Stochastic Volatility's Orderly Smiles

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Motivation

- Consider the following general dynamics for a diffusive stochastic volatility model:

$$
\begin{aligned}
dX_t &= -\frac{1}{2}\xi_t^t dt + \sqrt{\xi_t^t} dW_t^1, \qquad X_0 = x \\
d\xi_t^u &= \lambda(t, u, \xi_t^\cdot) \cdot dW_t, \qquad \xi_0^u = y^u
\end{aligned} \tag{1}
$$

- $X_t = \ln S_t$
- $\xi_t^\cdot \equiv (\xi_t^u, t \le u)$: instantaneous forward variance curve from t onwards. $\xi^u =$ driftless process; initial value y^u read on market prices of variance swap contracts: $\xi_0^u = \frac{d}{du}\left(\hat{\sigma}_u^2 u\right)$, where $\hat{\sigma}_u$ is the implied variance swap volatility for maturity u.
- $\lambda = (\lambda_1, \ldots, \lambda_d)$: volatility of forward instantaneous variances.
- $W = \left(W^1, \ldots, W^d\right) =$ a d-dimensional Brownian motion. W^1 drives the spot dynamics.
- No dividend. Zero rates and repos (for the sake of simplicity)

- **No closed-form formula** available **for the price of vanilla options** in Model (1).
- Approximations available in a few particular cases of "first generation" stochastic volatility models (e.g., Heston)
- Our goal: find a general approximation of the smile **which does not depend on a particular specification of the model**, i.e., on a particular choice of λ.

- $\Rightarrow$ We will derive **general asymptotic expansion of the smile, for small volatility of volatility, at second order**.
- Scaling factor ε: $\lambda \to \varepsilon\lambda$. X and ξ then depend on ε: $X \to X^\varepsilon$ and $\xi \to \xi^\varepsilon$.
- Two important assumptions: **no local volatility component**, and λ **does not depend on the asset value**.

- Smile produced by stochastic volatility models is generated by the covariance of forward variances with themselves and spot.
- Our goal: to **pinpoint exactly** which functionals of these covariances determine the vanilla smile

- Important to ensure, while varying ε, that implied volatilities of some specific payoffs are unchanged, so that the overall volatility level is not altered in the model.
- In our framework, **VS volatilities are unchanged** as ε is varied.

Expansion of the price of a vanilla option

- Consider the vanilla option delivering $g(X_T^\varepsilon)$ at time T.
- Price $P^\varepsilon\left(t, X_t^\varepsilon, \xi_t^{\cdot,\varepsilon}\right)$. We write $P^\varepsilon\left(t,x,y\right)$: **the variable** $y \equiv (y^u, t \le u \le T)$ **is a curve**.
- P^ε solves the PDE $\left(\partial_t + L^\varepsilon\right) P^\varepsilon = 0$ with terminal condition $P^\varepsilon(T,x,y) = g(x)$, where $L^\varepsilon = L_0 + \varepsilon L_1 + \varepsilon^2 L_2$ with

$$
\begin{aligned}
L_0 &= -\frac{1}{2} y^t \partial_x + \frac{1}{2} y^t \partial_x^2 \\
L_1 &= \int_t^T du\ \mu(t,u,y)\ \partial^2_{xy^u} \\
L_2 &= \frac{1}{2} \int_t^T du \int_t^T du'\ \nu(t,u,u',y)\ \partial^2_{y^u y^{u'}} \\
\mu(t,u,y) &= \sqrt{y^t} \lambda_1(t,u,y) = \frac{\mathbb{E}\left[dX_t d\xi_t^u | \xi_t = y\right]}{dt} = \frac{\mathbb{E}\left[\frac{dS_t}{S_t} d\xi_t^u | \xi_t = y\right]}{dt} \\
\nu(t,u,u',y) &= \sum_{i=1}^d \lambda_i(t,u,y)\lambda_i(t,u',y) = \frac{\mathbb{E}\left[d\xi_t^u d\xi_t^{u'} | \xi_t = y\right]}{dt}
\end{aligned}
$$

The perturbation equations

- Assume that $P^\varepsilon = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \varepsilon^3 P_3 + \cdots$

$$\begin{aligned} 0 &= \left(\partial_t + L_0 + \varepsilon L_1 + \varepsilon^2 L_2\right)\left(P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \varepsilon^3 P_3 + \cdots\right) \\ &= \left(\partial_t + L_0\right) P_0 + \varepsilon\left(\left(\partial_t + L_0\right) P_1 + L_1 P_0\right) \\ &\quad + \varepsilon^2\left(\left(\partial_t + L_0\right) P_2 + L_1 P_1 + L_2 P_0\right) \\ &\quad + \varepsilon^3\left(\left(\partial_t + L_0\right) P_3 + L_1 P_2 + L_2 P_1\right) + \cdots \end{aligned}$$

- $\Rightarrow$ We need to solve the following equations:

$$\begin{aligned} \left(\partial_t + L_0\right) P_0 &= 0, \quad P_0(T,x,y) = g(x) \\ \left(\partial_t + L_0\right) P_1 + L_1 P_0 &= 0, \quad P_1(T,x,y) = 0 \\ \left(\partial_t + L_0\right) P_n + L_1 P_{n-1} + L_2 P_{n-2} &= 0, \quad P_n(T,x,y) = 0, \qquad \forall n \geq 2 \end{aligned}$$

- L_0 = infinitesimal generator associated to X^0, the unperturbed diffusion for which $\varepsilon = 0$. L_0 = standard **one-dimensional Black-Scholes operator** with deterministic volatility $\sqrt{y^t}$ at time t.
- Each P_n = solution to the traditional **one-dimensional** diffusion equation **with a source term** $H_n = L_1 P_{n-1} + L_2 P_{n-2}$:

$$(\partial_t + L_0) P_n + H_n = 0$$

- Feynmann-Kac theorem $\Rightarrow$

$$\begin{aligned} P_0(t,x,y) &= \mathbb{E}\left[g\left(X_T^{0,t,x}\right)\right], \\ P_n(t,x,y) &= \mathbb{E}\left[\int_t^T H_n(s, X_s^{0,t,x}, y)ds\right], \qquad \forall n \geq 1 \end{aligned}$$

where $X^{0,t,x}$ is the unperturbed process where $\varepsilon = 0$, starting at log-spot x at time t:

$$dX_s^{0,t,x} = -\frac{1}{2} y^s ds + \sqrt{y^s} dW_s^1, \qquad X_t^{0,t,x} = x$$

The price at order 0

- P_0 is just the Black-Scholes price with time-dependent volatility $\sqrt{y^t}$:

$$P_0(t,x,y) = \mathbb{E}\left[g\left(x + \int_t^T \sqrt{y^s}dW_s^1 - \frac{1}{2}\int_t^T y^s ds\right)\right] = P_{BS}\left(x, \int_t^T y^s ds\right)$$

where

$$P_{BS}(x,v) = \mathbb{E}\left[g\left(x + \sqrt{v}G - \frac{1}{2}v\right)\right], \qquad G \sim \mathcal{N}(0,1) \tag{2}$$

- $v = \int_t^T y^s ds$ is the total variance of X^0 integrated from t to T.
- $P_0(t,x,y)$ depends on the curve $y \equiv (y^s, t \leq s \leq T)$ only through v.
- P_{BS} is solution to the PDE

$$\partial_v P_{BS} = \frac{1}{2}\left(\partial_x^2 - \partial_x\right)P_{BS}, \qquad P_{BS}(x,0) = g(x) \tag{3}$$

Links the vega and gamma of a vanilla option in the unperturbed state.

The price at order 0

An important observation:

- Because L_0 incorporates no local volatility, L_0 and ∂_x commute so $(\partial_t + L_0)\,\partial_x^p P_0 = \partial_x^p\,(\partial_t + L_0)\,P_0 = 0$.
- $\Rightarrow \partial_x^p P_{BS}\left(X_t^0, \int_t^T y^s ds\right) \equiv \partial_x^p P_0(t, X_t^0, y)$ is a martingale for all integer p.
- Equation (3) then shows that **for all integers** m, n, $\partial_v^m \partial_x^n P_{BS}\left(X_t^0, \int_t^T y^s ds\right)$ **is a martingale**.
- This is crucial in the computations of P_1 and P_2.

The price at order 1

- Let us define the **integrated spot-variance covariance** function $C_t^{X\xi}(y)$:

$$C_t^{X\xi}(y) = \int_t^T ds \int_s^T du \ \mu(s,u,y) = \int_t^T ds \int_s^T du \ \frac{\mathbb{E}\left[\frac{dS_s}{S_s} d\xi_s^u | \xi_s = y\right]}{ds}$$

- We then have

$$\begin{aligned}
P_1(t,x,y) &= \mathbb{E}\left[\int_t^T L_1 P_0(s, X_s^{0,t,x}, y) ds\right] \\
&= \mathbb{E}\left[\int_t^T ds \int_s^T du \ \mu(s,u,y) \ \partial_{y^u}\left(\partial_x P_{BS}\left(X_s^{0,t,x}, \int_s^T y^r dr\right)\right)\right] \\
&= \mathbb{E}\left[\int_t^T ds \int_s^T du \ \mu(s,u,y) \ \partial_{xv}^2 P_{BS}\left(X_s^{0,t,x}, \int_s^T y^r dr\right)\right] \\
&= \int_t^T ds \int_s^T du \ \mu(s,u,y) \mathbb{E}\left[\partial_{xv}^2 P_{BS}\left(X_s^{0,t,x}, \int_s^T y^r dr\right)\right] \\
&= C_t^{X\xi}(y) \ \partial_{xv}^2 P_{BS}\left(x, \int_t^T y^r dr\right)
\end{aligned}$$

The price at order 2

A similar result holds for the second order correction:

$$
\begin{aligned}
P_2 &= P_2^{L_2 P_0} + P_2^{L_1 P_1} \\
P_2^{L_2 P_0}(t,x,y) &= \frac{1}{2} C_t^{\xi\xi}(y)\ \partial_v^2 P_{BS}\left(x, \int_t^T y^r\, dr\right) \\
P_2^{L_1 P_1} &= P_{2,0}^{L_1 P_1} + P_{2,1}^{L_1 P_1} \\
P_{2,0}^{L_1 P_1}(t,x,y) &= \frac{1}{2} C_t^{X\xi}(y)^2\ \partial_x^2 \partial_v^2 P_{BS}\left(x, \int_t^T y^r\, dr\right) \\
P_{2,0}^{L_1 P_1}(t,x,y) &= C_t^{\mu}(y)\ \partial_x^2 \partial_v P_{BS}\left(x, \int_t^T y^r\, dr\right)
\end{aligned}
$$

$$
\begin{aligned}
C_t^{\xi\xi}(y) &= \int_t^T ds \int_s^T du \int_s^T du'\ \nu(s,u,u',y) = \int_t^T ds \int_s^T du \int_s^T du'\ \frac{\mathbb{E}\left[d\xi_s^u d\xi_s^{u'} | \xi_s = y\right]}{ds} \\
C_t^{\mu}(y) &= \int_t^T ds \int_s^T du\ \mu(s,u,y)\ \partial_{y^u}\left(C_s^{X\xi}(y)\right)
\end{aligned}
$$

$C_t^{\xi\xi}(y)$: **integrated variance-variance covariance** function

Expansion of the implied volatility

- We write $C^{X\xi} = C_0^{X\xi}(y)$, $C^{\xi\xi} = C_0^{\xi\xi}(y)$ and $C^{\mu} = C_0^{\mu}(y)$.
- **In the general diffusive stochastic volatility model (1), at second order in the vol of vol ε, the implied volatility for maturity T and strike K is quadratic in $L = \ln\left(\frac{K}{S_0}\right)$:**

$$\widehat{\sigma}^{\varepsilon}(T,K) \;=\; \widehat{\sigma}_T^{\text{ATM}} + \mathcal{S}_T \ln\left(\frac{K}{S_0}\right) + \mathcal{C}_T \ln^2\left(\frac{K}{S_0}\right) + O(\varepsilon^3) \qquad (4)$$

- Coefficients are

$$\widehat{\sigma}_T^{\text{ATM}} = \hat{\sigma}_T^{\text{VS}}\left[1 + \frac{\varepsilon}{4v}C^{X\xi} + \frac{\varepsilon^2}{32v^3}\left(12\left(C^{X\xi}\right)^2 - v\left(v+4\right)C^{\xi\xi} + 4v\left(v-4\right)C^{\mu}\right)\right]$$

$$\mathcal{S}_T = \hat{\sigma}_T^{\text{VS}}\left[\frac{\varepsilon}{2v^2}C^{X\xi} + \frac{\varepsilon^2}{8v^3}\left(4C^{\mu}v - 3\left(C^{X\xi}\right)^2\right)\right]$$

$$\mathcal{C}_T = \hat{\sigma}_T^{\text{VS}}\frac{\varepsilon^2}{8v^4}\left(4C^{\mu}v + C^{\xi\xi}v - 6\left(C^{X\xi}\right)^2\right)$$

- $v = \int_0^T \xi_0^s ds$ and $\hat{\sigma}_T^{\text{VS}} = \sqrt{\frac{v}{T}}$, the VS implied volatility for maturity T.

Comments

ATM implied volatility:

$$\widehat{\sigma}_T^{\text{ATM}} = \hat{\sigma}_T^{\text{VS}} \left[1 + \frac{\varepsilon}{4v} C^{X\xi} + \frac{\varepsilon^2}{32v^3} \left(12\left(C^{X\xi}\right)^2 - v(v+4)C^{\xi\xi} + 4v(v-4)C^{\mu}\right)\right]$$

- ATM implied volatility = variance swap volatility + spread. At first order, spread $= \frac{C^{X\xi}}{4\sqrt{vT}}\varepsilon$.
- Typically, on the equity market, $C^{X\xi} < 0$: ATM implied volatility lies below the variance swap volatility.
- When spot returns and forward variances are uncorrelated, $C^{X\xi} = C^{\mu} = 0$ so that

$$\widehat{\sigma}_T^{\text{ATM}} = \hat{\sigma}_T^{\text{VS}} \left(1 - \frac{\varepsilon^2}{32v^3} v(v+4) C^{\xi\xi}\right)$$

  Because $C^{\xi\xi} \geq 0$, ATM implied volatility lies again below variance swap volatility. The higher the volatility of variances, the smaller the ATM implied volatility.

Comments (continued)

ATM skew: $\mathcal{S}_T = \hat{\sigma}_T^{\text{VS}} \left[\frac{\varepsilon}{2v^2} C^{X\xi} + \frac{\varepsilon^2}{8v^3} \left(4C^{\mu}v - 3\left(C^{X\xi}\right)^2 \right) \right]$

- ATM skew $\mathcal{S}_T$ is of order ε. It has the sign of $C^{X\xi}$. $\mathcal{S}_T$ vanishes when spot returns and forward variances are uncorrelated, even at second order. ATM skew is produced only by the spot-variance correlation.
- Link ATM vol-VS vol-ATM skew:

$$\widehat{\sigma}_T^{\text{ATM}} = \hat{\sigma}_T^{\text{VS}} + \frac{\left(\hat{\sigma}_T^{\text{VS}}\right)^2 T}{2} \mathcal{S}_T$$

- At first order in ε, ATM skew has same sign as the difference between ATM implied volatility and variance swap volatility.

ATM convexity: $\mathcal{C}_T = \hat{\sigma}_T^{\text{VS}} \frac{\varepsilon^2}{8v^4} \left(4C^{\mu}v + C^{\xi\xi}v - 6\left(C^{X\xi}\right)^2 \right)$

- Curvature $\mathcal{C}_T$ is of order ε^2.
- Not only does it involve variance/variance covariance: spot/variance covariance (squared) contributes as well.
- If spot and variances are uncorrelated, $\mathcal{C}_T = \frac{C^{\xi\xi}}{8v^{5/2}\sqrt{T}}\varepsilon^2 \geq 0$.

Another derivation which stays at the level of operators

- Recall that the price P^ε of the vanilla option is solution to $\left(\partial_t + L_t^\varepsilon\right) P^\varepsilon = 0$ with $L_t^\varepsilon = L_{0,t} + \varepsilon L_{1,t} + \varepsilon^2 L_{2,t}$, and terminal condition $P^\varepsilon\left(T, x, y\right) = g(x)$.
- Price can be expressed in terms of the semigroup $(U_{st}^\varepsilon, 0 \le s \le t \le T)$ attached to the family of differential operators L_t^ε: $P^\varepsilon(t, \cdot) = U_{tT}^\varepsilon g$.
- The semigroup is defined by

$$U_{st}^\varepsilon = \lim_{n\to\infty} \left(1 - \delta t L_{t_0}^\varepsilon\right)\left(1 - \delta t L_{t_1}^\varepsilon\right)\cdots\left(1 - \delta t L_{t_{n-1}}^\varepsilon\right), \quad \delta t = \frac{t-s}{n}, \quad t_i = s + i\delta t$$

- It satisfies $U_{rt}^\varepsilon = U_{rs}^\varepsilon U_{st}^\varepsilon$ for $0 \le r \le s \le t \le T$, hence the notation $: \exp\left(\int_s^t L_\tau^\varepsilon d\tau\right) :$, where :: denotes time ordering.
- **We can directly expand U_{st}^ε in powers of ε.** Usual time-dependent perturbation technique in quantum mechanics. U_{st}^0 is called the free propagator.

- Consider the general situation where a differential operator L_t is perturbed by another operator H_t: $L_t^\varepsilon = L_t + \varepsilon H_t$
- From the definition of the semigroup, $U_{st}^\varepsilon = U_{st}^{(0)} + \varepsilon U_{st}^{(1)} + \varepsilon^2 U_{st}^{(2)} + \cdots$ with

$$
\begin{aligned}
U_{st}^{(1)} &= \int_s^t d\tau \; U_{s\tau}^0 H_\tau U_{\tau t}^0 \\
U_{st}^{(2)} &= \int_s^t d\tau_1 \int_{\tau_1}^t d\tau_2 \; U_{s\tau_1}^0 H_{\tau_1} U_{\tau_1\tau_2}^0 H_{t_2} U_{\tau_2 t}^0
\end{aligned}
$$

- $\Rightarrow P^\varepsilon = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \cdots$, with

$$
\begin{aligned}
P_1 &= \int_t^T d\tau \; U_{t\tau}^0 L_{1,\tau} U_{\tau T}^0 g \\
P_2 &= \int_t^T d\tau \; U_{t\tau}^0 L_{2,\tau} U_{\tau T}^0 g + \int_t^T d\tau_1 \int_{\tau_1}^T d\tau_2 \; U_{t\tau_1}^0 L_{1,\tau_1} U_{\tau_1\tau_2}^0 L_{1,\tau_2} U_{\tau_2 T}^0 g
\end{aligned}
$$

- We recover the expressions of P_1 and P_2.

Short maturity

- Assume $d\xi_t^t = \cdots dt + \varepsilon(\xi_t^t)^\varphi dB_t$
- Let ρ_{SV} be the correlation between S_t and instantaneous variance $V_t = \xi_t^t$
- Heston: $\varphi = \frac{1}{2}$, $\rho_{SV} = \rho$;
  Bergomi: $\varphi = 1$, $\rho_{SV} = \alpha_\theta\left((1-\theta)\rho_{SX} + \theta\rho_{SY}\right)$
- Then for short maturities

$$\mathcal{S}_0 \simeq \frac{\varepsilon}{4}\rho\left(\widehat{\sigma}^{\text{ATM}}\right)^{2\varphi-2} \quad (5)$$

$$\mathcal{C}_0 \simeq \varepsilon^2\left(\left(\frac{1}{12}\varphi - \frac{7}{48}\right)\rho^2 + \frac{1}{24}\right)\left(\widehat{\sigma}^{\text{ATM}}\right)^{4\varphi-5} \quad (6)$$

- $\Rightarrow$ Short-term ATM skew does not depend on short-term ATM vol iff $\varphi = 1$ (observed in equity markets)
- $\Rightarrow$ Short-term ATM convexity does not depend on short-term ATM vol iff $\varphi = \frac{5}{4}$. And $(\forall\rho_{SV},\ \mathcal{C}_0 \geq 0) \Longleftrightarrow \varphi \geq \frac{5}{4}$

Long-term asymptotics of implied volatility

- Assume the term-structure of variance swaps volatilities is flat: $\xi_0^t \equiv \xi$.
- Assume that for large $u-t$, $\mu(t,u,y) \propto (u-t)^{-\alpha}$, $\alpha > 0$.
  Then at higher order in ε, for long maturities,

$$\begin{aligned} \mathcal{S}_T &\propto T^{-\alpha} && \text{if } \alpha < 1 \\ \mathcal{S}_T &\propto T^{-1} && \text{if } \alpha > 1 \end{aligned}$$

  α is exactly a signature of the long-time decay of the spot/variance covariance function.
- Assume that for large $u-t$ and $u'-t$,
  $\nu(t,u,u',y) \propto (u-t)^{-\beta}(u'-t)^{-\beta}$, $\beta > 0$.
  Also assume that spots and volatilities are uncorrelated ($\mu \equiv 0$). Then at higher order in ε, for long maturities,

$$\begin{aligned} \mathcal{C}_T &\propto T^{-2\beta} && \text{if } \beta < 1 \\ \mathcal{C}_T &\propto T^{-2} && \text{if } \beta > 1 \end{aligned}$$

- Exponential decay $\leftrightarrow \beta > 1$.

First example: a Heston-like model

$$dX_t = -\frac{1}{2}V_t dt + \sqrt{V_t} dW_t^1, \qquad X_0 = x \tag{7}$$

$$dV_t = -k\left(V_t - V_\infty\right)dt + \lambda\left(V_t\right)^\varphi\left(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right), \qquad V_0 = V$$

- The instantaneous forward variance reads

$$\xi_t^u = \mathbb{E}\left[V_u | V_t\right] = V_\infty + \left(V_t - V_\infty\right)e^{-k(u-t)}$$

  and its dynamics is:

$$d\xi_t^u = \lambda e^{-k(u-t)}\left(\xi_t^t\right)^\varphi\left(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right)$$

- The initial term-structure of instantaneous forward variances is

$$y^u \equiv \xi_0^u = v_\infty + \left(v - v_\infty\right)e^{-ku}$$

- Like **in all classic "first generation" stochastic volatility models, this term-structure is determined by the model parameters**, and the current value of the instantaneous volatility.

- The volatility $\lambda(t,u,y)$ of instantaneous forward variances depends on the instantaneous forward variance curve $y = (y^s, t \le s \le T)$ only through the instantaneous spot variance y^t:

$$\begin{aligned} \lambda_1(t,u,y) &= \rho \left(y^t\right)^{\varphi} e^{-k(u-t)} \\ \lambda_2(t,u,y) &= \sqrt{1-\rho^2} \left(y^t\right)^{\varphi} e^{-k(u-t)} \end{aligned}$$

- As a consequence,

$$\begin{aligned} C^{X\xi} &= \frac{\rho}{k} \int_0^T ds \;\; (y^s)^{\varphi+\frac{1}{2}} \left(1 - e^{-k(T-s)}\right) \\ C^{\xi\xi} &= \sum_{i=1}^{2} \int_0^T ds \left(\int_s^T du \, \lambda_i(s,u,y) \right)^2 = \frac{1}{k^2} \int_0^T ds \;\; (y^s)^{2\varphi} \left(1 - e^{-k(T-s)}\right)^2 \\ C^{\mu} &= \left(\varphi + \frac{1}{2}\right) \frac{\rho^2}{k} \int_0^T ds \; (y^s)^{\varphi+\frac{1}{2}} \int_s^T du \; (y^u)^{\varphi-\frac{1}{2}} \, e^{-k(u-s)} \left(1 - e^{-k(T-u)}\right) \end{aligned}$$

- This coincides with Equations (3.7) to (3.10) in Lewis [7], where $J^{(1)} = C^{X\xi}$, $J^{(3)} = \frac{1}{2} C^{\xi\xi}$, and $J^{(4)} = C^{\mu}$

Second example: the Bergomi model

$$
\begin{aligned}
dx_t &= -\frac{1}{2}\xi_t^t dt + \sqrt{\xi_t^t} dW_t^S \\
d\xi_t^u &= \xi_t^u \alpha_\theta \omega \left((1-\theta)\, e^{-k_X (u-t)} dW_t^X + \theta e^{-k_Y (u-t)} dW_t^Y \right) \\
&= \lambda(t,u,\xi_t^{\cdot}) \cdot dW_t
\end{aligned}
$$

$$d\langle W^S, W^X\rangle_t = \rho_{SX} dt,\ d\langle W^S, W^Y\rangle_t = \rho_{SY} dt,\ d\langle W^X, W^Y\rangle_t = \rho_{XY} dt.$$

- The normalizing factor

$$\alpha_\theta = \left((1-\theta)^2 + 2\rho_{XY}\theta\,(1-\theta) + \theta^2 \right)^{-1/2}$$

is such that the very-short term variance $\xi_t^{t,\omega}$ has log-normal volatility ω.

- We pick $k_X > k_Y$, θ is a parameter which mixes the short-term factor W^X and the long-term factor W^Y.

- After a Cholesky transform, this can be restated using independent Brownian motions W^1, W^2 and W^3 as follows:

$$\begin{aligned} W^S &= W^1 \\ W^X &= \rho_{SX} W^1 + \sqrt{1-\rho_{SX}^2} W^2 \\ W^Y &= \rho_{SY} W^1 + \chi_{XY} \sqrt{1-\rho_{SY}^2} W^2 + \sqrt{\left(1-\chi_{XY}^2\right)\left(1-\rho_{SY}^2\right)} W^3 \end{aligned}$$

where $\chi_{XY} = \frac{\rho_{XY} - \rho_{SX}\rho_{SY}}{\sqrt{1-\rho_{SX}^2}\sqrt{1-\rho_{SY}^2}}$

- ρ_{SX}, ρ_{SY} and ρ_{XY} define a correlation matrix $\iff \chi_{XY} \in [-1,1]$.
- The volatility of variance $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ reads

$$\begin{aligned} \lambda_1(t,u,y) &= y^u \alpha_\theta \left((1-\theta)\, \rho_{SX} e^{-k_X(u-t)} + \theta \rho_{SY} e^{-k_Y(u-t)} \right) \\ \lambda_2(t,u,y) &= y^u \alpha_\theta \left((1-\theta) \sqrt{1-\rho_{SX}^2} e^{-k_X(u-t)} + \theta \chi_{XY} \sqrt{1-\rho_{SY}^2} e^{-k_Y(u-t)} \right) \\ \lambda_3(t,u,y) &= y^u \alpha_\theta \theta \sqrt{\left(1-\chi_{XY}^2\right)\left(1-\rho_{SY}^2\right)} e^{-k_Y(u-t)} \end{aligned}$$

- We write $\lambda_i(t,u,y) = y^u \alpha_\theta \left(w_{iX} e^{-k_X(u-t)} + w_{iY} e^{-k_Y(u-t)} \right)$

The covariance functions read

$$
\begin{aligned}
C^{X\xi} &= \int_0^T du \int_0^u dt \sqrt{y^t} \lambda_1(t,u,y) \\
&= \alpha_\theta (1-\theta) \rho_{SX} \int_0^T du\ y^u \int_0^u dt \sqrt{y^t} e^{-k_X(u-t)} \\
&\quad + \alpha_\theta \theta \rho_{SY} \int_0^T du\ y^u \int_0^u dt \sqrt{y^t} e^{-k_Y(u-t)} \\
C^{\xi\xi} &= \sum_{i=1}^3 \int_0^T ds \left(\int_s^T du\, \lambda_i(s,u,y) \right)^2 \\
&= \alpha_\theta^2 \sum_{i=1}^3 \int_0^T ds \left(w_{iX} \int_s^T du\ y^u e^{-k_X(u-s)} + w_{iY} \int_s^T du\ y^u e^{-k_Y(u-s)} \right)^2 \\
C^{\mu} &= \int_0^T ds \int_s^T du \sqrt{y^s} \lambda_1(s,u,y^u) \left(\frac{1}{2\sqrt{y^u}} \int_u^T dt \lambda_1(u,t,y^t) \right. \\
&\quad \left. + \int_s^u dr \sqrt{y^r} \frac{\partial \lambda_1}{\partial z}(r,u,z)|_{z=y^u} \right)
\end{aligned}
$$

In the case of a flat initial term structure of variance swaps ($y_0^t \equiv \xi$), this reads

$$
\begin{aligned}
C^{x\xi} &= \alpha_\theta \omega \xi^{3/2} T^2 \left(w_{1X} \mathcal{J}(k_X T) + w_{1Y} \mathcal{J}(k_Y T) \right) \\
C^{\xi\xi} &= \alpha_\theta^2 \omega^2 \xi^2 T^3 \left(w_0 + w_X \mathcal{I}(k_X T) + w_Y \mathcal{I}(k_Y T) \right. \\
&\quad \left. + w_{XX} \mathcal{I}(2k_X T) + w_{YY} \mathcal{I}(2k_Y T) + w_{XY} \mathcal{I}((k_X + k_Y) T) \right) \\
C^{\mu} &= \alpha_\theta^2 \omega^2 \xi^2 T^3 \left(C_1^\mu + C_2^\mu \right)
\end{aligned}
$$

with

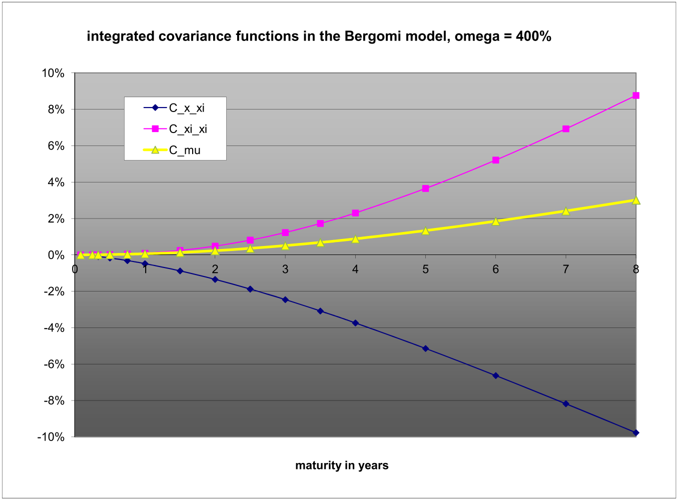
$$
\begin{aligned}
\mathcal{I}(\alpha) &= \frac{1 - e^{-\alpha}}{\alpha}, \quad \mathcal{J}(\alpha) = \frac{\alpha - 1 + e^{-\alpha}}{\alpha^2} \\
\mathcal{K}(\alpha) &= \frac{1 - e^{-\alpha} - \alpha e^{-\alpha}}{\alpha^2}, \quad \mathcal{H}(\alpha) = \frac{\mathcal{J}(\alpha) - \mathcal{K}(\alpha)}{\alpha} \\
w_0 &= \sum_{i=1}^{3} \left(\frac{w_{iX}}{k_X T} + \frac{w_{iY}}{k_Y T} \right)^2, \quad w_X = -2 \sum_{i=1}^{3} \frac{w_{iX}}{k_X T} \left(\frac{w_{iX}}{k_X T} + \frac{w_{iY}}{k_Y T} \right) \\
w_Y &= -2 \sum_{i=1}^{3} \frac{w_{iY}}{k_Y T} \left(\frac{w_{iX}}{k_X T} + \frac{w_{iY}}{k_Y T} \right), \\
w_{XX} &= \sum_{i=1}^{3} \frac{w_{iX}^2}{k_X^2 T^2}, \quad w_{YY} = \sum_{i=1}^{3} \frac{w_{iY}^2}{k_Y^2 T^2}, \quad w_{XY} = 2 \sum_{i=1}^{3} \frac{w_{iX} w_{iY}}{k_X k_Y T^2}
\end{aligned}
$$

and

$$\begin{aligned} C_1^\mu &= \frac{1}{2} w_{1X}^2 \mathcal{H}(k_X T) + \frac{1}{2} w_{1Y}^2 \mathcal{H}(k_Y T) - w_{1X} w_{1Y} \frac{\mathcal{J}(k_Y T) - \mathcal{J}(k_X T)}{(k_Y - k_X) T} \\ C_2^\mu &= w''_X \mathcal{J}(k_X T) + w''_Y \mathcal{J}(k_Y T) + w''_{XX} \mathcal{J}(2 k_X T) + w''_{YY} \mathcal{J}(2 k_Y T) \\ &\quad + w''_{XY} \mathcal{J}((k_X + k_Y) T) \end{aligned}$$

with

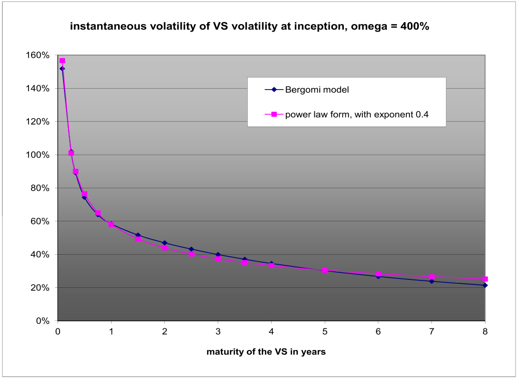
$$\begin{aligned} w''_X &= \frac{w_{1X}^2}{k_X T} + \frac{w_{1X} w_{1Y}}{k_Y T}, \qquad w''_Y = \frac{w_{1Y}^2}{k_Y T} + \frac{w_{1X} w_{1Y}}{k_X T} \\ w''_{XX} &= -\frac{w_{1X}^2}{k_X T}, \qquad w''_{YY} = -\frac{w_{1Y}^2}{k_Y T}, \qquad w''_{XY} = -\frac{w_{1X} w_{1Y}}{k_X T} - \frac{w_{1X} w_{1Y}}{k_Y T} \end{aligned}$$


integrated covariance functions in the Bergomi model, omega = 400%
C_x_xi
C_xi_xi
C_mu
10%
8%
6%
4%
2%
0%
-2%
-4%
-6%
-8%
-10%
0
1
2
3
4
5
6
7
8
maturity in years

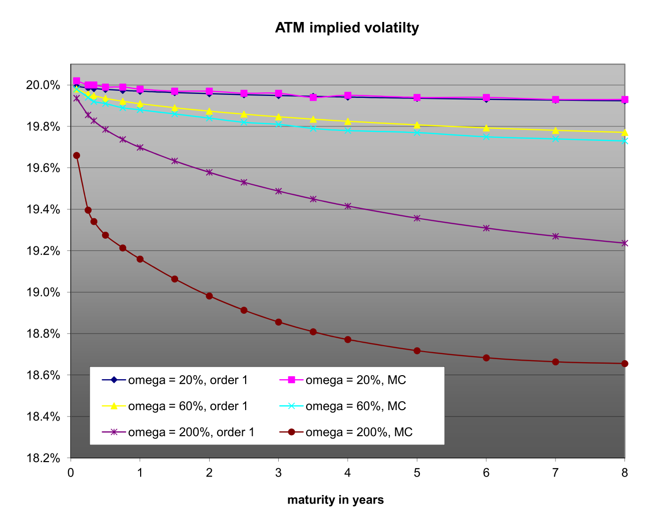
Numerical experiments

We pick the Bergomi model with a flat initial term structure of variance swap prices and

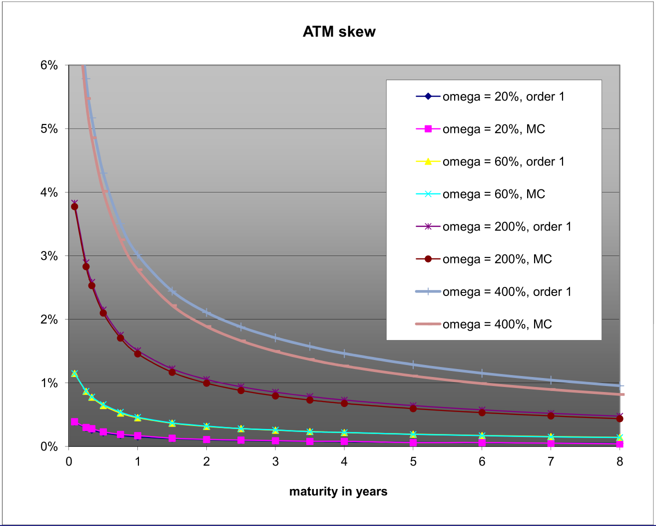
θ	k_X	k_Y	ρ_{SX}	ρ_{SY}	ρ_{XY}	χ_{XY}	ξ
0.25	8	0.35	-0.8	-0.48	0	-0.73	$(0.2)^2$



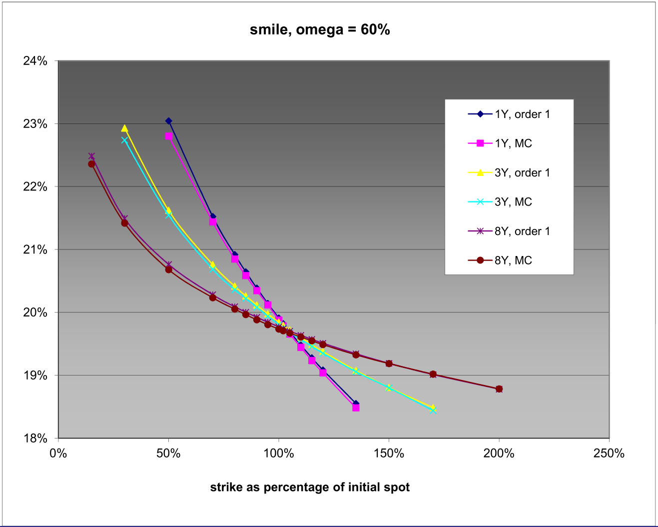
First order


ATM implied volatilty
20.0%
19.8%
19.6%
19.4%
19.2%
19.0%
18.8%
18.6%
18.4%
18.2%
0
1
2
3
4
5
6
7
8
maturity in years
omega = 20%, order 1
omega = 20%, MC
omega = 60%, order 1
omega = 60%, MC
omega = 200%, order 1
omega = 200%, MC

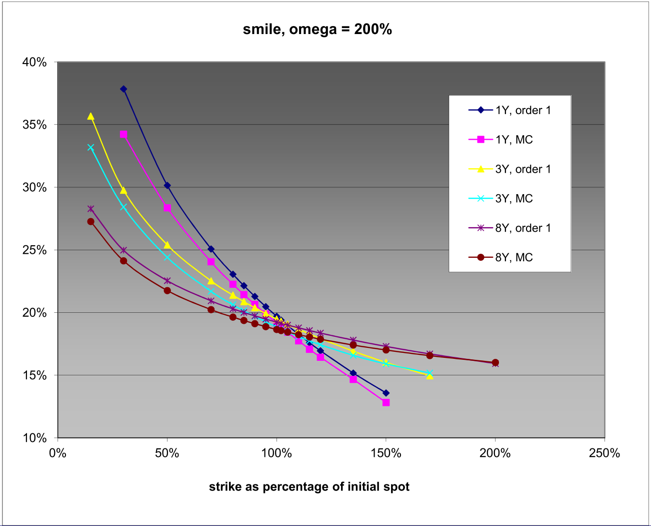
First order


ATM skew
6%
5%
4%
3%
2%
1%
0%
0
1
2
3
4
5
6
7
8
maturity in years
omega = 20%, order 1
omega = 20%, MC
omega = 60%, order 1
omega = 60%, MC
omega = 200%, order 1
omega = 200%, MC
omega = 400%, order 1
omega = 400%, MC

First order


smile, omega = 60%
24%
23%
22%
21%
20%
19%
18%
0%
50%
100%
150%
200%
250%
strike as percentage of initial spot
1Y, order 1
1Y, MC
3Y, order 1
3Y, MC
8Y, order 1
8Y, MC

First order


smile, omega = 200%
40%
35%
30%
25%
20%
15%
10%
0%
50%
100%
150%
200%
250%
strike as percentage of initial spot
1Y, order 1
1Y, MC
3Y, order 1
3Y, MC
8Y, order 1
8Y, MC

First order

- **ATM skew very sharply estimated by the first order expansion**, even for large values of the volatility of variance ω.
- **ATM volatility well captured by the expansion at first order in ω only for small values of ω** (say, up to 60%).
- True ATM implied volatilities are below their first order approximates $\Rightarrow$ ATM volatility is a very concave function of ω, around $\omega = 0$. In view of the expression for $\widehat{\sigma}_T^{\text{ATM}}$, this means that, for the set of parameters picked,

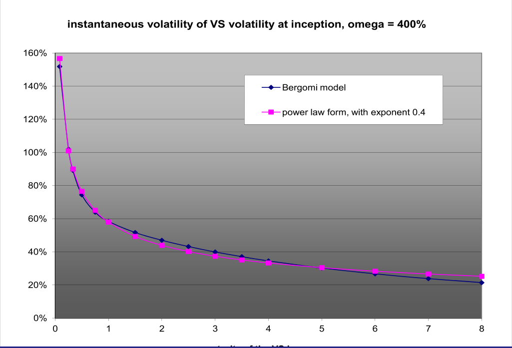
$$12C^{X\xi 2} - C^{\xi\xi}v\,(v+4) + 4C^{\mu}v\,(v-4) \leq 0$$

- **Global shape of smile well captured by first order expansion**: the true implied volatility for strike K is indeed approximately affine in $\ln(K/S_0)$.
- But level of smile well captured only for small values of ω.

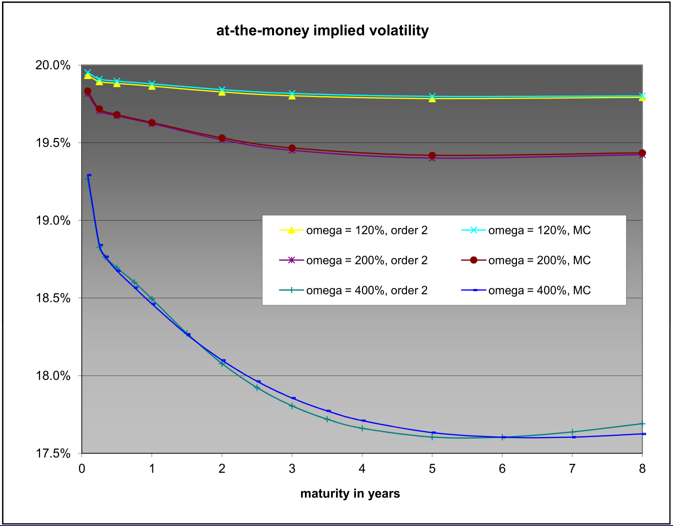
Second order

We first consider the situation when spot returns and forward variances are uncorrelated. In this case, the ATM skew vanishes, and so does its expansion at second order in ω. We pick

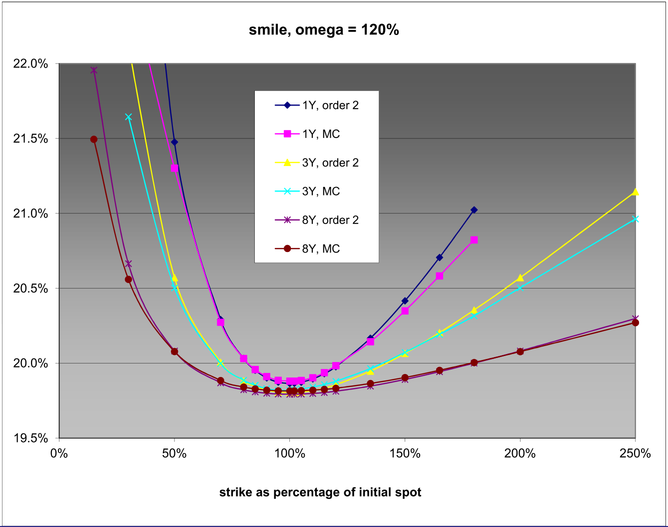
θ	k_X	k_Y	ρ_{SX}	ρ_{SY}	ρ_{XY}	ξ
0.25	8	0.35	0	0	0	$(0.2)^2$



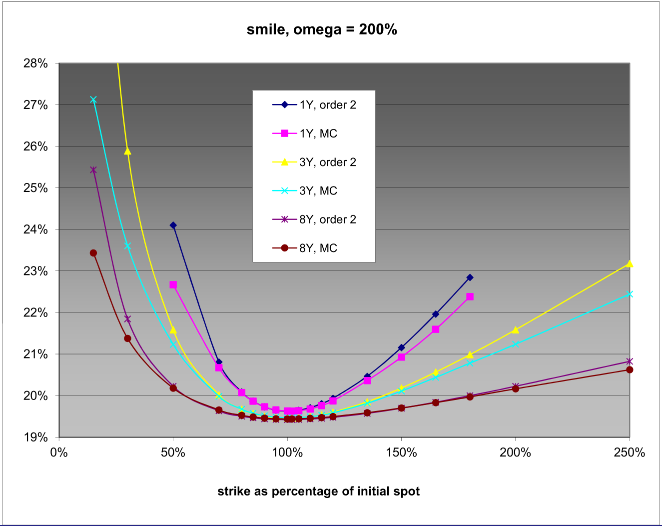
Second order



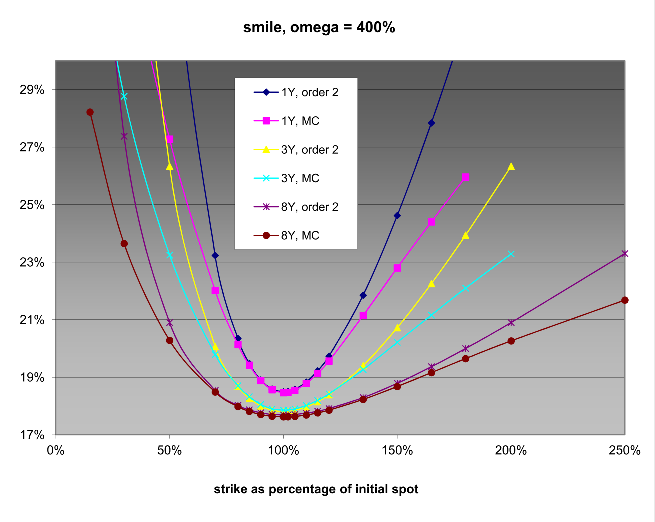
Second order



Second order



Second order

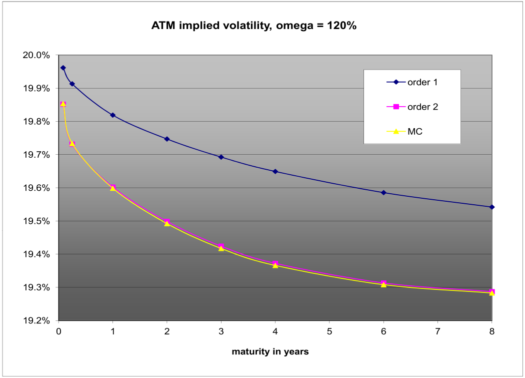


Second order

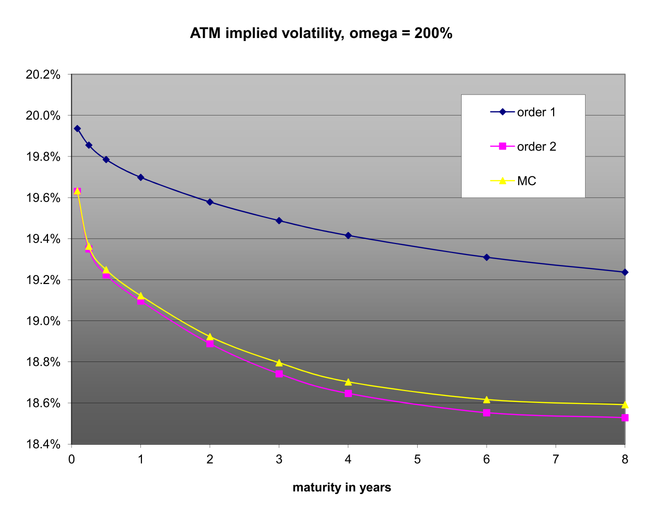
- **ATM implied volatility very sharply estimated by the second order expansion**, even up to $\omega = 400\%$ and to long maturities. For $T = 15$ years, estimate is less than 15 bps above true ATM volatility.
- Looking at the whole smile: second order expansion of the implied volatility is excellent around the money, but becomes too large for strikes far from the money.
- Not surprising: No arbitrage $\Rightarrow$ for very small and very large strikes, $\widehat{\sigma}(T,K)^2$ grows at most linearly with $\ln(K/S_0)$ (see Lee [6]), whereas second order estimate for $\widehat{\sigma}(T,K)^2$ grows like $\ln^4(K/S_0)$, see (4). Remainder $O(\omega^3) = R(\omega, T, K)$ is large for large K, for finite ω.
- Nevertheless, even for $\omega = 400\%$, a maturity of 8 years and an out-the-money strike of 250%, the error is only 1.5 point of volatility.

Second order

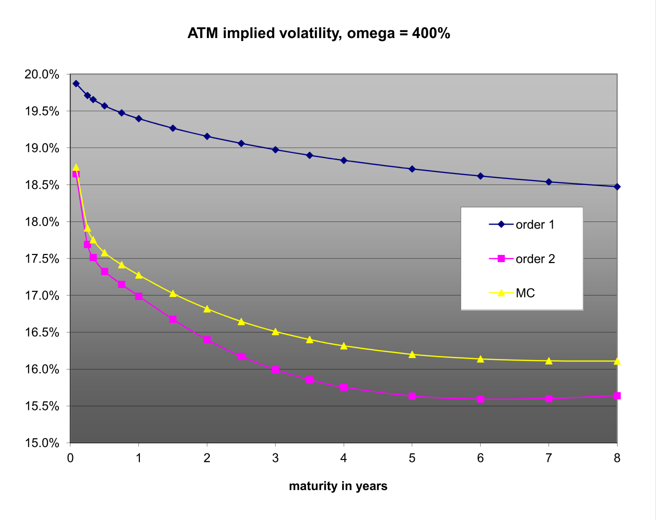
We now check numerically the accuracy of the second order expansion of the smile in the general case of correlated spot returns and variances.



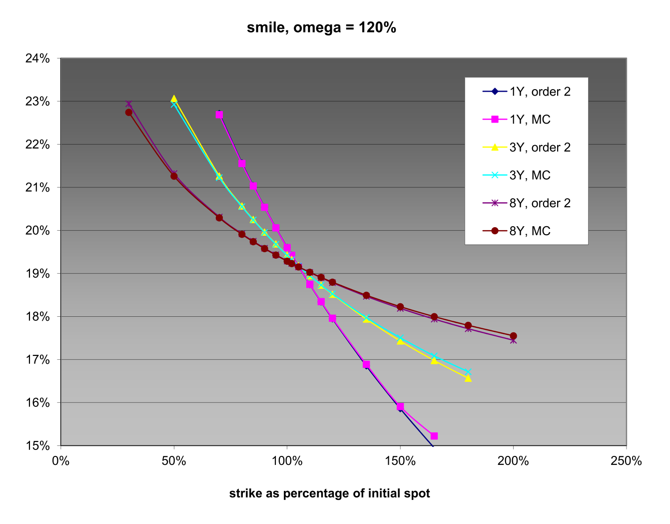
Second order



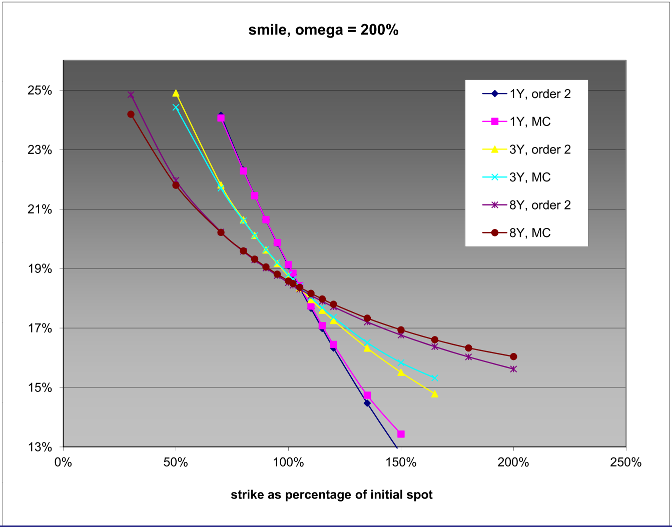
Second order



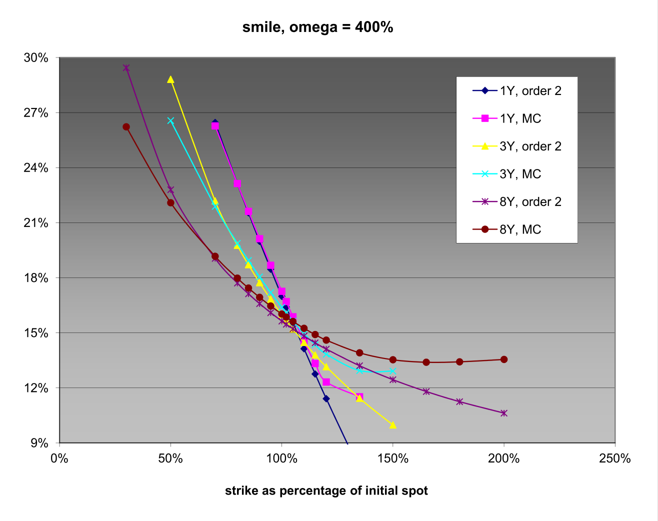
Second order



Second order



Second order



- Remember $\mathcal{S}_T = \frac{C^{X\xi}}{2v^{3/2}\sqrt{T}}\varepsilon + O\left(\varepsilon^2\right)$
- Let us now compute the skewness s_T of log-returns:

$$s_T = \frac{\mathbb{E}\left[\mathcal{X}_T^3\right]}{\mathbb{E}\left[\mathcal{X}_T^2\right]^{3/2}}, \qquad \mathcal{X}_T = X_T - \mathbb{E}\left[X_T\right] = \int_0^T \sqrt{\xi_t^{t,\varepsilon}} dW_t^1$$

- We have $\mathbb{E}\left[\mathcal{X}_T^2\right] = \int_0^T \xi_0^t dt + O(\varepsilon)$ and

$$\mathbb{E}\left[\mathcal{X}_T^3\right] = 3\varepsilon C^{X\xi} + O\left(\varepsilon^2\right)$$

- At first order in the vol of vol, the skewness of (the distribution of) $\ln\left(S_T/S_0\right)$ is thus

$$s_T = \frac{3\varepsilon C^{X\xi}}{\left(\int_0^T \xi_0^t dt\right)^{3/2}}$$

- The ATM skew $\mathcal{S}_T$ simply reads

$$\mathcal{S}_T = \frac{s_T}{6\sqrt{T}} + O(\varepsilon^2)$$

Conclusion

- We provide an expansion at order two in volatility-of-volatility for **general stochastic volatility models** based on a forward variance formulation.
- VS volatilities for all maturities are unchanged as ε is varied.
- At order two in ε, **the smile is exactly quadratic in log-moneyness** and **depends on only three model-dependent dimensionless quantities**:
  - $C^{X\xi}$, the integrated spot/variance covariance function,
  - $C^{\xi\xi}$, the integrated variance/variance covariance function,
  - C^{μ}, which, like $C^{x\xi}$, depends only on instantaneous spot/variance covariances.
- We shed light on the significance of $C^{X\xi}$ by establishing a simple link between the ATM skew and the skewness of $\ln S_T$.

- From our general expression we derive the short-maturity limits of ATM volatility, skew, curvature: we give structural dependencies of the ATM skew and curvature on ATM volatility.
- We also link the long-term decay of the ATM skew and curvature to the decay of spot/variance and variance/variance covariance functions.
- Numerical experiments in the case of a two-factor version of the Bergomi model show good agreement of the order one expression for the ATM skew, and of the order two expression for the ATM volatility, for values of the volatility of short-dated variance (around 400%) that are typical of implied levels of equity indices.

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