mathfrak{g}) = U_q^- \oplus U_q(\mathfrak{h}) \oplus U_q^+$$

U_q^+ has the structure of a bialgebra: try to categorify the bialgebra U_q^+

The plan: define a new algebra R

$$R-\mathrm{mod} - \left(\begin{array}{l} \text{category of finitely generated} \\ \text{graded projective modules} \end{array} \right)$$

Decategorification (Grothendieck group)

$$K_0(R-\mathrm{mod}) \cong U_q^+(\mathfrak{g})$$

Why categorify quantum groups?

Categorified representation theory should provide new insights for ordinary representation theory, especially relating to positivity and integrality properties.

- Algebraic/combinatorial analog of perverse sheaves.

Conjectured applications to low-dimensional topology

- Representation theoretic explanation of Khovanov homology
- Categorification of the Reshetikhin-Turaev quantum knot invariants.
- Crane-Frenkel conjectured categorified quantum groups would give 4-dimensional TQFTs

$U_q^+ \subset U_q(\mathfrak{g})$

$\mathfrak{g} = \mathfrak{sl}_n$

$$E_i = e_{i,i+1} = \begin{array}{cc} & \begin{array}{cccc} & & i+1 & \end{array} \\ i & \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & & \ddots & \\ 0 & & \dots & 0 \end{pmatrix} \end{array}$$

Lie algebra relations:

$$[E_i, E_j] = 0 \quad |i-j|>1 \qquad\qquad [E_i,[E_i,E_j]] = 0 \quad |i-j| = 1$$

Enveloping algebra relations for $U^+(\mathfrak{sl}_n)$

$$E_iE_j = E_jE_i \qquad |i-j|>1$$

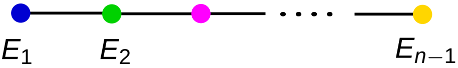
$$2E_iE_jE_i = E_i^2E_j + E_jE_i^2 \qquad j = i\pm 1$$

Quantum enveloping algebra $U_q^+(\mathfrak{sl}_n)$

$$E_iE_j = E_jE_i \qquad |i-j|>1$$

quantum 2 $\rightarrow (q+q^{-1})E_iE_jE_i = E_i^2E_j + E_jE_i^2 \qquad j = i\pm 1$

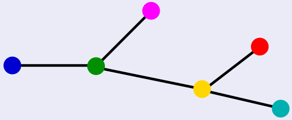
$U_q^+(\mathfrak{sl}_n)$ has a generator E_i for each vertex of the Dynkin graph



U_q^+ for any Γ

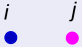
Let Γ be an unoriented graph with set of vertices I.

Γ =



U_q^+ is the $\mathbb{Q}(q)$-algebra with:

- generators: $E_i \quad i \in I$
- relations: $E_iE_j = E_jE_i$ if



$(q+q^{-1})E_iE_jE_i = E_i^2E_j + E_jE_i^2$ if

U_q^+ is $\mathbb{N}[I]$ graded with $\deg(E_i) = i$.

Integral form of U_q^+

Define quantum integers and quantum factorials:

$$[a] := \frac{q^a - q^{-a}}{q - q^{-1}} \qquad [a]! := [a][a-1]\dots[1]$$

Example

- $[1] \;=\; 1$
- $[2] \;=\; \frac{q^2-q^{-2}}{q-q^{-1}} \;=\; q+q^{-1}$
- $[3] \;=\; \frac{q^3-q^{-3}}{q-q^{-1}} \;= q^2+1+q^{-2}$

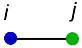
The algebra $U_{\mathbb{Z}}^+$ is the $\mathbb{Z}[q,q^{-1}]$-subalgebra of U_q^+ generated by all products of quantum divided powers:

$$E_i^{(a)} := \frac{E_i^a}{[a]!}$$

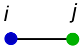
Since

$$E_i^{(2)} = \frac{E_i^2}{q+q^{-1}}$$

we can write the U_q^+ relation

$$(q+q^{-1})E_iE_jE_i = E_i^2E_j + E_jE_i^2$$ if 

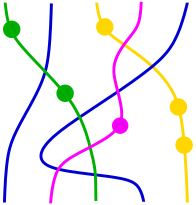
as

$$E_iE_jE_i = E_i^{(2)}E_j + E_jE_i^{(2)}$$ if 

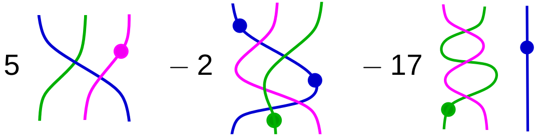
Categorification of U_q^+

Associated to graph Γ consider braid-like diagrams with dots whose strands are labelled by the vertices $i \in I$ of the graph Γ.

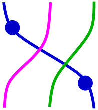
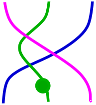
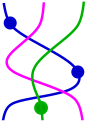
Let $\nu = \sum_{i \in I} \nu_i \cdot i$, for $\nu_i = 0, 1, 2, \ldots$
ν keeps track of how many strands of each color occur in a diagram

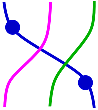
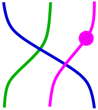


Form an abelian group by taking $\mathbb{Z}$-linear (or $\Bbbk$-linear) combinations of diagrams:



Multiplication is given by stacking diagrams on top of each other when the colors match:

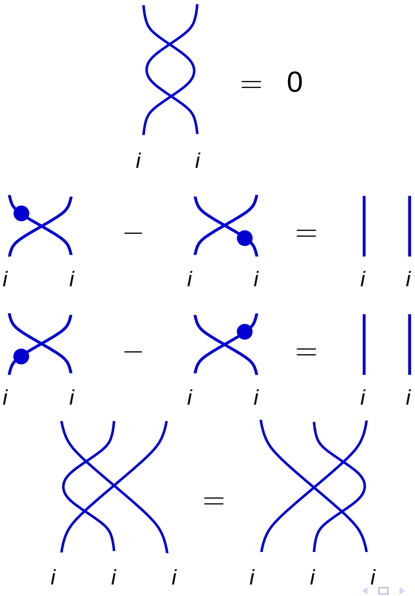
 $*$  $=$ 

 $*$  $= 0$

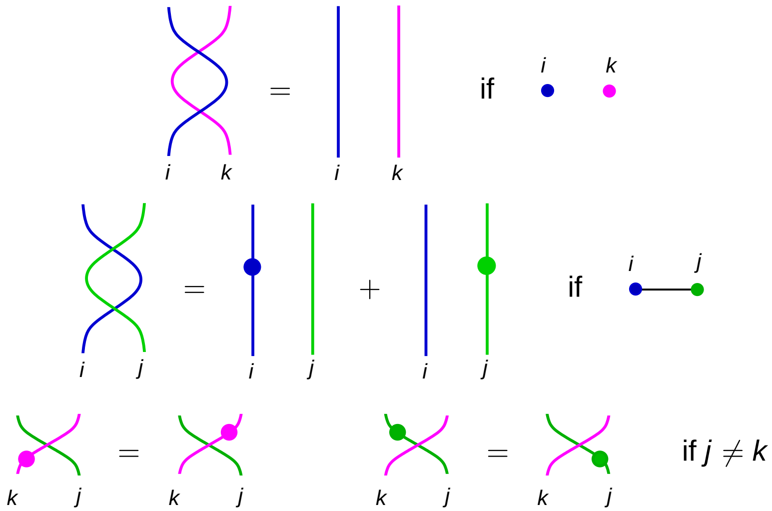
Definition

Given $\nu \in \mathbb{N}[I]$ define the ring $R(\nu)$ as the set of planar diagrams colored by ν, modulo planar braid-like isotopies and the following local relations:

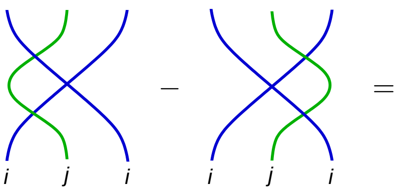
Local relations I



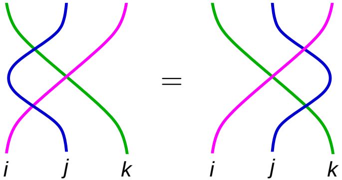
Local relations II



Local relations III



if 



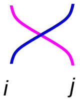
otherwise,

some of i, j, k may be equal

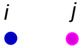
Grading

$q \longrightarrow$ grading shift

deg () $=$ 2

deg (

) $=$

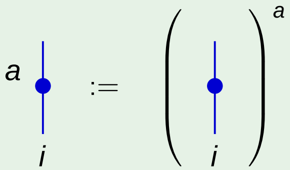
- -2 if $i = j$
- 0 if 

- 1 if 

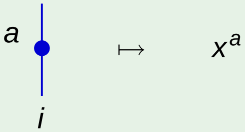
The $R(\nu)$ relations are homogeneous with respect to this grading.

Example

- If $\nu = 0$ then $R(0) = \mathbb{Z}$ with unit element given by the empty diagram.
- If $\nu = i$ for some vertex i, then a diagram is a line with some number $a \geq 0$ of dots on it.



Hence, $R(i) \cong \mathbb{Z}[x]$ where the isomorphism maps



R_ν is the associative, F-algebra on generators $1_{\underline{i}}$, $x_{a,\underline{i}}$, ; $\psi_{b,\underline{i}}$ for $1 \leq a \leq m$, $1 \leq b \leq m-1$ and $\underline{i} \in \mathrm{Seq}(\nu)$ subject to the following relations for $\underline{i}, \underline{j} \in \mathrm{Seq}(\nu)$:

$$1_{\underline{i}} 1_{\underline{j}} = \delta_{\underline{i},\underline{j}} 1_{\underline{i}}, \qquad x_{a,\underline{i}} = 1_{\underline{i}} x_{a,\underline{i}} 1_{\underline{i}},$$

$$\psi_{a,\underline{i}} = 1_{s_a(\underline{i})} \psi_{a,\underline{i}} 1_{\underline{i}}, \qquad x_{a,\underline{i}} x_{b,\underline{i}} = x_{b,\underline{i}} x_{a,\underline{i}},$$

$$\psi_{a,s_a(\underline{i})}\psi_{a,\underline{i}} = \begin{cases} 0 & \text{if } i_r = i_{r+1} \\ 1_{\underline{i}} & \text{if } (\alpha_{i_a}, \alpha_{i_{a+1}}) = 0 \\ \left(x_{a,\underline{i}}^{-\langle i_a, i_{a+1} \rangle} + x_{a+1,\underline{i}}^{-\langle i_{a+1}, i_a \rangle} \right) 1_{\underline{i}} & \text{if } (\alpha_{i_a}, \alpha_{i_{a+1}}) \neq 0 \text{ and } i_a \neq i_{a+1} \end{cases},$$

$$\psi_{b,s_a(\underline{i})}\psi_{a,\underline{i}} = \psi_{a,s_b(\underline{i})}\psi_{b,\underline{i}} \qquad \text{if } |a-b| > 1,$$

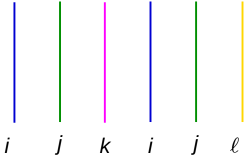
$$\psi_{a,s_{a+1}s_a(\underline{i})}\psi_{a+1,s_a(\underline{i})}\psi_{a,\underline{i}} - \psi_{a+1,s_a s_{a+1}(\underline{i})}\psi_{a,s_{a+1}(\underline{i})}\psi_{a+1,\underline{i}} =$$

$$= \begin{cases} \displaystyle\sum_{r=0}^{-\langle i_a, i_{a+1} \rangle - 1} x_{a,\underline{i}}^r x_{a+2,\underline{i}}^{-\langle i_a, i_{a+1} \rangle - 1 - r} & \text{if } i_a = i_{a+2} \text{ and } (\alpha_{i_a}, \alpha_{i_{a+1}}) \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{a,\underline{i}} x_{b,\underline{i}} - x_{s_a(b),s_a(\underline{i})}\psi_{a,\underline{i}} = \begin{cases} 1_{\underline{i}} & \text{if } a = b \text{ and } i_a = i_{a+1} \\ -1_{\underline{i}} & \text{if } a = b+1 \text{ and } i_a = i_{a+1} \\ 0 & \text{otherwise.} \end{cases}$$

Let $R = \bigoplus_{\nu} R(\nu)$. For each product of E_i's in U_q^+ we have an idempotent in R:

$$E_iE_jE_kE_iE_jE_\ell \qquad \mapsto \qquad 1_{ijkij\ell} \;:=$$

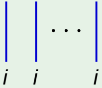


This gives rise to a projective module

$$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_k\mathcal{E}_i\mathcal{E}_j\mathcal{E}_\ell \quad := \quad R1_{ijkij\ell} \quad = \quad R(2i+2j+k+\ell)1_{ijkij\ell}$$

corresponding to the idempotent $1_{ijkij\ell}$ above.

Example

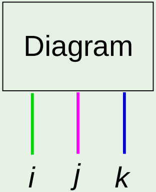
For a given $i \in I$ we write $\mathcal{E}_i^m$ for the projective module $R(mi) \cong \mathrm{NH}_m$ corresponding to the idempotent $1_{i^m} =$  , where $i^m := i \dots i$.

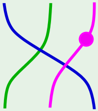
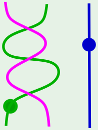
Example

Consider

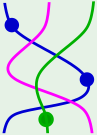
$$R1_{ijk} \quad = \quad R(i+j+k)1_{ijk}$$

The projective module $\mathcal{E}_i\mathcal{E}_j\mathcal{E}_k := R(i+j+k)1_{ijk}$ consists of linear combinations of diagrams that have the sequence ijk at the bottom



i.e.  and  $\in R(i+j+k)1_{ijk}$

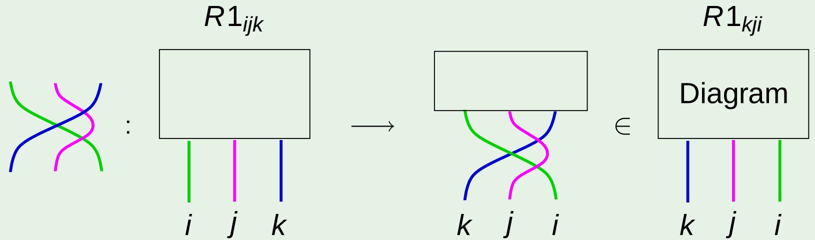
But

 $\notin R(i+j+k)1_{ijk}$

We can construct maps between projective modules by adding diagrams at the *bottom*

Example

We get a module map from $\mathcal{E}_i\mathcal{E}_j\mathcal{E}_k := R(i+j+k)1_{ijk}$ to $\mathcal{E}_k\mathcal{E}_j\mathcal{E}_i := R(i+j+k)1_{kji}$ as follows:



Given a graded module M and a Laurent polynomial $f = \sum f_a q^a \in \mathbb{Z}[q,q^{-1}]$ write

$$M^{\oplus f} \qquad \text{or} \qquad \bigoplus_f M$$

to denote the direct sum over $a \in \mathbb{Z}$ of f_a copies of $M\{a\}$

Example

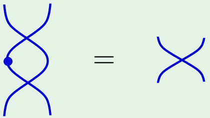
Since $[3] = q^2 + 1 + q^{-2} \in \mathbb{Z}[q,q^{-1}]$, for a graded module M

$$\bigoplus_{[3]} M \quad = \quad M\{2\} \oplus M\{0\} \oplus M\{-2\}$$

Example ($n = 2$)

$E_i^{(2)} = \frac{E_i^2}{q+q^{-1}}$ or $E_i^2 = (q+q^{-1})E_i^{(2)}$

Recall that



so that $e_2 =$  is an idempotent.

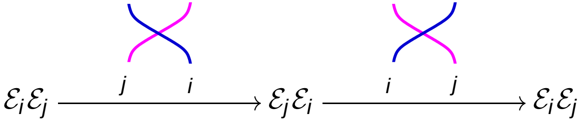
$\mathcal{E}_i^{(2)}$ is the projective module for this idempotent

$$\mathcal{E}_i^{(2)} := R(2i)e_2\{1\}$$

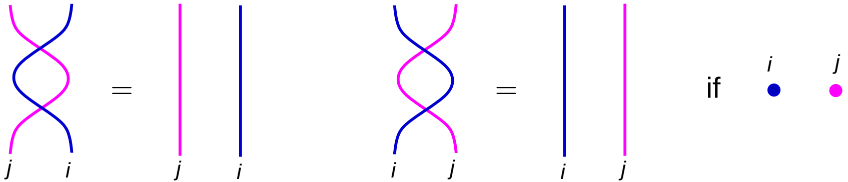
$$\mathcal{E}_i^2 \cong \mathcal{E}_i^{(2)}\{1\} \oplus \mathcal{E}_i^{(2)}\{-1\}$$

Categorification of $E_iE_j = E_jE_i$

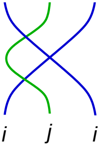
$$E_iE_j = E_jE_i \quad \text{if} \quad \overset{i}{\bullet} \qquad \overset{j}{\bullet} \quad \rightsquigarrow \quad \mathcal{E}_i\mathcal{E}_j \cong \mathcal{E}_j\mathcal{E}_i \quad \text{if} \quad \overset{i}{\bullet} \qquad \overset{j}{\bullet}$$



These maps are isomorphisms since



Categorification of $E_iE_jE_i = E_i^{(2)}E_j + E_jE_i^{(2)}$



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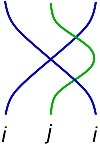

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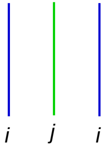
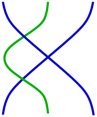
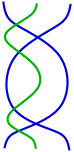
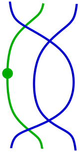
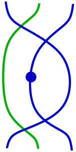
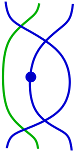
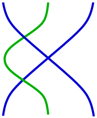
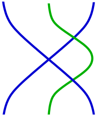


if

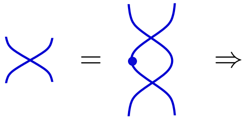

Let $e' =$ 

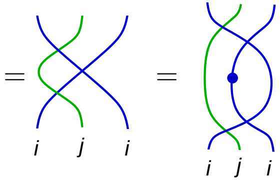
$(e')^2 =$  $=$  $+$  $=$  $=$  $= \; e'$

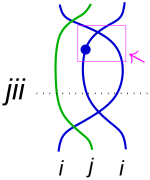
$e'' = 1_{iji} - e' = -$  is idempotent too $\quad (e'')^2 = e''$

Orthogonality $e'e'' = e''e' = 0$ and $1_{iji} = e' + e''$ imply

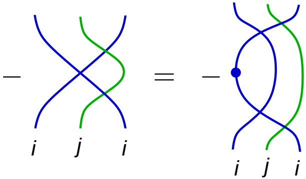
$$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i \cong \mathcal{E}_i\mathcal{E}_j\mathcal{E}_i e' \oplus \mathcal{E}_i\mathcal{E}_j\mathcal{E}_i e''$$

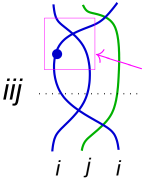
But 

$e' =$ 



idempotent e_2 of projection onto $\mathcal{E}_i^{(2)}$

$e'' = -$ 

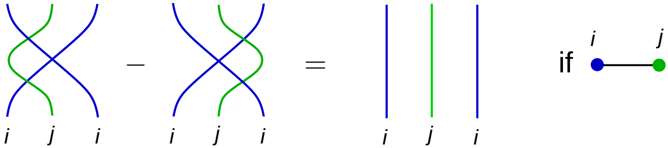


idempotent e_2 of projection onto $\mathcal{E}_i^{(2)}$

Therefore,

$$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i e' \cong \mathcal{E}_j\mathcal{E}_i^{(2)} \qquad\qquad \mathcal{E}_i\mathcal{E}_j\mathcal{E}_i e'' \cong \mathcal{E}_i^{(2)}\mathcal{E}_j$$

so that the relation



together with the other relations imply

$$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i \cong \mathcal{E}_j\mathcal{E}_i^{(2)} \oplus \mathcal{E}_i^{(2)}\mathcal{E}_j$$

Grothendieck groups

$$R = \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu) \qquad K_0(R) := \bigoplus_{\nu \in \mathbb{N}[I]} K_0(R(\nu))$$

where $K_0(R(\nu))$ is the Grothendieck group of the category $R(\nu)-\mathrm{pmod}$ of graded projective finitely-generated $R(\nu)$-modules.

$K_0(R(\nu))$ has generators $[M]$ over all objects of $R(\nu)-\mathrm{pmod}$ and defining relations

$$\begin{aligned} [M] &= [M_1] + [M_2] && \text{if } M \cong M_1 \oplus M_2 \\ [M\{s\}] &= q^s [M] && s \in \mathbb{Z} \end{aligned}$$

$K_0(R(\nu))$ is a $\mathbb{Z}[q, q^{-1}]$-module.

There are induction and restriction functors corresponding to inclusions $R(\nu) \otimes R(\nu') \subset R(\nu + \nu')$

$$\mathrm{Ind}_{\nu,\nu'}^{\nu+\nu'} : R(\nu) \otimes R(\nu')\mathrm{-pmod} \to R(\nu+\nu')\mathrm{-pmod}$$

$$\mathrm{Res}_{\nu,\nu'}^{\nu+\nu'} : R(\nu+\nu')\mathrm{-pmod} \to R(\nu) \otimes R(\nu')\mathrm{-pmod}$$

Summing over all ν, ν' gives functors

$$\mathrm{Ind}\colon (R \otimes R)\mathrm{-pmod} \to R\mathrm{-pmod} \qquad \mathrm{Res}\colon R\mathrm{-pmod} \to (R \otimes R)\mathrm{-pmod}$$

These map projectives to projectives $\Rightarrow$

$$[\mathrm{Ind}]\colon K_0(R) \otimes K_0(R) \to K_0(R) \qquad [\mathrm{Res}]\colon K_0(R) \to K_0(R) \otimes K_0(R)$$

Write $[\mathrm{Ind}](x_1, x_2)$ for $x_1, x_2 \in K_0(R)$ as $x_1 x_2$

Work over a field $\Bbbk$.

Theorem (M.Khovanov, A. L. arXiv:0803.4121)

There is an isomorphism of twisted bialgebras:

$$\begin{aligned} \gamma\colon U_{\mathbb{Z}}^{+} &\longrightarrow K_0(R) \\ E_{i_1}^{(a_1)}E_{i_2}^{(a_2)}\dots E_{i_k}^{(a_k)} &\mapsto \left[\mathcal{E}_{i_1}^{(a_1)}\mathcal{E}_{i_2}^{(a_2)}\dots \mathcal{E}_{i_k}^{(a_k)}\right] \end{aligned}$$

$$\begin{aligned} \text{multiplication} &\mapsto \text{multiplication given by } [\mathrm{Ind}] \\ \text{comultiplication} &\mapsto \text{comultiplication given by } [\mathrm{Res}] \end{aligned}$$

The semilinear form on $U_{\mathbb{Z}}^{+}$ maps to the HOM form on $K_0(R)$

$$(x,y) \quad = \quad (\gamma(x),\gamma(y))$$

Injectivity of γ

Injectivity of the map $\gamma \colon U_{\mathbb{Z}}^{+} \to K_0(R)$ uses that U_q^{+} is the quotient of a free associative algebra by the radical of the semilinear form. This follows from the quantum version of the Gabber-Kac theorem (proof, due to Lusztig for an arbitrary graph, uses perverse sheaves).

Surjectivity of γ

Surjectivity follows by mirroring the work of Grojnowski and Vazirani.

M.Khovanov, A. L. (arXiv:0804.2080)

This theorem has an extension to the non-simply laced case. The basis of indecomposable gives a new basis for $U_{\mathbb{Z}}^{+}$ where structure constants are necessarily positive.

Conjecture (Proven in simply-laced case)

$$U_{\mathbb{Z}}^{+} \xrightarrow{\quad\quad\sim\quad\quad} K_0(R)$$

Lusztig-Kashiwara canonical basis $\longmapsto$ indecomposable projective $[P]$

arXiv:0901.4450

Brundan and Kleshchev gave an algebraic proof when Γ is a chain or a cycle.

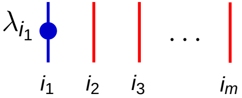
arXiv:0901.3992

The general case (over $\mathbb{C}$) was proven by Varagnolo and Vasserot who showed that rings $R(\nu)$ in the simply-laced case were isomorphic to certain Ext-algebras of Perverse sheaves on Lusztig quiver varieties.

Cyclotomic quotients

For a given weight $\lambda = \sum_{i \in I} \lambda_i \cdot \Lambda_i$ define the cyclotomic quotient R_ν^λ of $R(\nu)$ by imposing the additional relations: for any sequence $i_1 i_2 \cdots i_m$ of vertices of Γ

λ_{i_1} dots on the first strand of any sequence is zero $\longrightarrow$



$= \quad 0$

This is analogous to taking the Ariki-Koike cyclotomic quotient of the affine Hecke algebra:

$$H_d^\lambda := H_d \Big/ \left\langle \prod_{i \in I} (X_1 - q^i)^{\lambda_i} \right\rangle$$

Cyclotomic quotient conjecture

The category of finitely-generated graded modules over the ring

$$R^\lambda = \bigoplus_{\nu \in \mathbb{N}[I]} R^\lambda_\nu$$

categorifies the integrable version of the representation V_λ of $U_q(\mathfrak{g})$ of highest weight λ.

$$V(\lambda) \xrightarrow{\ \sim\ } K_0(R^\lambda)$$

$$\text{Lusztig-Kashiwara canonical basis} \longmapsto \text{indecomposable projective } [P]$$

Theorem (Brundan-Kleshchev, arXiv:0808.2032)

There is an isomorphism $R_\nu^\lambda \longrightarrow H_\nu^\lambda$ where H_ν^λ is a single block of the cyclotomic Hecke algebra H_d^λ. Using this isomorphism they proved the cyclotomic quotient conjecture in type A and affine type A.

For level 2 quotients the result follows from earlier work of Brundan and Stroppel.

Corollary

There is a $\mathbb{Z}$-grading on blocks H_ν^λ of affine Hecke algebras.

Implies there is a new $\mathbb{Z}$-grading on blocks of the symmetric group. Leads to graded Specht module theory, see Brundan-Kleshchev-Wang, arXiv:0901.0218.

Leads to a graded version of the generalized LLT-conjecture.

Generalizations

A.L (arXiv:0803.3652)

There is a graphical 2-category categorifying the integral form of the idempotent completion of the entire quantum group $U_q(\mathfrak{sl}_2)$

- $\dot{\mathbf{U}}_{\mathbb{Z}} \cong K_0(\dot{\mathcal{U}})$ the Grothendieck ring/category of this 2-category
- Indecomposable 1-morphisms $\Leftrightarrow$ Lusztig canonical basis element
- The 2-category $\dot{\mathcal{U}}$ acts on cohomology of iterated flag varieties, categorifying the irreducible N-dimensional rep of $\mathbf{U}_q(\mathfrak{sl}_2)$

M. Khovanov, A.L. (arXiv:0807.3250)

- 2-category $\dot{\mathcal{U}}$ has an extension to a categorification of $\dot{\mathbf{U}}(\mathfrak{sl}_n)$.
- Conjectural categorification of the integral form of $\dot{\mathbf{U}}(\mathfrak{g})$ for any Kac-Moody algebra.

arXiv:0812.5023

Closely related 2-categories were recently studied by Rouquier.