# Market Microstructure Invariants

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# Overview

Our goal is to explain how **order size**, **order frequency**, **and trading costs** vary across stocks with different **trading activity**.

- We develop a **model of market microstructure invariance** that generates predictions concerning cross-sectional variations of these variables.
- These predictions are tested using a data set of portfolio transitions and find a strong support in the data.
- The model implies simple formulas for order size, order frequency, market impact, and bid-ask spread as functions of observable dollar trading volume and volatility.

# A Framework

We think of trading a stock as playing a trading game:

- **Long-term traders** buy and sell shares to implement "bets."
- **Intermediaries with short-term strategies**–market makers, high frequency traders, and other arbitragers–clear markets.

The intuition behind a trading game was first described by Jack Treynor (1971). In that game informed traders, noise traders and market makers traded with each other.

Since managers trade many different stocks, we can think of them as playing many different trading games simultaneously.

# MAIN IDEA: Trading Games Across Stocks Are Played in "Business Time."

Stocks are different in terms of their trading activity: dollar trading volume, volatility etc. Trading games look different across stocks only at first sight!

**Our intuition** is that trading games are the same across stocks, except for the length of time over which these games are played or the speed with which they are played.

"Business time" passes faster for more actively traded stocks.

## Games Across Stocks

Only the speed with which business time passes varies as trading activity varies:

- **For active stocks** (high trading volume and high volatility), trading games are played at a **fast pace**, i.e. the length of trading day is small and business time passes quickly.
- **For inactive stocks** (low trading volume and low volatility), trading games are played at a **slow pace**, i.e. the length of trading day is large and business time passes slowly.

# Reduced Form Approach

As a rough approximation, we assume that bets arrive according to a compound Poisson process with **bet arrival rate** $\gamma$ bets per day and **bet size** having a distribution represented by $\tilde{Q}$ shares, $E(\tilde{Q}) = 0$.

Both $\tilde{Q}$ and $\gamma$ vary across stocks.

## Bet Volume and Bet Volatility

We define **bet volume** $\bar{V} := \gamma \cdot E|\tilde{Q}| = V/(\zeta/2)$.

We define **bet volatility** $\bar{\sigma} := \psi \cdot \sigma$.

$\zeta$ is "intermediation multiplier" and $\psi$ is "volatility multiplier". We might assume $\zeta$ and $\psi$ are constant, e.g., $\zeta = 2$ and $\psi = 1$.

# Market Microstructure Invariance-1

Business time passes at a rate proportional to bet arrival rate $\gamma$, which measures market "velocity."

**"Market Microstructure Invariance"** is the hypothesis that *the dollar distribution of these gains or losses is the same across all markets when measured in units of business time*, i.e., the distribution of the random variable

$$\tilde{I} := P \cdot \tilde{Q} \cdot \left( \frac{\sigma}{\gamma^{1/2}} \right)$$

is invariant across stocks or across time.

# Market Microstructure Invariance-2

**"Market Microstructure Invariance"** is also the hypothesis that *the dollar cost of risk transfers is the same function of their size across all markets, when size of risk transfer is measured in units of business time*, i.e., trading costs of a risk transfer of size $\tilde{I}$,

$$C_B(\tilde{I})$$

is invariant across stocks or across time.

## Trading Activity

Stocks differ in their **"trading activity"** $W$, or a measure of gross risk transfer, defined as dollar volume adjusted for volatility:

$$\bar{W} = \bar{\sigma} \cdot P \cdot \bar{V} = \bar{\sigma} \cdot P \cdot \gamma \cdot E|\tilde{Q}|.$$

Observable trading activity is a product of unobservable number of bets $\gamma$ and bet size $\bar{\sigma} \cdot P \cdot E|\tilde{Q}|$.

## Key Results

Since $\tilde{I} := P \cdot \tilde{Q} \cdot [\sigma/\gamma^{1/2}]$ and $\bar{W} = \bar{\sigma} \cdot P \cdot \gamma \cdot E|\tilde{Q}|$, we get

$$\gamma = \bar{W}^{2/3} \cdot \{E|\tilde{I}|\}^{-2/3}.$$

$$\frac{\tilde{Q}}{\bar{V}} \sim \bar{W}^{-2/3} \cdot \{E|\tilde{I}|\}^{-1/3} \cdot \tilde{I}.$$

Frequency increases twice as fast as size, as trading speeds up.

# Key Results

Let $C(\tilde{Q})$ be the percentage costs of executing a bet $P|\tilde{Q}|$. Then,

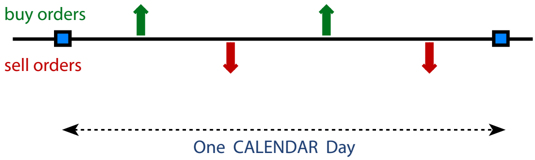
$$C(\tilde{Q}) = \frac{C_B(\tilde{I})}{P|\tilde{Q}|} = \bar{\sigma}\bar{W}^{-1/3}\{E|\tilde{I}|\}^{1/3} \cdot f(\tilde{I}) = \frac{1}{L} \cdot f(\tilde{I}),$$

where

- $L := \frac{\bar{W}^{1/3}}{\bar{\sigma}} \cdot E|\tilde{I}|^{1/3} = \left[\frac{P\bar{V}}{\bar{\sigma}^2}\right]^{1/3} \cdot E|\tilde{I}|^{1/3} =$ is **asset-specific measure of liquidity**;
- $f(\tilde{I}) := C_B(\tilde{I})/\tilde{I}$ is **invariant price impact function**.

# A Benchmark Stock

**Benchmark Stock** - daily volatility $\sigma = 200$ bps, price $P^* = \$40$, volume $V^* = 1$ million shares. Trades over a calendar day:




**Arrival Rate** $\gamma^* = 4$

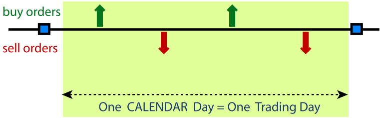
**Avg. Order Size** $\bar{Q}^*$ **as fraction of** $V^* = 1/4$

**Market Impact of 1/4** $V^* = 200$ bps / $4^{1/2} = 100$ bps

# Market Microstructure Invariance - Intuition

**Benchmark Stock with Volume $V^*$**

$(\gamma^*, \quad \tilde{Q}^*)$




**Avg. Order Size $\tilde{Q}^*$ as fraction of $V^*$**

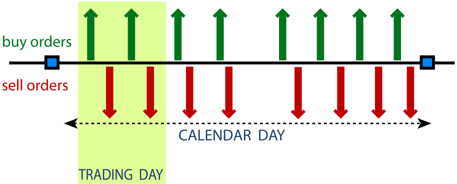
$= 1/4$

**Market Impact of a Bet ($1/4 \ V^*$)**

$= 200$ bps $/ \ 4^{1/2} = 100$ bps

**Stock with Volume $V = \mathbf{8} \cdot V^*$**

$(\gamma = \gamma^* \cdot 4, \quad \tilde{Q} = \tilde{Q}^* \cdot 2)$




**Avg. Order Size $\tilde{Q}$ as fraction of $V$**

$= 1/16 = 1/4 \cdot \mathbf{8}^{-2/3}$

**Market Impact of a Bet ($1/16 \ V$)**

$= 200$ bps $/ \ (4 \cdot \mathbf{8}^{2/3})^{1/2} = 50$ bps

$= 100$ bps $\cdot \mathbf{8}^{-1/3}$

# Invariance Satisfies Theoretical Irrelevance Principles

**1. Modigliani-Miller Irrelevance:** The trading game involving a financial security issued by a firm is **independent** of its **capital structure:**

- Stock Split Irrelevance,
- Leverage Irrelevance.

**2. Time-Clock Irrelevance:** The trading game is **independent** of the **time clock**.

# Meta Model

We outline a steady-state meta-model of trading, from which various invariance relationships are derived results.

- **Informed traders** face given costs of acquiring information of given precision, then place informed bets which incorporate a given fraction of the information into prices.
- **Noise traders** place bets which turn over a constant fraction of the stocks float,mimicking the size distribution of bets placed by informed trades.
- **Market makers** offer a residual demand curve of constant slope, lose money from being "run over" by informed bets, but make up the losses from bid ask spreads, temporary impact, or other trading costs imposed on informed and noise traders.

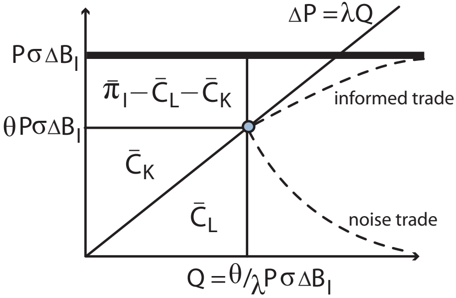
# Meta Model - Outline

- The unobserved "fundamental value" of the asset follows an exponential martingale: $V(t) := \exp[\sigma \cdot B(t) - \sigma^2 t/2]$;
- The market's conditional estimate of $B(t)$ is distributed approximately $N[\bar{B}(t), \Sigma(t)]$.
- Informed traders ($\gamma_I$) get signals $\tilde{i}_n = \tau^{1/2} \cdot [B - \bar{B}] + \tilde{Z}_{I,n}$ and submit $\tilde{Q} = \theta/\lambda \cdot P \cdot \sigma \cdot \Delta B_I$, where $\Delta B_I$ is the update of his estimate of $B(t)$.
- Noise traders ($\gamma_U$) turn over a constant percentage of market cap and mimic the size distribution of informed bets $\tilde{Q}$.

# Meta Model - Outline

- **"Market efficiency"**: The permanent price impact of anonymous trades by informed and noise traders reveals on average the information in the order flow.
- **"Break-even condition" for market makers**: losses on trading with informed traders are equal to total gains on trading with noise traders, $\gamma_I \cdot (\bar{\pi}_I - \bar{C}_B) = \gamma_U \cdot \bar{C}_B$.
- **"Break-even condition" for informed**: Profits of informed are equal to the cost of acquiring private information $c_i$ and trading costs $C_B$, $\bar{\pi}_I = \bar{C}_B + c_i$.

# Meta Model - Intuition




There is price continuation after an informed trade and mean reversion after a noise trade. The losses on trading with informed traders are equal to total gains on trading with noise traders, $\gamma_I \cdot (\bar{\pi}_I - \bar{C}_B) = \gamma_U \cdot \bar{C}_B$.

## Meta Model - Results

The meta-model generates invariance relationships:

$$\gamma = \left(\frac{\lambda \cdot V}{\sigma P}\right)^2 = \left(\frac{E\{|\tilde{Q}|\}}{V}\right)^{-1} = \left(\frac{\sigma}{L}\right)^2 = \frac{1}{\Sigma^2 \cdot \theta^2 \cdot \tau} = \left(\frac{W}{m \cdot \bar{C}_B}\right)^{2/3}.$$

$$\tilde{I} := \frac{P \cdot \tilde{Q} \cdot \sigma}{\gamma^{1/2}} = \frac{\tilde{Q}}{V} \cdot W^{2/3} \cdot (m \cdot \bar{C}_B)^{1/3} = \bar{C}_B \cdot \tilde{i} = \bar{\pi}_B \cdot \tilde{i}.$$

The meta-model reveals that microstructure invariance is ultimately related to **granularity of information flow**.

# Invariance and Previous Literature

Microstructure invariance does not undermine or contradict other theoretical models of market microstructure. It builds a bridge from theoretical models to empirical tests of those models.

- **Theoretical models** usually suggest that order flow imbalances move prices, but do not provide a unified framework for mapping the theoretical concept of an order flow imbalance into empirically observed variables.
- **Empirical tests** often use “wrong” proxies for unobserved order imbalances such as volume or square root of volume.

Microstructure invariance is **a modeling principle** making it possible to test theoretical models empirically.

## Example: Invariance and Kyle (1985)

Kyle (1985) and other models imply a linear price impact formula

$$\lambda = \frac{\sigma_V}{\sigma_U}$$

where $\sigma_V$ is the standard deviation of dollar price change per share resulting from price impact, and $\sigma_U$ is the standard deviation of "order imbalances".

- Market depth invariance identifies $\sigma_V$: $\sigma_V = \psi \cdot \sigma \cdot P$
- Microstructure invariance identifies $\sigma_U$: $\sigma_U = \left(\gamma \cdot E\{\tilde{Q}^2\}\right)^{1/2} \sim W^{2/3}/(P\sigma)$.

# Testing - Portfolio Transition Data

The empirical implications of the three proposed models are tested using a proprietary dataset of **portfolio transitions**.

- Portfolio transition occurs when an old (legacy) portfolio is replaced with a new (target) portfolio during replacement of fund management or changes in asset allocation.
- Our data includes 2,550+ portfolio transitions executed by a large vendor of portfolio transition services over the period from 2001 to 2005.
- Dataset reports executions of 400,000+ orders with average size of about 4% of ADV.

# Portfolio Transitions and Trades

We use the data on **transition orders** to examine which model makes the most reasonable assumptions about how the **size of trades** varies with **trading activity**.

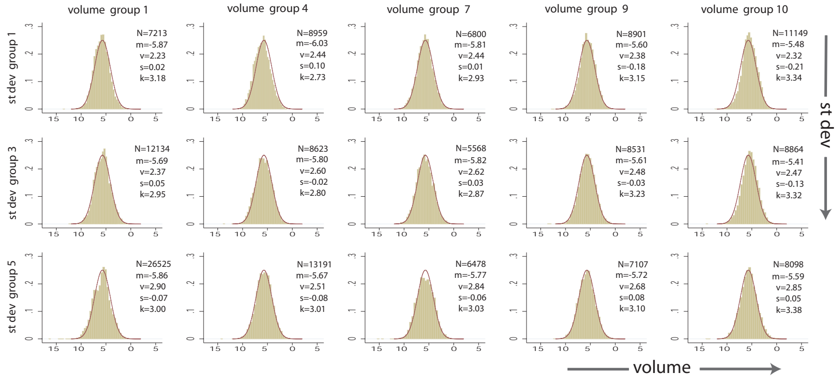
## Distribution of Order Sizes

Microstructure invariance predicts that **distributions** of order sizes $X$, adjusted for differences in trading activity $W$, are the same across different stocks:

$$\ln\left(\frac{|\tilde{Q}|}{V} \cdot \left[\frac{W}{W^*}\right]^{-2/3}\right).$$

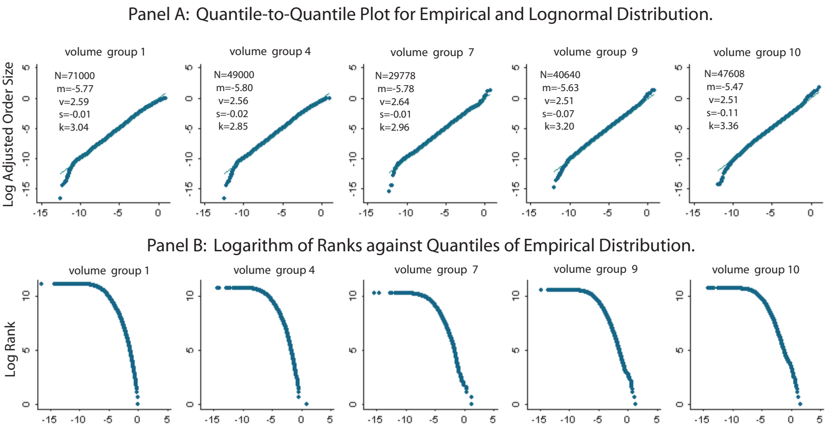
We compare distributions across **10 volume/5 volatility groups**.

# Distributions of Order Sizes




Microstructure invariance works well for **entire distributions** of order sizes. These distributions are approximately **log-normal** with log-variance of 2.53.

# Log-Normality of Order Size Distributions




Microstructure invariance works well for **entire distributions** of order sizes. These distributions are approximately **log-normal**.

# Tests for Orders Size - Design

In regression equation that relates **trading activity** $W$ and the trade size $\tilde{Q}$, proxied by **a transition order of** $X$ **shares**, as a fraction of average daily volume $V$:

$$\ln\left[\frac{X_i}{V_i}\right] = ln[\bar{q}] + a_0 \cdot \ln\left[\frac{W_i}{W_*}\right] + \tilde{\epsilon}$$

**Microstructure Invariance predicts** $a_0 = -2/3$.

The variables are scaled so that $\bar{q}$ is (assuming log-normal distribution) the median size of liquidity trade as a fraction of daily volume for **a benchmark stock** with daily standard deviation of 2%, price of \$40 per share, trading volume of 1 million shares per day, ($W_* = 0.02 \cdot 40 \cdot 10^6$).

# Tests for Order Size: Results

| | | NYSE | | NASDAQ | |
|---|---|---|---|---|---|
| | **All** | Buy | Sell | Buy | Sell |
| $\ln\left[\bar{q}\right]$ | **-5.67** | -5.68 | -5.63 | -5.75 | -5.65 |
| | (0.017) | (0.023) | (0.018) | (0.035) | (0.032) |
| $\alpha_0$ | **-0.62** | **-0.63** | **-0.59** | **-0.71** | **-0.59** |
| | (0.009) | (0.011) | (0.008) | (0.019) | (0.015) |

- **Microstructure Invariance:** $a_0 = -2/3$.

# Why Coefficients for Sells Different from Buys

- Since asset managers are "long only," buys are related to current value of $W$, while sells are related to value of $W$ when stocks were bought.
- Since increases in $W$ result from positive returns, higher values of $W$ are correlated with higher past returns.
- Implies sell coefficients smaller in absolute value than buy coefficients, consistent with empirical results.
- Adding lagged returns or lagged trading activity $W$ may improve results.

# Percentiles Tests for Order Size: Results

| | p1 | p5 | p25 | p50 | p75 | p95 | p99 |
|---|---|---|---|---|---|---|---|
| $\ln\left[\bar{q}\right]$ | -9.37 | -8.31 | -6.73 | -5.66 | -4.59 | -3.05 | -2.05 |
| | (0.008) | (0.006) | (0.004) | (0.003) | (0.004) | (0.006) | (0.009) |
| $\alpha_0$ | -0.65 | -0.64 | -0.61 | -0.62 | -0.61 | -0.64 | -0.63 |
| | (0.005) | (0.003) | (0.002) | (0.002) | (0.002) | (0.003) | (0.005) |

- **Microstructure Invariance:** $a_0 = -2/3$.

# Tests for Orders Size - $R^2$

| | All | NYSE Buy | NYSE Sell | NASDAQ Buy | NASDAQ Sell |
|---|---|---|---|---|---|
| | **Unrestricted Specification:** $\alpha_0 = -2/3$ | | | | |
| $R^2$ | 0.3229 | 0.2668 | 0.2739 | 0.4318 | 0.3616 |
| | **Restricted Specification:** $b_1 = b_2 = b_3 = b_4 = 0$ | | | | |
| $R^2$ | 0.3167 | 0.2587 | 0.2646 | 0.4298 | 0.3542 |
| | **Microstructure Invariance:** $\alpha_0 = -2/3$, $b_1 = b_2 = b_3 = b_4 = 0$ | | | | |
| $R^2$ | 0.3149 | 0.2578 | 0.2599 | 0.4278 | 0.3479 |

$$\ln\left[\frac{X_i}{V_i}\right] = \ln\left[\bar{q}\right] - \alpha_0 \cdot \ln\left[\frac{W_i}{W*}\right] + b_1 \cdot \ln\left[\frac{\sigma_i}{0.02}\right] + b_2 \cdot \ln\left[\frac{P_{0,i}}{40}\right] + b_3 \cdot \ln\left[\frac{V_i}{10^6}\right] + b_4 \cdot \ln\left[\frac{\nu_i}{1/12}\right] + \tilde{\epsilon}.$$

# Tests for Orders Size - Summary

**Microstructure Invariance predicts:** An increase of one percent in trading activity $W$ leads to a decrease of 2/3 of one percent in bet size as a fraction of daily volume (for constant returns volatility).

**Results:** The estimates provide strong support for microstructure invariance. The coefficient predicted to be -2/3 is estimated to be -0.62.

**Discussion:**

- The assumptions made in our model match the data economically.
- F-test rejects our model statistically because of small standard errors.
- Invariance explains data for buys better than data for sells.
- Estimating coefficients on $P$, $V$, $\sigma$, $\nu$ improves $R^2$ very little compared with imposing coefficient value of $-2/3$.

# Portfolio Transitions and Trading Costs

We use data on the **implementation shortfall** of portfolio transition trades to test predictions of the three proposed models concerning how **transaction costs,** both market impact and bid-ask spread, vary with **trading activity.**

# Portfolio Transitions and Trading Costs

**"Implementation shortfall"** is the difference between actual trading prices (average execution prices) and hypothetical prices resulting from "paper trading" (price at previous close).

There are **several problems** usually associated with using implementation shortfall to estimate transactions costs. Portfolio transition orders avoid most of these problems.

# Problem I with Implementation Shortfall

Implementation shortfall is a **biased estimate** of transaction costs when it is based on price changes and executed quantities, because these quantities themselves are often correlated with price changes in a manner which biases transactions costs estimates.

**Example A:** Orders are often canceled when price runs away. Since these non-executed, high-cost orders are left out of the sample, we would underestimate transaction costs.

**Example B:** When a trader places an order to buy stock, he has in mind placing another order to buy more stock a short time later.

For **portfolio transitions**, this problem does not occur: Orders are not canceled. The timing of transitions is somewhat exogenous.

# Problems II with Implementation Shortfall

The second problem is **statistical power**.

**Example:** Suppose that 1% ADV has a transactions cost of 20 bps, but the stock has a volatility of 200 bps. Order adds only 1% to the variance of returns. A properly specified regression will have an R squared of 1% only!

For **portfolio transitions**, this problem does not occur: Large and numerous orders improve statistical precision.

# Tests For Transaction Costs - Design

In the regression specification that relates **trading activity** $W$ and **implementation shortfall C** for a transition order for $X$ shares:

$$\mathbb{I}_{BS,i} \cdot C(X_i) \cdot \frac{(0.02)}{\sigma_i} = a \cdot R_{mkt} \cdot \frac{(0.02)}{\sigma_i} + \mathbb{I}_{BS,i} \cdot \left[\frac{W_i}{W^*}\right]^{\alpha} \cdot C^*(I_i) + \epsilon_i.$$

**Microstructure invariance predicts that $\alpha = -1/3$ and function $C^*(I)$ does not vary across stocks and time.** Function $C^*(I) = L^* \cdot f(I)$ quantifies the trading costs for a benchmark stock.

- Implementation shortfall is adjusted for market changes.
- Implementation shortfall is adjusted for differences in volatility.

# Percentiles Tests for Quoted Spread: Results

| | | NYSE | | NASDAQ | |
|---|---|---|---|---|---|
| | **All** | Buy | Sell | Buy | Sell |
| $\ln\left[k^*/(40\cdot 0.02)\right]$ | **-3.07** | -3.09 | -3.08 | -3.04 | -3.04 |
| | (0.008) | (0.008) | (0.008) | (0.013) | (0.012) |
| $\alpha_1$ | **-0.35** | **-0.31** | **-0.32** | **-0.40** | **-0.39** |
| | (0.003) | (0.003) | (0.003) | (0.004) | (0.004) |

- **Microstructure Invariance:** $a_1 = -1/3$.

$$\ln\left[\frac{\kappa_i}{P_{0,i}\sigma_i}\right] = \ln\left[\frac{k^*}{40\cdot 0.02}\right] + \alpha_1 \cdot \ln\left[\frac{W_i}{W^*}\right] + \tilde{\epsilon}.$$

## Results Related to Quoted Spread

Regression of log of spread on log of trading activity $W$:

- Predicted coefficient is $-1/3$.
- Estimated coefficient is $-0.35$, being different for NYSE ($-0.31$)and for NASDAQ ($-0.40$).

Using quoted spread rather than implicit realized spread cost in transactions cost regression, we get estimated coefficient of 0.71, with puzzling variation across buys (0.61) and sells (0.75).

# Tests For Market Impact and Spread: Results

| | All | NYSE Buy | NYSE Sell | NASDAQ Buy | NASDAQ Sell |
|---|---|---|---|---|---|
| $a$ | 0.66 | 0.63 | 0.62 | 0.76 | 0.78 |
| | (0.013) | (0.016) | (0.016) | (0.037) | (0.036) |
| $^1/_2\bar{\lambda}^* \times 10^4$ | **10.69** | 12.08 | 9.56 | 12.33 | 9.34 |
| | (1.376) | (2.693) | (2.254) | (2.356) | (2.686) |
| $z$ | **0.57** | 0.54 | 0.56 | 0.44 | 0.63 |
| | (0.039) | (0.056) | (0.062) | (0.051) | (0.086) |
| $\alpha_2$ | **-0.32** | **-0.40** | **-0.33** | **-0.41** | **-0.29** |
| | (0.015) | (0.037) | (0.029) | (0.035) | (0.037) |
| $^1/_2\bar{\kappa}^* \times 10^4$ | **1.77** | -0.27 | 1.14 | 0.77 | 3.55 |
| | (0.837) | (2.422) | (1.245) | (4.442) | (1.415) |
| $\alpha_3$ | **-0.49** | **-0.37** | **-0.50** | **0.53** | **-0.44** |
| | (0.050) | (1.471) | (0.114) | (1.926) | (0.045) |

- **Microstructure Invariance:** $\alpha_2 = 1/3, \alpha_3 = -1/3$.

$$\mathbb{I}_{BS,i} \cdot C(X_i) \cdot \frac{(0.02)}{\sigma_i} = a \cdot R_{mkt} \cdot \frac{(0.02)}{\sigma_i} + \frac{\bar{\lambda}^*}{2}\mathbb{I}_{BS,i} \cdot \left[\frac{\phi I_i}{0.01}\right]^z \cdot \left[\frac{W_i}{W^*}\right]^{\alpha_2} + \frac{\bar{\kappa}^*}{2}\mathbb{I}_{BS,i} \cdot \left[\frac{W_i}{W^*}\right]^{\alpha_3} + \tilde{\epsilon}.$$

## Discussion

- Estimated coefficient $a = 0.66$ suggests that most orders are executed within one day.
- In a non-linear specification, $\alpha_3$ is often different from predicted -1/3, but spread cost $\bar{\kappa}$ is insignificant.
- Scaled cost functions are non-linear with the estimated exponent $z = 0.57$.
- Buys have higher price impact $\bar{\lambda}^*$ than sells, since buys may be more informative whereas price reversals after sells makes their execution cheaper.

# Tests for Transaction Costs - $R^2$

| | All | NYSE | | NASDAQ | |
|---|---|---|---|---|---|
| | | Buy | Sell | Buy | Sell |
| | **Unrestricted Specification, 12 Degrees of Freedom:** $\alpha_2 = \alpha_3 = -1/3$ | | | | |
| $R^2$ | 0.1016 | 0.1121 | 0.1032 | 0.0957 | 0.0944 |
| | **Restricted Specification:** $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_7 = \beta_8 = 0$ | | | | |
| $R^2$ | 0.1010 | 0.1118 | 0.1029 | 0.0945 | 0.0919 |
| | **Microstructure Invariance, SQRT Model:** $z = 1/2, \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_7 = \beta_8 = 0, \alpha_2 = \alpha_3 = -1/3$ | | | | |
| $R^2$ | 0.1007 | 0.1116 | 0.1027 | 0.0941 | 0.0911 |
| | **Microstructure Invariance, Linear Model:** $z = 1, \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_7 = \beta_8 = 0, \alpha_2 = \alpha_3 = -1/3$ | | | | |
| $R^2$ | 0.0991 | 0.1102 | 0.1012 | 0.0926 | 0.0897 |

$$\mathbb{I}_{BS,i} \cdot C(X_i) \cdot \frac{(0.02)}{\sigma_i} = a \cdot R_{mkt} \cdot \frac{(0.02)}{\sigma_i} + \frac{\bar{\lambda}^*}{2} \mathbb{I}_{BS,i} \cdot \left[\frac{\phi I_i}{0.01}\right]^z \cdot \left[\frac{W_i}{W*}\right]^{\alpha_2} \cdot \frac{\sigma_i^{\beta_1} \cdot P_{0,i}^{\beta_2} \cdot V_i^{\beta_3} \cdot \nu_i^{\beta_4}}{(0.02)(40)(10^6)(1/12)}$$

$$+ \frac{\bar{\kappa}^*}{2} \mathbb{I}_{BS,i} \cdot \left[\frac{W_i}{W*}\right]^{\alpha_3} \cdot \frac{\sigma_i^{\beta_5} \cdot P_{0,i}^{\beta_6} \cdot V_i^{\beta_7} \cdot \nu_i^{\beta_8}}{(0.02)(40)(10^6)(1/12)} + \tilde{\epsilon}.$$

# Tests for Trading Costs - Summary

**Microstructure Invariance predicts:** An increase of one percent in trading activity $W$ leads to a decrease of $1/3$ of one percent in transaction costs (for constant returns volatility).

**Results:** The estimates provide strong support for microstructure invariance. The coefficient predicted to be -1/3 is estimated to be -0.32.

**Discussion:**

- Invariance matches the data economically.
- F-test rejects invariance statistically because of small standard errors.
- Price impact cost is better described by a non-linear function with exponent of 0.57.
- Estimating coefficients on $P$, $V$, $\sigma$, $\nu$ improves $R^2$ very little comparing with imposing coefficient of $-1/3$, especially comparing to a square root model.

# Transactions Costs Across Volume Groups

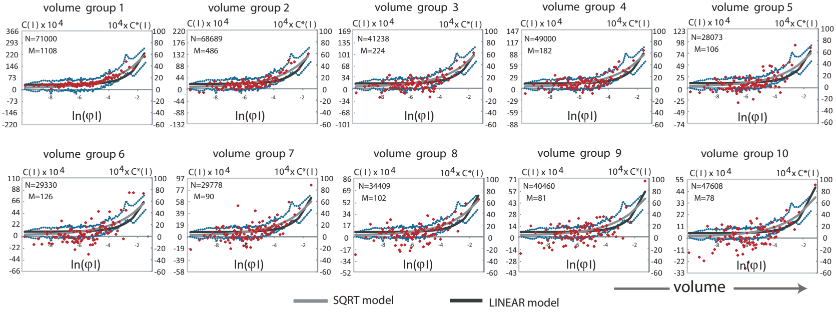
For each of **10 volume groups/100 order size groups**, we estimate dummy coefficients from regression:

$$\mathbb{I}_{BS,i} \cdot C(X_i) \cdot \frac{(0.02)}{\sigma_i} = a \cdot R_{mkt} \cdot \frac{(0.02)}{\sigma_i} + \mathbb{I}_{BS,i} \cdot \left[\frac{W_i}{W^*}\right]^{-1/3} \cdot \sum_{j=1}^{100} \mathbb{I}_{i,j,k} \cdot c^*_{k,j}.$$

- Indicator variable $\mathbb{I}_{i,j,k}$ is one if $i$th order is in the $k$th volume groups and $j$th size group.
- The dummy variables $c^*_{k,j}, j = 1, ..100$ track the shape of scaled transaction costs function $C^*(I)$ for $k$th volume group.

If invariance holds, then all estimated functions should be the same across volume groups.

# Transactions Costs Across Volume Groups




For each of 10 volume groups, 100 estimated dummy variables $c^*_{k,j}, j = 1, ..100$ track scaled cost functions $C^*(I)$ for a benchmark stock on the left axis. Actual costs functions $C(I)$ are on the right axis. Group 1 contains stocks with the lowest volume. Group 10 contains stocks with the highest volume. The volume thresholds are 30th, 50th, 60th, 70th, 75th, 80th, 85th, 90th, and 95th percentiles for NYSE stocks.

# Invariance of Cost Functions - Discussion

- Cost functions scaled by $\sigma W^{-1/3}$ with argument $X$ scaled by $W^{2/3}/V$ seem to be stable across volume groups.
- The estimates are more "noisy" in higher volume groups, since transitions are usually implemented over one calendar day, i.e., over longer horizons in business time for larger stocks.
- The square-root specification fits the data slightly better than the linear specification, particularly for large orders in size bins from 90th to 99th.
- The linear specification fits better costs for very large orders in active stocks.

# Calibration: Bet Sizes

Our estimates imply that portfolio transition orders $|\tilde{X}|/V$ are approximately distributed as **a log-normal** with the log-variance of 2.53 and the number of bets per day $\gamma$ is defined as,

$$\ln\gamma = \ln 85 + \frac{2}{3}\ln\left[\frac{W}{(0.02)(40)(10^6)}\right].$$

$$\ln\left[\frac{|\tilde{X}|}{V}\right] \approx -5.71 - \frac{2}{3}\cdot\ln\left[\frac{W}{(0.02)(40)(10^6)}\right] + \sqrt{2.53}\cdot N(0,1)$$

For a benchmark stock, there are 85 bets with the median size of 0.33% of daily volume. Buys and sells are symmetric.

# Calibration: Transactions Cost Formula

Our estimates imply two simple formulas for expected trading costs for any order of $X$ **shares** and for any security. The linear and square-root specifications are:

$$C(X) = \left(\frac{W}{(0.02)(40)(10^6)}\right)^{-1/3} \frac{\sigma}{0.02}\left(\frac{2.50}{10^4}\cdot\frac{X}{0.01\,V}\left[\frac{W}{(0.02)(40)(10^6)}\right]^{2/3} + \frac{8.21}{10^4}\right).$$

$$C(X) = \left(\frac{W}{(0.02)(40)(10^6)}\right)^{-1/3} \frac{\sigma}{0.02}\left(\frac{12.08}{10^4}\cdot\sqrt{\frac{X}{0.01\,V}\left[\frac{W}{(0.02)(40)(10^6)}\right]^{2/3}} + \frac{2.08}{10^4}\right).$$

# More Practical Implications

- **Trading Rate:** If it is reasonable to restrict trading of the benchmark stock to say 1% of average daily volume, then a smaller percentage would be appropriate for more liquid stocks and a larger percentage would be appropriate for less liquid stocks.
- **Components of Trading Costs:** For orders of a given percentage of average daily volume, say 1%, bid-ask spread is a relatively larger component of transactions costs for less active stocks, and market impact is a relatively larger component of costs for more active stocks.
- **Comparison of Execution Quality:** When comparing execution quality across brokers specializing in stocks of different levels of trading activity, performance metrics should take account of nonlinearities documented in our paper.

# Conclusions

- Predictions of microstructure invariance largely hold in portfolio transitions data for equities.
- We conjecture that invariance predictions can be found to hold as well in other datasets and may generalize to other markets and other countries.
- We conjecture that market frictions such as wide tick size and minimum round lot sizes may result in deviations from the invariance predictions. Invariance provides a benchmark for measuring the importance of those frictions.
- Microstructure invariance has numerous implications.

## Calibration: Bet Size and Trading Activity

For **a benchmark stock** with \$40 million daily volume and 2% daily returns standard deviation, empirical results imply:

- Median bet size is \$132,500 or 0.33% of daily volume.
- Average bet size is \$469,500 or 1.17% of daily volume.
- Benchmark stock has about 85 bets per day.
- Order imbalances are 38% of daily volume.
- Half price impact is 2.50 and half spread is 8.21 basis points.
- Expected cost of a bet is about \$2,000.

Invariance allows to extrapolate these estimates to other assets.

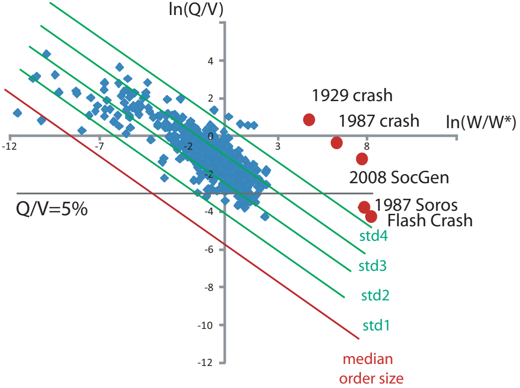
## Calibration: Implications of Log-Normality for Volume and Volatility

Standard deviation of log of bet size is $2.53^{1/2}$ implies:

- a one-standard-deviation increase in bet size is a factor of about 4.90.
- 50% of trading volume generated by largest 5.39% of bets.
- 50% of returns variance generated by largest 0.07% of bets (linear model).

# Implication for Market Crashes

Order of 5% of daily volume is “normal” for a typical stock. Order of 5% of daily volume is “unusually large” for the market.




Conventional intuition that order equal to 5% of average daily volume will not trigger big price changes in indices is wrong!

## Calibration of Market Crashes

| | Actual | Predicted Invariance | Predicted Conventional | %ADV | %GDP |
|---|---|---|---|---|---|
| 1929 Market Crash | **25%** | **44.35%** | 1.36% | 241.52% | 1.136% |
| 1987 Market Crash | **32%** | **16.77%** | 0.63% | 66.84% | 0.280% |
| 1987 Soros's Trades | **22%** | **6.27%** | 0.01% | 2.29% | 0.007% |
| 2008 SocGén Trades | **9.44%** | **10.79%** | 0.43% | 27.70% | 0.401% |
| 2010 Flash Crash | **5.12%** | **0.61%** | 0.03% | 1.49% | 0.030% |

Table shows the actual price changes, predicted price changes, orders as percent of average daily volume and GDP, and implied frequency.

## Discussion

- **Price impact predicted by invariance is large and similar to actual price changes.**
- **The financial system in 1929 was remarkably resilient.** The 1987 portfolio insurance trades were equal to about 0.28% of GDP and triggered price impact of 32% in cash market and 40% in futures market. The 1929 margin-related sales during the last week of October were equal to 1% of GDP. They triggered price impact of 24% only.

## Discussion - Cont'd

- **Speed of liquidation magnifies short-term price effects.** The 1987 Soros trades and the 2010 flash-crash trades were executed rapidly. Their actual price impact was greater than predicted by microstructure invariance, but followed by rapid mean reversion in prices.

- **Market crashes happen too often.** The three large crash events were approximately 6 standard deviation bet events, while the two flash crashes were approximately 4.5 standard deviation bet events. Right tail appears to be fatter than predicted. The true standard deviation of underlying normal variable is not 2.53 but 15% bigger, or far right tail may be better described by a power law.

# Early Warning System

**Early warning systems** may be useful and practical. Invariance can be used as a practical tool to help quantify the systemic risks which result from sudden liquidations of speculative positions.

# "Time Change" Literature

"Time change" is the idea that a larger than usual number of independent price fluctuations results from business time passing faster than calendar time.

- Mandelbrot and Taylor (1967): Stable distributions with kurtosis greater than normal distribution implies infinite variance for price changes.
- Clark (1973): Price changes result from log-normal with time-varying variance, implying finite variance to price changes.
- Econophysics: Gabaix et al. (2006); Farmer, Bouchard, Lillo (2009). Right tail of distribution might look like a power law.
- **Microstructure invariance**: Kurtosis in returns results from rare, very large bets, due to high variance of log-normal. Caveat: Large bets may be executed very slowly, e.g., over weeks.

## Market Temperature

Derman (2002): "Market Temperature" $\chi = \sigma \cdot \gamma^{1/2}$. Standard deviation of order imbalances is $P \cdot \sigma_U = P \cdot [\gamma \cdot E\{\tilde{Q}^2\}]^{1/2}$.

- Product of temperature and order imbalances proportional to trading activity: $P\sigma_U \cdot \chi \propto W$
- Invariance implies temperature $\propto (PV)^{1/3}\sigma^{4/3} = \sigma \cdot W$.
- Invariance implies expected market impact cost of an order $\propto (PV)^{1/3}\sigma^{4/3} = \sigma \cdot W$.

Therefore invariance implies temperature proportional to market impact cost of an order.

## Invariance-Implied Liquidity Measures

- "Velocity":

$$\gamma = \mathrm{const} \cdot W^{2/3} = \mathrm{const} \cdot [P \cdot V \cdot \sigma]^{2/3}$$

- Cost of Converting Asset to Cash (basis points) $= 1/L_{\$}$:

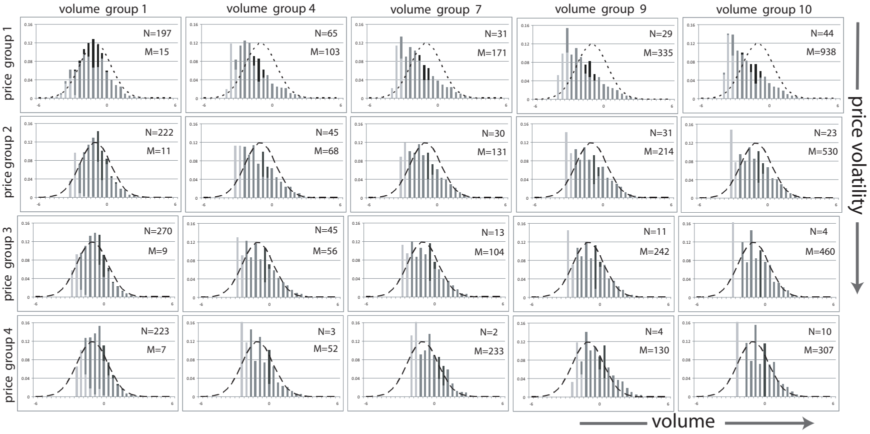
$$L_{\$} = \mathrm{const} \cdot \cdot \left[ \frac{P \cdot V}{\sigma^2} \right]^{1/3}$$

- Cost of Transferring a Risk (Sharpe ratio) $= 1/L_{\sigma}$

$$L_{\sigma} = \mathrm{const} \cdot W^{1/3} = \mathrm{const} \cdot [P \cdot V \cdot \sigma]^{1/3}$$

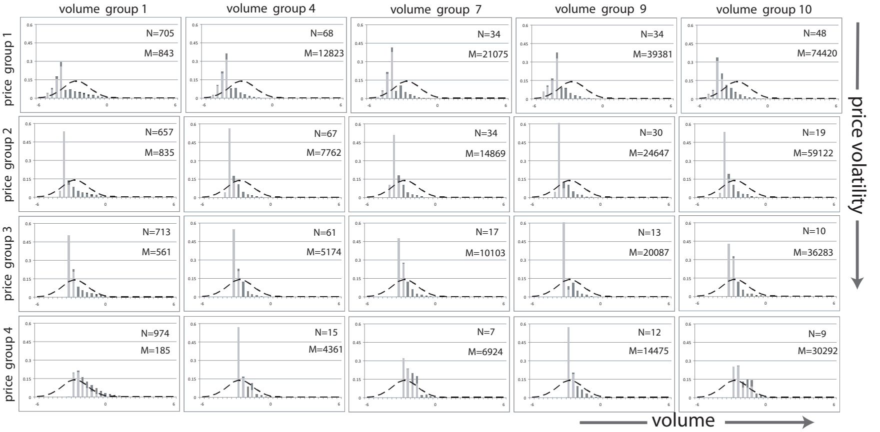
# Evidence From TAQ Dataset Before 2001

Trading game invariance seems to **work** in TAQ **before 2001,** subject to market frictions (Kyle, Obizhaeva and Tuzun (2010)).



# Evidence From TAQ Dataset After 2001

Trading game invariance is **hard to test** in TAQ **after 2001.**



# News Articles and Trading Game Invariance

Data on the number of Reuters news items $N$ is consistent with trading game invariance (Kyle, Obizhaeva, Ranjan, and Tuzun (2010)).