Quiver varieties and cluster algebras

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Definition of cluster algebras (with coefficients) (Fomin-Zelevinsky 2001)

$Q=(I,\Omega)\subset \tilde{Q}=(\tilde{I},\tilde{\Omega})$: pair of quivers

s.t. • $\Omega = \tilde{\Omega}\cap (I\times I)$

• no edge loops $\cancel{\circlearrowleft}$, nor 2-cycles $\cancel{\bullet \rightleftarrows \bullet}$

$I_{fr} = \tilde{I}\setminus I$: frozen vertex

assume no $\cancel{i \to j}$, $i \in I_{fr}$, $j \in I_{fr}$

Ex.

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[d] & \bullet \arrow[l] & I \\ \bullet \arrow[u] & \bullet & \bullet \arrow[u] & I_{fr} \end{tikzcd}$$

quiver mutation at $k \in I$ (not frozen vertex)

$$\tilde{\Omega} : \begin{tikzcd} i \arrow[rr, "r"] \arrow[dr, "s"'] & & j \\ & k \arrow[ur, "t"'] & \end{tikzcd} \quad \overset{}{\underset{\mu_k}{\Longrightarrow}} \quad \mu_k(\tilde{\Omega}) : \begin{tikzcd} i \arrow[rr, "r+st"] & & j \arrow[dl, "t"] \\ & k \arrow[ul, "s"] & \end{tikzcd}$$

$(s,t \geqq 0)$

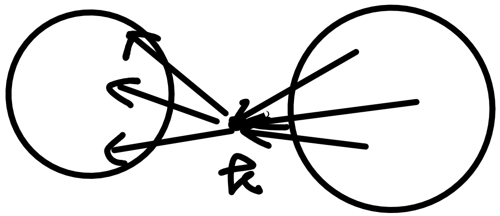
"$i \xrightarrow{\ell} j$" means "$i \underset{\ell \text{ arrows}}{\overset{\vdots}{\rightrightarrows}} j$" or "$i \underset{(-\ell) \text{ arrows}}{\overset{\vdots}{\leftleftarrows}} j$"

$\mathcal{F} = \mathbb{Q}(x_i)_{i\in\tilde{I}}$ $\qquad \mathbb{X} = (x_i)_{i\in\tilde{I}}$: $\tilde{I}$-indexed subset of $\mathcal{F}$

variable mutation

$$\mu_k(\mathbb{X}) = (x_i)_{i\neq k} \cup \{x_k^*\}$$: exchange x_k by x_k^*

$$\boxed{x_k^* := \frac{\prod_{k\to i} x_i^{\#\{k\to i\}} + \prod_{k\leftarrow i} x_i^{\#\{k\leftarrow i\}}}{x_k}}$$



mutation $\quad \mu_k(\mathbb{X}, \tilde{\Omega}) = (\mu_k(\mathbb{X}), \mu_k(\tilde{\Omega}))$

We can iterate this mutation recursively.

seed : a pair $(\mathbb{Y}, \tilde{\Omega})$: obtained in this manner

cluster : a collection $\mathbb{Y}$ of variables in a seed

cluster variable : a variable in a cluster.

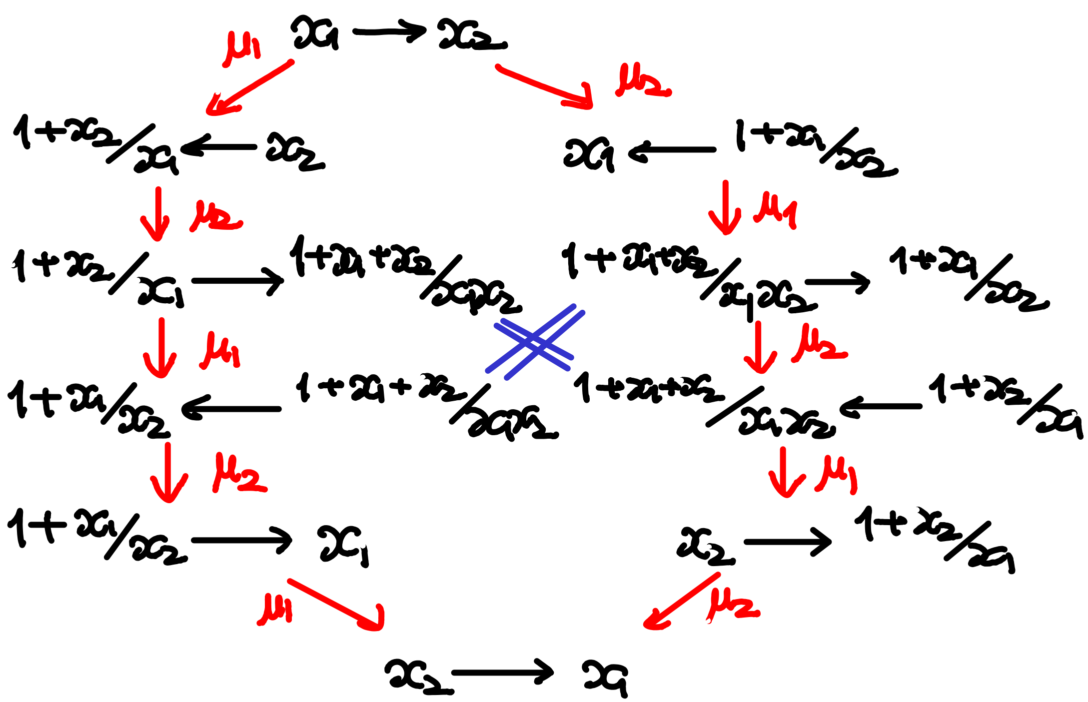
cluster monomial : a monomial in cluster variables in a single cluster

$\boxed{\text{cluster algebra } \mathcal{A}(\tilde{B}) \subset \mathcal{F} : \text{subalgebra generated by cluster variables from various clusters}}$

Example

$I = 1 \to 2$

$I_{fr} = \phi$



cluster variables : $x_1,\ x_2,\ \frac{1+x_2}{x_1},\ \frac{1+x_1}{x_2},\ \frac{1+x_1+x_2}{x_1x_2}$

denominators $\quad *,\ *,\ \underbrace{x_1,\ x_2,\ x_1x_2}_{\text{positive roots } \Delta_+(A_2)}$

Remark. mutations are performed only at $k \in I = \tilde{I} \setminus I_{fr}$.

$\Longrightarrow$ x_i $(i \in I_{fr})$ is always in a cluster.

$\rightsquigarrow$ we call a **frozen variable**.

Th (1) (Fomin–Zelevinsky, **Finite type classification**)

Suppose (I, Ω) : type ADE

$$\{\text{cluster variables}\} \setminus \{\text{frozen variables}\} \xleftrightarrow[\text{bijective}]{} \text{almost positive roots } \Delta_{\geq -1} = \Delta_+ \sqcup \{-\alpha_i \mid i \in I\}$$

$$x_i \longleftrightarrow -\alpha_i$$

$$x[\alpha] = \frac{*}{x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}} \longleftrightarrow \underset{\in \Delta_+}{\alpha} = \sum m_i \alpha_i \qquad m_i \in \mathbb{Z}_{\geq 0}$$

(2) (Caldero–Keller using [BMRRT: cluster category])

More generally

$$\{\text{cluster variables}\} \setminus \{\text{frozen}\} \longleftrightarrow \text{real "Schur" roots} \sqcup \{-\alpha_i\}$$

$$\underset{\text{initial, frozen}}{\text{cluster variable}} = \frac{\text{polynomial in } x_i}{\text{monomial in } x_i} \leftarrow \text{root}$$

(FZ **Laurent phenomenon**)

Conjecture (FZ)

numerator : positive coefficients

Fomin–Zelevinsky motivation:

To understand "dual canonical bases" of coordinate rings $\mathbb{C}[X]$ of representation theoretic origins

variant of $\mathbb{C}[N^-]$

$= (U_q^-|_{q=1})^*$

Conjecture (Working hypothesis?)

$\exists$? "dual canonical bases" (variant of Lusztig's) such that

$\{\text{cluster monomials}\} \subset \text{"dual canonical base"}|_{q=1}$

Remark. This implies the positivity conjecture.

Main Result:

The cluster algebra can be realised via perverse sheaves on vector spaces of quiver representations which Lusztig used to define canonical bases so that

$$\{\text{cluster monomials}\} \subset \text{dual canonical base}.$$

(When the initial quiver is (bipartite)

$$\begin{tikzcd} I & \bullet \arrow[r] & \bullet \arrow[d] & \bullet \arrow[l] \\ I_{fr}=I & \bullet \arrow[u] & \bullet & \bullet \arrow[u] \end{tikzcd}$$

hoped to be generalised)

More precisely,

cluster algebra $=$ subquotient of $(U^-_{q=1})^*$

s.t. Lusztig's canonical base elements descend.

Remark. This result is motivated by [Hernandez-Leclerc].

cluster algebra $\overset{?}{\subset}$ Grothendieck ring of f.d. representations of quantum affine algebras

control

vector spaces of quiver representations $\longleftarrow$ special case of graded quiver varieties

Now we change the view point. We are interested in dual canonical base.

Q. What are clusters from dual canonical base side?

A. Take $b \in \{\text{cluster monomials}\} \subset \{\text{dual canonical base elements}\}$

$$\Longrightarrow \quad b = \text{a cluster monomial} = b_1^{k_1} b_2^{k_2} \cdots b_n^{k_n} \quad (k_i \in \mathbb{Z}_{\geq 0})$$

s.t. $\{b_1, \cdots, b_n\}$ form a cluster.

$\boxed{\text{factorization of dual canonical base elements}}$

This is very interesting also from the canonical base side, as
the multiplication of $U_q^- \longleftrightarrow \begin{cases} \text{tensor product of quantum affine alg.} \\ \text{or restriction functor of affine Hecke alg.} \end{cases}$

It is a difficult problem in general:
when $\begin{cases} \text{tensor products of simples} \\ \text{restrictions} \end{cases}$ remain simple.

Following was known before:

Lemma. (Berenstein-Zelevinsky) Reinke? b, b': dual canonical base element s.t. $bb'|_{q=1} \overset{\exists}{=} \tilde{b}|_{q=1}$

$\Longrightarrow$ $bb' = q^n \tilde{b}$ (and hence $b'b = q^m \tilde{b}$)

$\therefore$ b & b' : q-commute

But q-commute $\not\Longrightarrow$ $bb'|_{q=1}$: dual canonical base element (Leclerc)

(proof) a simple consequence of the positivity.

Thus we understand that the cluster alg. structure is useful.

The proof of Main Result is an answer to the following:

Q. Why does "dual canonical base" have the cluster algebra structure?

A. The cluster algebra structure is shadow of quiver representation theory.

multiplication of dual canonical base $\longleftarrow$ extensions of quiver representations

Therefore (very roughly) a factorization of a dual canonical element corresponds to a direct sum decomposition of quiver representations.

(recall cluster variables $\longleftrightarrow$ real Schur roots
$\longleftrightarrow$ indecomposable representations with special properties

Today I do not explain results in the quiver representation side (in particular, cluster category: Buan–Marsh–Reineke–Reiten–Todorov).
I only explain the definition of the subquotient.

Construction

- (I,Ω) : bipartite quiver (i.e. $I = I_{\text{sink}} \sqcup I_{\text{source}}$)
- $\tilde{I} = I \sqcup \underset{\substack{\| \\ I_{\text{fr}}}}{I}$ $\tilde{\Omega}$:

$$\begin{tikzcd}
i \circ & \circ \arrow[l] \arrow[r] \arrow[d] & \circ \\
i' \circ \arrow[u, "*"] & \circ & \circ \arrow[u]
\end{tikzcd}$$

- W : $\tilde{I}$-graded vector space / $\mathbb{C}$
- $E_W = \bigoplus_{h \in \tilde{\Omega}} \mathrm{Hom}(W_{o(h)}, W_{i(h)})$
- $\mathcal{D}(E_W)$: bdd derived cat. of constructible sheaves on E_W
- $\mathcal{P}_W$: a certain class of simple perverse sheaves on E_W
  $\subsetneqq$ Lusztig's class
- $Q_W \subset \mathcal{D}(E_W)$ finite direct sums of various $L[k]$
  $L \in \mathcal{P}_W$, $k \in \mathbb{Z}$

$\Longrightarrow K(Q_W)$: Grothendieck group of Q_W
$\mathbb{Z}[t, t^{-1}]$-module with base $\{L \mid L \in \mathcal{P}_W\}$

Definition of $\mathcal{P}_W$

S: another collection of vector spaces indexed by I

- i: sink. $\quad S_i \subset W_i$
- i: source $\quad S_i \subset \bigoplus_{h_i : o(h_i)=i} S_{i(h_i)} \oplus W_i$

- $\mathcal{F}(V,W)$ = variety of such subspaces $\quad \dim S_i$: fixed

  $V=(V_i)_{i\in\tilde{I}}$: underlying vector spaces of $S=(S_i)_{i\in\tilde{I}}$

- $\tilde{\mathcal{F}}(V,W) = \{ (y, S) \mid y \in E_W, \ \overset{i:\text{sink}}{\mathrm{Im}(W_i \leftarrow \bullet)} \subset S_i, \ \overset{i:\text{source}}{\mathrm{Im}(W_i \rightarrow \bullet)} \subset S_i \}$

$$\begin{tikzcd} & \tilde{\mathcal{F}}(V,W) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ E_W & & \mathcal{F}(V,W) \end{tikzcd}$$

projective p_1 $\qquad$ p_2: vector b'dle

$\Rightarrow$ $\mathcal{P}_W$ = simple perverse sheaves L on E_W whose shifts appear in $p_{1!}(\mathbb{C}_{\tilde{\mathcal{F}}(V,W)})$ for some V

constant sheaf

restriction functor

$W^2 \subset W$: $\widetilde{I}$-graded subspace

$W^1 = W/W^2$

$$\mathbb{E}_{W^1} \times \mathbb{E}_{W^2} \xleftarrow{\kappa} \mathfrak{Z}_0^{\bullet}(W^1, W^2; W) \xrightarrow{\iota} \mathbb{E}_W$$

$$\mathfrak{Z}_0^{\bullet}(W^1, W^2; W) = \{ y \in \mathbb{E}_W \mid y(W^2) \subset W^2 \}$$

$\widetilde{\mathrm{Res}} = \kappa_! \iota^* : \mathcal{Q}_W \longrightarrow \mathcal{Q}_{W^1} \otimes \mathcal{Q}_{W^2}$ $\qquad K(\mathcal{Q}_W) \longrightarrow K(\mathcal{Q}_{W^1}) \otimes K(\mathcal{Q}_{W^2})$

This defines a comultiplication on $\bigoplus_W K(\mathcal{Q}_W)$

Res : a shift of $\widetilde{\mathrm{Res}}$

Remark. This is exactly Lusztig's comultiplication, except the shift is different.

$$\boxed{\text{subalgebra of } \prod_{W} \mathrm{Hom}(K(Q_W), \mathbb{Z}[t,t^{-1}]) = \Big(\bigoplus_{W} K(Q_W)\Big)^*}$$

new part!

Want : equiv. rel. $\sim$ on $\coprod_{W} Q_W$ so that

$$\text{the subalgebra} = \{ f \in \prod_{W} \mathrm{Hom}(\cdots) \mid f(x) = f(y) \text{ if } x \sim y \} =: R_t$$

$\Longrightarrow$ dual canonical base : characteristic functions of equiv. classes.

Remark. The equiv. relation appears in the graded quiver varieties v.s. quantum loop algebras

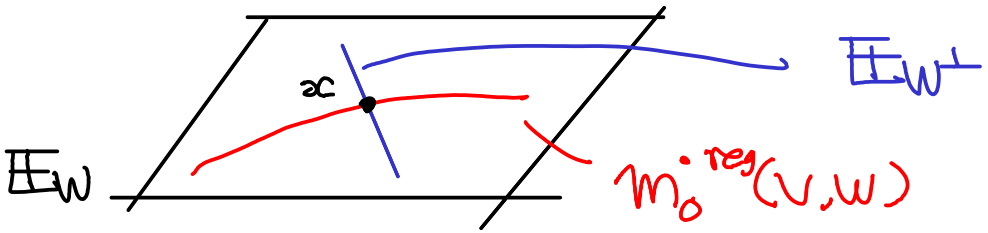
Definition of $\sim$

- $\mathcal{P}_W = \{ IC_W(\mathcal{V}) := IC(\overline{\mathfrak{M}_0^{\bullet\,reg}(V,W)}) \}$ $\quad\mathfrak{M}_0^{\bullet\,reg}(V,W) \subset \mathbb{E}_W$ certain loc. closed subvar.

Take $x \in \mathfrak{M}_0^{\bullet\,reg}(V,W)$

"transversal slice"

locally $(x, \mathbb{E}_W) \cong (0, \mathbb{E}_{W^\perp}) \times$ vector space



compatible with $\mathcal{P}_W$: $\mathcal{P}_W \longrightarrow \mathcal{P}_{W^\perp} \sqcup \{0\}$ restriction to $\mathbb{E}_{W^\perp}$

$IC_W(\mathcal{V}) \longmapsto IC_{W^\perp}(0) =$ skyscraper sheaf of $0 \in \mathbb{E}_{W^\perp}$

$IC_W(\mathcal{V}') \longmapsto IC_{W^\perp}(\mathcal{V}'^\perp)$

$\sim$: generated by $IC_W(\mathcal{V}') \sim IC_{W^\perp}(\mathcal{V}'^\perp)$

Remark In any equivalence class $\ni \exists 1$ skyscraper sheaf at the origin

Main Theorem.

(1) The subalgebra $\mathbb{R}_t\big|_{t=1} \cong$ cluster algebra for the quiver

where $x_i =$ skyscraper of E_W with W s.t. $\begin{cases} \mathbb{C} & \text{at vertex } i' \\ 0 & \text{otherwise} \end{cases}$

$$\begin{tikzcd} 0 & 0 \arrow[l] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[u] & \mathbb{C} & 0 \arrow[u] \end{tikzcd}$$

$f_i =$ 〃 of E_W with W s.t. $\begin{cases} \mathbb{C} & \text{at vertices } i \text{ and } i' \\ 0 & \text{otherwise} \end{cases}$

$$\begin{tikzcd} 0 & \mathbb{C} \arrow[l] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[u] & \mathbb{C} & 0 \arrow[u] \end{tikzcd}$$

(2) $\{$cluster monomials$\} \subset$ canonical base $\{L(W)\}$

$=$ if $Q = (I, \Omega)$: Dynkin type