Imaginary Verma modules and Kashiwara algebras for $U_q(\widehat{\mathfrak{sl}(2)})$.

Ben Cox,
Vyacheslav Futorny,
Kailash C. Misra

Let $\mathbb{F}$ be a field of characteristic 0. The algebra $A_1^{(1)}$ is the affine Kac-Moody algebra over field $\mathbb{F}$ with generalized Cartan matrix $A = (a_{ij})_{0\leq i,j\leq 1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. The algebra $A_1^{(1)}$ has a Chevalley-Serre presentation with generators $e_0, e_1, f_0, f_1, h_0, h_1, d$ and relations

$$\begin{aligned}
&[h_i, h_j] = 0, \quad [h_i, d] = 0, \\
&[e_i, f_j] = \delta_{ij} h_i, \\
&[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \\
&[d, e_j] = \delta_{0,j} e_j, \quad [d, f_j] = -\delta_{0,j} f_j, \\
&(\operatorname{ad} e_i)^3 e_j = (\operatorname{ad} f_i)^3 f_j = 0, \quad i \neq j.
\end{aligned}$$

Alternatively, we may realize $A_1^{(1)}$ through the loop algebra construction

$$A_1^{(1)} \cong \mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}] \oplus \mathbb{F}c \oplus \mathbb{F}d$$

with Lie bracket relations

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n\delta_{n+m,0}(x, y)c,$$
$$[x \otimes t^n, c] = 0 = [d, c], \qquad [d, x \otimes t^n] = nx \otimes t^n,$$

for $x, y \in \mathfrak{sl}_2$, $n, m \in \mathbb{Z}$, where $(\ , \ )$ denotes the Killing form on $\mathfrak{sl}_2$. For $x \in \mathfrak{sl}_2$ and $n \in \mathbb{Z}$, we write $x(n)$ for $x \otimes t^n$.

Let Δ denote the root system of $A_1^{(1)}$, and let $\{\alpha_0, \alpha_1\}$ be a basis for Δ. Let $\delta = \alpha_0 + \alpha_1$, the minimal imaginary root. Then

$$\Delta = \{\pm\alpha_1 + n\delta \mid n \in \mathbb{Z}\} \cup \{k\delta \mid k \in \mathbb{Z} \setminus \{0\}\}.$$

A subset S of the root system Δ is called *closed* if $\alpha, \beta \in S$ and $\alpha + \beta \in \Delta$ implies $\alpha + \beta \in S$. The subset S is called a *closed partition* of the roots if S is closed, $S \cap (-S) = \emptyset$, and $S \cup -S = \Delta$. The classification of closed partitions of the root system for affine Lie algebras was obtained by Jakobsen and Kac and also indepently by Futorny.

The set

$$S = \{\alpha_1 + k\delta \mid k \in \mathbb{Z}\} \cup \{l\delta \mid l \in \mathbb{Z}_{>0}\}$$

is a closed partition of Δ and is $W \times \{\pm 1\}$-inequivalent to the standard partition of the root system into positive and negative roots.

For $\mathfrak{g} = A_1^{(1)}$, let $\mathfrak{g}_\pm^{(S)} = \sum_{\alpha \in S} \mathfrak{g}_{\pm\alpha}$. In the loop algebra formulation of $\mathfrak{g}$, we have that $\mathfrak{g}_+^{(S)}$ is the subalgebra generated by $e(k) = e \otimes t^k$ ($k \in \mathbb{Z}$) and $h(l) = h \otimes t^l$ ($l \in \mathbb{Z}_{>0}$) and $\mathfrak{g}_-^{(S)}$ is the subalgebra generated by $f(k) = f \otimes t^k$ ($k \in \mathbb{Z}$) and $h(-l)$ ($l \in \mathbb{Z}_{>0}$). Since S is a partition of the root system, the algebra has a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_-^{(S)} \oplus \mathfrak{h} \oplus \mathfrak{g}_+^{(S)}.$$

Let $U(\mathfrak{g}_\pm^{(S)})$ be the universal enveloping algebra of $\mathfrak{g}_\pm^{(S)}$. Then, by the PBW theorem, we have

$$U(\mathfrak{g}) \cong U(\mathfrak{g}_-^{(S)}) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{g}_+^{(S)}),$$

where $U(\mathfrak{g}_+^{(S)})$ is generated by $e(k)$ ($k \in \mathbb{Z}$), $h(l)$ ($l \in \mathbb{Z}_{>0}$), $U(\mathfrak{g}_-^{(S)})$ is generated by $f(k)$ ($k \in \mathbb{Z}$), $h(-l)$ ($l \in \mathbb{Z}_{>0}$) and $U(\mathfrak{h})$, the universal enveloping algebra of $\mathfrak{h}$, is generated by h, c and d.

Let $\lambda \in \Lambda$, the weight lattice of $\mathfrak{g} = A_1^{(1)}$. A $U(\mathfrak{g})$-module V is called a *weight* module if $V = \oplus_{\mu \in P} V_\mu$, where

$$V_\mu = \{v \in V \mid h \cdot v = \mu(h)v, c \cdot v = \mu(c)v, d \cdot v = \mu(d)v\}.$$

Any submodule of a weight module is a weight module. A $U(\mathfrak{g})$-module V is called an *S-highest weight module* with highest weight λ if there is a non-zero $v_\lambda \in V$ such that (i) $u^+ \cdot v_\lambda = 0$ for all $u^+ \in U(\mathfrak{g}_+^{(S)}) \setminus \mathbb{F}^*$, (ii) $h \cdot v_\lambda = \lambda(h)v_\lambda$, $c \cdot v_\lambda = \lambda(c)v_\lambda$, $d \cdot v_\lambda = \lambda(d)v_\lambda$, (iii) $V = U(\mathfrak{g}) \cdot v_\lambda = U(\mathfrak{g}_-^{(S)}) \cdot v_\lambda$. An S-highest weight module is a weight module.

For $\lambda \in \Lambda$, let $I_S(\lambda)$ denote the ideal of $U(A_1^{(1)})$ generated by $e(k)$ $(k \in \mathbb{Z})$, $h(l)$ $(l > 0)$, $h - \lambda(h)1$, $c - \lambda(c)1$, $d - \lambda(d)1$. Then we define $M(\lambda) = U(A_1^{(1)})/I_S(\lambda)$ to be the *imaginary Verma module* of $A_1^{(1)}$ with highest weight λ. Imaginary Verma modules have many structural features similar to those of standard Verma modules, with the exception of the infinite-dimensional weight spaces. In particular, $M(\lambda)$ has a unique maximal submodule and it is irreducible if and only if $\lambda(c) \neq 0$.

The *quantum group* $U_q(A_1^{(1)})$ is the $\mathbb{F}(q^{1/2})$-algebra with 1 generated by

$$e_0, e_1, f_0, f_1, K_0^{\pm 1}, K_1^{\pm 1}, D^{\pm 1}$$

with defining relations:

$$\begin{aligned}
&DD^{-1} = D^{-1}D = K_iK_i^{-1} = K_i^{-1}K_i = 1,\\
&e_if_j - f_je_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}},\\
&K_ie_iK_i^{-1} = q^2e_i, \quad K_if_iK_i^{-1} = q^{-2}f_i,\\
&K_ie_jK_i^{-1} = q^{-2}e_j, \quad K_if_jK_i^{-1} = q^2f_j, \quad i \neq j,\\
&K_iK_j - K_jK_i = 0, \quad K_iD - DK_i = 0,\\
&De_iD^{-1} = q^{\delta_{i,0}}e_i, \quad Df_iD^{-1} = q^{-\delta_{i,0}}f_i,\\
&e_i^3e_j - [3]e_i^2e_je_i + [3]e_ie_je_i^2 - e_je_i^3 = 0, \quad i \neq j,\\
&f_i^3f_j - [3]f_i^2f_jf_i + [3]f_if_jf_i^2 - f_jf_i^3 = 0, \quad i \neq j,
\end{aligned}$$

where, $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$.

The quantum group $U_q(A_1^{(1)})$ can be given a Hopf algebra structure with a comultiplication given by

$$\Delta(K_i) = K_i \otimes K_i, \Delta(D) = D \otimes D,$$

$$\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i,$$

and an antipode given by

$$s(e_i) = -e_iK_i^{-1}, s(f_i) = -K_if_i, s(K_i) = K_i^{-1}, s(D) = D^{-1}.$$

We also need the Drinfeld realization for $U_q(A_1^{(1)})$, which is as follows. Let U_q be the associative algebra with 1 over $\mathbb{F}(q^{1/2})$ generated by the elements $x^{\pm}(k)$ $(k \in \mathbb{Z})$, $a(l)$ $(l \in \mathbb{Z} \setminus \{0\})$, $K^{\pm 1}$, $D^{\pm 1}$, and $\gamma^{\pm \frac{1}{2}}$ with the following defining relations:

$$DD^{-1} = D^{-1}D = KK^{-1} = K^{-1}K = 1, \tag{1}$$

$$[\gamma^{\pm\frac{1}{2}}, u] = 0 \quad \forall u \in U, \tag{2}$$

$$[a(k), a(l)] = \delta_{k+l,0} \frac{[2k]}{k} \frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}, \tag{3}$$

$$[a(k), K] = 0, \quad [D, K] = 0, \tag{4}$$

$$Da(k)D^{-1} = q^k a(k), \tag{5}$$

$$Dx^{\pm}(k)D^{-1} = q^k x^{\pm}(k), \tag{6}$$

$$Kx^{\pm}(k)K^{-1} = q^{\pm 2} x^{\pm}(k), \tag{7}$$

$$[a(k), x^{\pm}(l)] = \pm \frac{[2k]}{k} \gamma^{\mp \frac{|k|}{2}} x^{\pm}(k+l), \tag{8}$$

$$\begin{aligned} x^{\pm}(k+1)x^{\pm}(l) - q^{\pm 2}x^{\pm}(l)x^{\pm}(k+1) \\ = q^{\pm 2}x^{\pm}(k)x^{\pm}(l+1) - x^{\pm}(l+1)x^{\pm}(k), \end{aligned} \tag{9}$$

$$[x^{+}(k), x^{-}(l)] = \frac{1}{q - q^{-1}} \left(\gamma^{\frac{k-l}{2}} \psi(k+l) - \gamma^{\frac{l-k}{2}} \phi(k+l) \right), \tag{10}$$

where $$\sum_{k=0}^{\infty} \psi(k) z^{-k} = K \exp\left((q - q^{-1}) \sum_{k=1}^{\infty} a(k) z^{-k} \right), \tag{11}$$

$$\sum_{k=0}^{\infty} \phi(-k) z^{k} = K^{-1} \exp\left(-(q - q^{-1}) \sum_{k=1}^{\infty} a(-k) z^{k} \right). \tag{12}$$

The algebras $U_q(A_1^{(1)})$ and U_q are isomorphic. The action of the isomorphism, which we call the *Drinfeld Isomorphism,* on the generators of $U_q(A_1^{(1)})$ is:

$$\begin{aligned} e_0 &\mapsto x^-(1)K^{-1}, \quad f_0 \mapsto Kx^+(-1), \\ e_1 &\mapsto x^+(0), \quad f_1 \mapsto x^-(0), \\ K_0 &\mapsto \gamma K^{-1}, \quad K_1 \mapsto K, \quad D \mapsto D. \end{aligned}$$

We use the formal sums

$$\phi(u) = \sum_{p\in\mathbb{Z}} \phi(p)u^{-p}, \;\; \psi(u) = \sum_{p\in\mathbb{Z}} \psi(p)u^{-p}, \;\; x^{\pm}(u) = \sum_{p\in\mathbb{Z}} x^{\pm}(p)u^{-p} \tag{13}$$

Then it follows from Drinfeld's relations (3), (8)-(10) that:

$$[\phi(u), \phi(v)] = 0 = [\psi(u), \psi(v)] \tag{14}$$

$$\phi(u)x^{\pm}(v)\phi(u)^{-1} = g(uv^{-1}\gamma^{\mp 1/2})^{\pm 1}x^{\pm}(v) \tag{15}$$

$$\psi(u)x^{\pm}(v)\psi(u)^{-1} = g(vu^{-1}\gamma^{\mp 1/2})^{\mp 1}x^{\pm}(v) \tag{16}$$

$$(u - q^{\pm 2}v)x^{\pm}(u)x^{\pm}(v) = (q^{\pm 2}u - v)x^{\pm}(v)x^{\pm}(u) \tag{17}$$

$$[x^+(u), x^-(v)] = (q - q^{-1})^{-1}(\delta(u/v\gamma)\psi(v\gamma^{1/2}) - \delta(u\gamma/v)\phi(u\gamma^{1/2})) \tag{18}$$

where $g(t) = g_q(t)$ is the Taylor series at $t = 0$ of the function $(q^2t - 1)/(t - q^2)$ and $\delta(z) = \sum_{k\in\mathbb{Z}} z^k$ is the formal Dirac delta function. Writing $g(t) = g_q(t) = \sum_{p\geq 0} g(p)t^p$ we have

$$g(0) = q^{-2}, \quad g(p) = (1 - q^4)q^{-2p-2}, \quad p > 0.$$

Note that $g_q(t)^{-1} = g_{q^{-1}}(t)$.

Using the root partition $S = \{\alpha_1 + k\delta \mid k \in \mathbb{Z}\} \cup \{l\delta \mid l \in \mathbb{Z}_{>0}\}$, we define:
$U_q^+(S)$ to be the subalgebra of U_q generated by $x^+(k)$ $(k \in \mathbb{Z})$ and $a(l)$ $(l > 0)$;
$U_q^-(S)$ to be the subalgebra of U_q generated by $x^-(k)$ $(k \in \mathbb{Z})$ and $a(-l)$ $(l > 0)$, and
$U_q^0(S)$ to be the subalgebra of U_q generated by $K^{\pm 1}$, $\gamma^{\pm 1/2}$, and $D^{\pm 1}$.

Then we have the following PBW theorem due to Cox, Futorny, Kang and Melville.

Theorem: A basis for U_q is the set of monomials of the form

$$x^- a^- K^\alpha D^\beta \gamma^{\mu/2} a^+ x^+$$

where

$$x^\pm = x^\pm(m_1)^{n_1} \cdots x^\pm(m_k)^{n_k}, \qquad m_i < m_{i+1}, \quad m_i \in \mathbb{Z},$$
$$a^\pm = a(r_1)^{s_1} \cdots a(r_l)^{s_l}, \qquad \pm r_i < \pm r_{i+1}, \quad \pm r_i \in \mathbb{N}^*,$$

and $\alpha, \beta, \mu \in \mathbb{Z}$, $n_i, s_i \in \mathbb{N}$. In particular, $U_q \cong U_q^-(S) \otimes U_q^0(S) \otimes U_q^+(S)$.

Let $\mathbb{N}^{\mathbb{N}^*}$ denote the set of all functions from $\{k\delta \mid k \in \mathbb{N}^*\}$ to $\mathbb{N}$ with finite support. Then we can write

$$a^+ = a_+^{(s_k)} := a(r_1)^{s_1} \cdots a(r_l)^{s_l}, a^- := a_-^{(s_k)} = a(-r_1)^{s_1} \cdots a(-r_l)^{s_l}$$

for $f = (s_k) \in \mathbb{N}^{\mathbb{N}^*}$ where $f(r_k) = s_k$ and $f(t) = 0$ for $t \neq r_i, 1 \leq i \leq l$.

Now consider the subalgebra $\mathcal{N}_q^-$, generated by $\gamma^{\pm 1/2}$, and $x^-(l)$, $l \in \mathbb{Z}$. Then any element in $\mathcal{N}_q^-$ is a sum of products of elements of the form

$$P = \gamma^{l/2} x^-(m_1) \cdots x^-(m_k),$$

where $m_i \in \mathbb{Z}, m_1 \leq m_2 \leq \cdots \leq m_k, k \geq 0$, $l \in \mathbb{Z}$

and such a product is a summand of

$$P = P(v_1, \ldots, v_k) := \gamma^{l/2} x^-(v_1) \cdots x^-(v_k).$$

Set $\bar{P} = x^-(v_1) \cdots x^-(v_k)$ and
$\bar{P}_l = x^-(v_1) \cdots x^-(v_{l-1}) x^-(v_{l+1}) \cdots x^-(v_k)$.

Note that by Drinfeld relations (15) and (16) we have:

$$x^-(v_1) \cdots x^-(v_{l-1}) \psi(v_l \gamma^{1/2}) = \prod_{j=1}^{l-1} g(v_j v_l^{-1})^{-1} \psi(v_l \gamma^{1/2}) x^-(v_1) \cdots x^-(v_{l-1})$$

$$x^-(v_1) \cdots x^-(v_{l-1}) \phi(u \gamma^{1/2}) = \prod_{j=1}^{l-1} g(u \gamma v_j^{-1}) \phi(u \gamma^{1/2}) x^-(v_1) \cdots x^-(v_{l-1}).$$

So by Drinfeld relation (18) we have

$$[x^+(u),\bar{P}]=\sum_{l=1}^{k}x^-(v_1)\cdots[x^+(u),x^-(v_l)]\cdots x^-(v_k)$$

$$=\sum_{l=1}^{k}x^-(v_1)\cdots\left(\frac{\delta(u/v_l\gamma)\psi(v_l\gamma^{1/2})-\delta(u\gamma/v_l)\phi(u\gamma^{1/2})}{q-q^{-1}}\right)\cdots x^-(v_k)$$

$$=\frac{\psi(u\gamma^{-1/2})}{q-q^{-1}}\sum_{l=1}^{k}\prod_{j=1}^{l-1}g_{q^{-1}}(v_j/v_l)\bar{P}_l\delta(u/v_l\gamma)$$

$$-\frac{\phi(u\gamma^{1/2})}{q-q^{-1}}\sum_{l=1}^{k}\prod_{j=1}^{l-1}g(v_l/v_j)\bar{P}_l\delta(u\gamma/v_l)$$

Now we have the following lemma.

Lemma: Fix $k\in\mathbb{Z}$. Then for any $P\in\mathcal{N}_q^-$, there exists unique

$$Q(a,(q_k)),R(c,(r_l))\in\mathcal{N}_q^-,\quad a,b\in\mathbb{Z},(q_l),(r_m)\in\mathbb{N}^{\mathbb{N}^*},$$

such that

$$[x^+(k),P]=\sum\frac{a_+^{(q_l)}K^aQ(a,(q_l))}{q-q^{-1}}+\sum\frac{a_-^{(r_m)}K^b\ R(b,(r_m))}{q-q^{-1}}.$$

This Lemma motivates the definition of a family of operators as follows. Set

$$G_l = G_l^{1/q} := \prod_{j=1}^{l-1} g_{q^{-1}}(v_j/v_l), \quad G_l^q = \prod_{j=1}^{l-1} g(v_l/v_j)$$

where $G_1 := 1$. Now define a collection of operators $\Omega_\psi(k), \Omega_\phi(k) : \mathcal{N}_q^- \to \mathcal{N}_q^-$, $k \in \mathbb{Z}$, in terms of the generating functions

$$\Omega_\psi(u) = \sum_{l\in\mathbb{Z}} \Omega_\psi(l)u^{-l}, \quad \Omega_\phi(u) = \sum_{l\in\mathbb{Z}} \Omega_\phi(l)u^{-l}$$

by

$$\Omega_\psi(u)(\bar{P}) := \gamma^m \sum_{l=1}^{k} G_l \bar{P}_l \delta(u/v_l\gamma)$$

$$\Omega_\phi(u)(\bar{P}) := \gamma^m \sum_{l=1}^{k} G_l^q \bar{P}_l \delta(u\gamma/v_l).$$

Note that $\Omega_\psi(u)(1) = \Omega_\phi(u)(1) = 0$. Then we have:

$$[x^+(u), \bar{P}] = (q-q^{-1})^{-1}\left(\psi(u\gamma^{-1/2})\Omega_\psi(u)(\bar{P}) - \phi(u\gamma^{1/2})\Omega_\phi(u)(\bar{P})\right).$$

The following Proposition lists the relations among Ω operators.

Proposition Consider $x^-(v) = \sum_m x^-(m)v^{-m}$ as a formal power series of left multiplication operators $x^-(m) : \mathcal{N}_q^- \to \mathcal{N}_q^-$. Then

$$\begin{aligned}
\Omega_\psi(u)x^-(v) &= \delta(v\gamma/u) + g_{q^{-1}}(v\gamma/u)x^-(v)\Omega_\psi(u), \\
\Omega_\phi(u)x^-(v) &= \delta(u\gamma/v) + g(u\gamma/v)x^-(v)\Omega_\phi(u) \\
(q^2u_1 - u_2)\Omega_\psi(u_1)\Omega_\psi(u_2) &= (u_1 - q^2u_2)\Omega_\psi(u_2)\Omega_\psi(u_1) \\
(q^2u_1 - u_2)\Omega_\phi(u_1)\Omega_\phi(u_2) &= (u_1 - q^2u_2)\Omega_\phi(u_2)\Omega_\phi(u_1) \\
(q^2\gamma^2u_1 - u_2)\Omega_\phi(u_1)\Omega_\psi(u_2) &= (\gamma^2u_1 - q^2u_2)\Omega_\psi(u_2)\Omega_\phi(u_1)
\end{aligned}$$

In terms of components and as operators on $\mathcal{N}_q^-$ we have:

$$\Omega_\psi(k)x^-(m) = \delta_{k,-m}\gamma^k + \sum_{r\geq 0} g_{q^{-1}}(r)x^-(m+r)\Omega_\psi(k-r)\gamma^r.$$

and

$$\Omega_\psi(k)\Omega_\phi(m) = \sum_{r\geq 0} g(r)\gamma^{2r}\Omega_\phi(r+m)\Omega_\psi(k-r).$$

Note that the sum on the right hand side turns into a finite sum when applied to an element in $\mathcal{N}_q^-$.

We define the Kashiwara algebra $\mathcal{K}_q$ to be the $\mathbb{F}(q^{1/2})$-algebra with generators $\Omega_\psi(m), x^-(n), \gamma^{\pm 1/2}$, $m, n \in \mathbb{Z}$ where $\gamma^{\pm 1/2}$ are central and the defining relations are:

$$q^2\gamma\Omega_\psi(m)x^-(n+1) - \Omega_\psi(m+1)x^-(n)$$

$$= (q^2\gamma - 1)\delta_{m,-n-1} + \gamma x^-(n+1)\Omega_\psi(m) - q^2 x^-(n)\Omega_\psi(m+1),$$

$$q^2\Omega_\psi(k+1)\Omega_\psi(l) - \Omega_\psi(l)\Omega_\psi(k+1) = \Omega_\psi(k)\Omega_\psi(l+1) - q^2\Omega_\psi(l+1)\Omega_\psi(k),$$

$$x^-(k+1)x^-(l) - q^{-2}x^-(l)x^-(k+1) = q^{-2}x^-(k)x^-(l+1) - x^-(l+1)x^-(k)$$

and

$$\gamma^{1/2}\gamma^{-1/2} = 1 = \gamma^{-1/2}\gamma^{1/2}.$$

We have the following Lemmas:

Lemma: The $\mathbb{F}(q^{1/2})$-linear map $\bar{\alpha} : \mathcal{K}_q \to \mathcal{K}_q$ given by

$$\bar{\alpha}(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2}, \quad \bar{\alpha}(x^-(m)) = \Omega_\psi(-m), \quad \bar{\alpha}(\Omega_\psi(m)) = x^-(-m)$$

for all $m \in \mathbb{Z}$ is an involutive anti-automorphism.

Lemma: $\mathcal{N}_q^-$ is a left $\mathcal{K}_q$-module and $\mathcal{N}_q^- \cong \mathcal{K}_q / \sum_{k\in\mathbb{Z}} \mathcal{K}_q\Omega_\psi(k)$.

Lemma: There is a unique symmetric form $(\ ,\ )$ defined on $\mathcal{N}_q^-$ satisfying

$$(x^-(m)a, b) = (a, \Omega_\psi(-m)b), \quad (1,1) = 1.$$

Let $\lambda \in \Lambda$, the weight lattice of $A_1^{(1)}$. Denote by $I^q(\lambda)$ the ideal of $U_q = U_q(\hat{\mathfrak{s}l}(2))$ generated by $x^+(k)$, $k \in \mathbb{Z}$, $a(l), l > 0$, $K^{\pm 1} - q^{\lambda(h)}1$, $\gamma^{\pm\frac{1}{2}} - q^{\pm\frac{1}{2}\lambda(c)}1$ and $D^{\pm 1} - q^{\pm\lambda(d)}1$. The imaginary Verma module with highest weight λ is defined to be

$$M_q(\lambda) = U/I^q(\lambda).$$

Cox, Futorny, Kang and Melville showed that the imaginary Verma module $M(\lambda)$ over the affine $\hat{\mathfrak{s}l}(2)$ admits a quantum deformation to the imaginary Verma module $M_q(\lambda)$ over U_q in such a way that the dimensions of the weight spaces are invariant under this deformation. They also proved:

Theorem: Imaginary Verma module $M_q(\lambda)$ is simple if and only if $\lambda(c) \neq 0$.

Suppose now that $\lambda(c) = 0$. Then $\gamma^{\pm\frac{1}{2}}$ acts on $M_q(\lambda)$ by 1. Consider an ideal $J^q(\lambda)$ of U_q generated by $I^q(\lambda)$ and $a(l)$ for all l. Denote

$$\tilde{M}_q(\lambda) = U_q/J^q(\lambda).$$

Then $\tilde{M}_q(\lambda)$ is a homomorphic image of $M_q(\lambda)$ which we call *reduced imaginary Verma module*. Module $\tilde{M}_q(\lambda)$ has a Λ-gradation:

$$\tilde{M}_q(\lambda) = \sum_{\xi\in\Lambda} \tilde{M}_q(\lambda)_\xi.$$

If α denotes a simple root of $\mathfrak{s}l(2)$ and δ denotes an indivisible imaginary root then $\tilde{M}_q(\lambda)_{\lambda-\xi} \neq 0$ if and only if $\xi = 0$ or $\xi = -n\alpha + m\delta$ with $n > 0$, $m \in \mathbb{Z}$.

If $\xi = -n\alpha + m\delta$ then we set $|\xi| = n$. Note that $\mathcal{N}_q^-$ has also a Λ-grading: $x^-(n_1)x^-(n_2)\ldots x^-(n_k) \in (\mathcal{N}_q^-)_\xi$, where $\xi = -k\alpha + (n_1 + \ldots + n_k)\delta$, $|\xi| = k$.

Theorem: Let $\lambda \in \Lambda$ such that $\lambda(c) = 0$. Then module $\tilde{M}_q(\lambda)$ is simple if and only if $\lambda(h) \neq 0$.

Using this Theorem we can prove:

Lemma: Let $P \in \mathcal{N}_q^-$. If $\Omega_\psi(s)P = 0$ for any $s \in \mathbb{Z}$, then P is a constant multiple of 1.

This in turn gives our main theorem:

Theorem: The algebra $\mathcal{N}_q^-$ is simple as a $\mathcal{K}_q$-module.