Incorporating Managerial Cash-Flow Estimates and Risk Aversion to Value Real Options Projects

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November 26, 2014

Agenda

- Introduction
    - Motivation
    - Real Options
- Matching cash-flows
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    - Normal distribution (analytical solution)
- Indifference pricing
    - General approach (numerical solution)
    - Normal distribution (analytical solution)
- Results
- Practical implementation
- Conclusions

Motivation

- To develop a theoretically consistent real options approach to value R&D type projects
- **Theoretical Approaches**: Cash-flow determined by GBM

$$df_t = \mu f_t dt + \sigma f_t dW_t$$

- **Practice**: Managerial supplied cash-flow estimates consist of low, medium and high values

	F'0	F'1	F'2	F'3	F'4
Economic Profit (Optimistic)	80	120	150	180	200
Economic Profit (Likely)	50	70	75	80	90
Economic Profit (Pessimistic)	20	25	25	20	20

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Valuation of R&D Projects: Managerial Sales and Cost Estimates

- Managers provide sales and cost estimates

Table : Managerial Supplied Cash-Flow (Millions $).

	3	4	5	6	7	8	9	10
Sales	10.00	30.00	50.00	100.00	100.00	80.00	50.00	30.00
COGS	6.00	18.00	30.00	60.00	60.00	48.00	30.00	18.00
GM	4.00	12.00	20.00	40.00	40.00	32.00	20.00	12.00
SG&A	0.50	1.50	2.50	5.00	5.00	4.00	2.50	1.50
EBITDA	3.50	10.50	17.50	35.00	35.00	28.00	17.50	10.50
CAPEX	1.00	3.00	5.00	10.00	10.00	8.00	5.00	3.00
Cash-Flow	2.50	7.50	12.50	25.00	25.00	20.00	12.50	7.50

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Standard NPV Approach Using CAPM

- Ryan and Ryan (2002) report that 83% of businesses apply the WACC to value discounted cash-flows (DCF)
- CAPM: $\mathbb{E}[r_E] = r_f + \beta_C(\mathbb{E}[r_M] - r_f)$
- Use of CAPM implies beta: $\beta_C = \dfrac{\rho_{M,C}\sigma_C}{\sigma_M}$

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- Use of CAPM implies beta: $\beta_C = \dfrac{\rho_{M,C}\sigma_C}{\sigma_M}$
- Some assumptions regarding β when using WACC
  - Market volatility, σ_M, is known . . . . ✓
  - Cash-flow volatility: $\sigma_{project} = \sigma_C$ . . . . ?
  - Correlation of the cash-flows: $\rho_{project} = \rho_C$ . . . . ?

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  - No managerial flexibility / optionality imbedded in the project
  - Financial risk profile of the value of the cash-flows matches that of the average project of the company

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- Proper beta: $\beta_{project} = \dfrac{\rho_{M,project}\sigma_{project}}{\sigma_M}$
- **Matching method uses managerial supplied cash-flow estimates to determine $\sigma_{project}$**

Real Options

- Why real options?
  - Superior to discounted cash flow (DCF) analysis for capital budgeting / project valuation
  - Accounts for the inherent value of managerial flexibility
  - Adoption rate ∼12% in industry (Block (2007))
- What is required?
  - Consistency with financial theory
  - Intuitively appealing
  - Practical to implement

Introduction: Real Options Approaches

	Intuitive	Practical / Easy to Implement	Financially Consistent	Minimal Subjectivity
Classic Approach	✓	✗	✓	-
Subjective Approach	✓	✓	-	✗
Market Asset Disclaimer	✓	✓	✗	✗
Revised Classic Approach	✓	✗	✓	-
Integrated Approach	✓	✗	✓	✓

* As classified by Borison (2005)

Relevant Literature - Utility Based Models

- Berk et al.[1] developed a real options framework for valuing early stage R&D projects
  - Accounts for: technical uncertainty, cash-flow uncertainty, obsolescence, cost uncertainty
  - Value of the project is a function of a GBM process representing the cash-flows
  - **Main issue: how to fit real managerial cash-flow estimates to a GBM process**
- Miao and Wang[2], and Henderson[3]
  - Present incomplete market real options models that show standard real options, which assume complete markets, can lead to contradictory results

[1] See Berk, Green, and Naik (2004).
[2] See Miao and Wang (2007).
[3] See Henderson (2007).

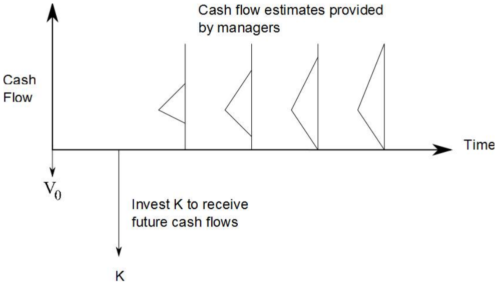
Matching Method Advantages

- The approach utilizes managerial cash-flow estimates
- The approach is theoretically consistent
  - Provides a mechanism to account for systematic versus idiosyncratic risk
  - Provides a mechanism to properly correlate cash-flows from period to period
- The approach requires little subjectivity with respect to parameter estimation
- The approach provides a **missing link** between practical estimation and theoretical frameworks

RO in R&D Applications: Managerial Cash-Flow Estimates

- Managers provide cash flow estimates

	Expected Cash Flows per Year								
Scenario	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Optimistic	0	0	80	120	150	180	200	220	250
Most likely	0	0	50	70	75	80	90	100	110
Pessimistic	0	0	20	25	25	20	20	20	20
Investment		450							



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RO in R&D Type Applications: Two Approaches

- Managers supply low, medium and high sales and cost estimates (numerical solution)
- Managers supply $\pm$ sales and cost estimates from which a standard deviation can be determined for a normal distribution (analytical solution)

RO in R&D Type Applications: Low, Medium and High Sales and Cost Estimates

- Managers supply revenue and GM% estimates

Scenario	End of Year Sales (Margin%)						
	3	4	5	6	7	8	9
Optimistic	80 (50%)	116 (60%)	153 (65%)	177 (60%)	223 (60%)	268 (55%)	314 (55%)
Most Likely	52 (30%)	62 (40%)	74 (40%)	77 (40%)	89 (35%)	104 (35%)	122 (35%)
Pessimistic	20 (20%)	23 (20%)	24 (20%)	18 (20%)	20 (15%)	20 (10%)	22 (10%)
SG&A*	10%	5%	5%	5%	5%	5%	5%
Fixed Costs	30	25	20	20	20	20	20

* Sales / General and Administrative Costs

RO in R&D Type Applications: ± Sales and Cost Estimates

	End of Year Sales (Margin)			
	3	4	5	6
Sales	52 ± 10	62 ± 12	74 ± 15	77 ± 15
COGS	(31 ± 6)	(37 ± 7)	(44 ± 9)	(46 ± 10)
SG&A	10%	5%	5%	5%
CAPEX	(30 ± 6)	(25 ± 5)	(20 ± 4)	(20 ± 14)

	3	4	5	6
σ_S (Sales)	5.20	6.20	7.40	7.70
σ_C (COGS)	3.12	3.72	4.44	4.62
σ_{EX} (CAPEX)	3.00	2.50	2.00	2.00
σ_{CF} (Cash-Flow)	4.61	4.94	5.55	5.75

$$\sigma_{CF} = \sqrt{\sigma_S^2 + \sigma_C^2 + \sigma_{EX}^2 - 2\rho_{S,C}\sigma_S\sigma_C - 2\rho_{S,EX}\sigma_S\sigma_{EX} + 2\rho_{S,C}\rho_{S,EX}\sigma_C\sigma_{EX}}$$

Real Options in R&D Type Applications

- Problem:
  - How should we value the cash flows?
  - How should we account for managerial risk aversion?
- Approach:
  - Apply "matching method" with MMM to value cash flows
  - Apply indifference pricing to determine the value with manager's risk aversion
- Why Account for Risk Aversion:
  - MMM assumes investors are fully diversified
  - Impact of managerial risk aversion on the valuation of a real options project can enhance decision making

Market Stochastic Driver

- Traded index / asset

$$dI_t = \mu I_t dt + \sigma I_t dW_t$$

- Assume there exists a *Market Stochastic Driver / Indicator* correlated to the traded index

$$dS_t = \nu S_t dt + \eta S_t(\rho dW_t + \sqrt{1-\rho^2} dW_t^{\perp})$$

- Market stochastic driver
  - does not need to be traded
  - could represent market size / revenues
  - is not constrained to a GBM process
- Risk-neutral MMM

$$dI_t = r I_t dt + \sigma I_t d\widetilde{W}_t$$

$$dS_t = \widehat{\nu} S_t dt + \rho\eta S_t \left(d\widetilde{W}_t + \sqrt{1-\rho^2} dW_t^{\perp} \right)$$

$$\widehat{\nu} = \nu - \frac{\rho\eta}{\sigma}(\mu - r)$$

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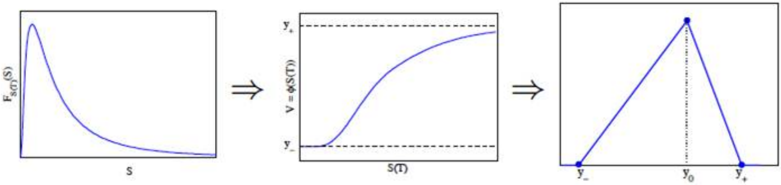
Match Cash Flow Payoff



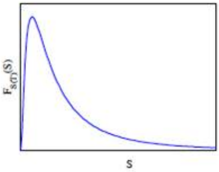
Match Cash Flow Payoff

- Each cash flow is effectively an option on the *market stochastic driver*, $V_T = \varphi(S_T)$, and so, we match probabilities

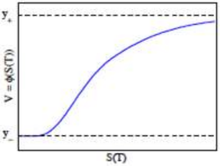
$$
\begin{aligned}
\mathbb{P}(\varphi(S_T) < v) &= F^*(v) \\
\mathbb{P}(S_T < \varphi^{-1}(v)) &= F^*(\varphi(S)) \\
\mathbb{P}(S_0 e^{(\nu - \frac{\eta^2}{2})T + \eta\sqrt{T}Z} < S) &= F^*(\varphi(S)), \;\; Z \underset{\mathbb{P}}{\sim} N(0,1) \\
\mathbb{P}\left(Z < \frac{\ln\frac{S}{S_0} - (\nu - \frac{\eta^2}{2})T}{\eta\sqrt{T}} \right) &= F^*(\varphi(S)) \\
\Phi\left(\frac{\ln\frac{S}{S_0} - (\nu - \frac{\eta^2}{2})T}{\eta\sqrt{T}} \right) &= F^*(\varphi(S))
\end{aligned}
$$

Match Cash Flow Payoff

$$\varphi(S) = F^{*-1}\left\{\Phi\left(\frac{\ln\frac{S}{S_0} - (\nu - \frac{\eta^2}{2})T}{\eta\sqrt{T}}\right)\right\}$$



$\Rightarrow$



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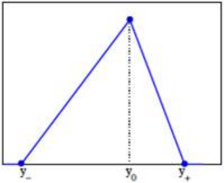
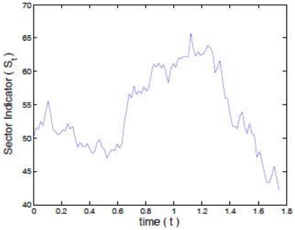
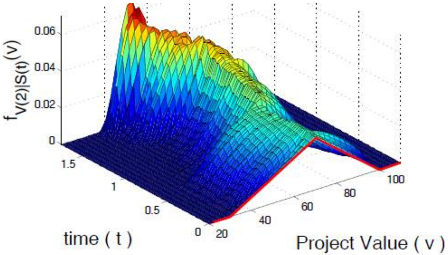


Information Distortion



(a) Market Indicator Flow



(b) Distorted Distribution Flow

Risk-Neutral Measure

Theorem

The GBM Risk-Neutral Distribution. *The conditional distribution function $\widehat{F}_{v_k|S_t}(v)$ of v_k conditional on S_t at t, for $0 < t < T_k$, under the measure $\mathbb{Q}$ is given by*

$$\widehat{F}_{v_k|S_t}(v) = \Phi\left(\sqrt{\tfrac{T_k}{T_k - t}}\Phi^{-1}\left(F_k^*(v)\right) - \widehat{\lambda}_k(t, S_t)\right)$$

where the pseudo-market-price-of-risk

$$\widehat{\lambda}_k(t, S) = \frac{1}{\eta\sqrt{T_k - t}} \ln \frac{S}{S_0} + \frac{\widehat{\nu} - \frac{1}{2}\eta^2}{\eta}\sqrt{T_k - t} - \frac{\nu - \frac{1}{2}\eta^2}{\eta}\frac{T_k}{\sqrt{T_k - t}}.$$

- Note that as $t \downarrow 0$ and $S \downarrow S_0$ then $\widehat{\lambda}_k(t, S) \downarrow -\rho\frac{\mu - r}{\sigma}\sqrt{T_k}$, i.e. **the valuation is independent of ν and η.**

Option Pricing

- Value of the cash flows

$$\begin{aligned} V_t &= \sum_{i=1}^{n} e^{-r(t_i-t)}\mathbb{E}^Q\left[V_{t_i} \middle| \mathcal{F}_t \right] \\ &= \sum_{i=1}^{n} e^{-r(t_i-t)}\mathbb{E}^Q\left[\varphi_i(S_{t_i}) \middle| \mathcal{F}_t \right] \end{aligned}$$

- Value of the project with option

$$\begin{aligned} V &= e^{-rt}\mathbb{E}^Q\left[\max\left(V_t - K, 0\right)\right] \\ &= e^{-rt_K}\int_{-\infty}^{\infty}\left(\sum_{i=1}^{n}\left(e^{-r(t_i-t_K)}\int_{-\infty}^{\infty}\varphi_i\left(S_{t_i}\right)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}dy\right) - K\right)_+ \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \end{aligned}$$

$$S_{t_i} = S_0 e^{(\bar{\nu}-\frac{1}{2}\eta^2)t_i+\eta(\sqrt{t_K}x+\sqrt{t_i-t_K}y)}$$

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Matching Cash-Flows for Normally Distributed Estimates

- Assume that the managers have provided cash-flow estimates of the form $N(\mu_k, \sigma_k^2)$
- Assume the *Market Stochastic Driver* to be a Brownian motion
- Assume that there exists a cash-flow process: F_t
- Introduce a collection of functions $\varphi_k(S_t)$ such that at each T_k, $F_{T_k} = \varphi_k(S_{T_k})$

Theorem

The Replicating Cash-Flow Payoff. *The cash-flow payoff function* $\varphi_k(s)$ *which produces the managerial specified distribution* $\Phi\left(\frac{s-\mu_k}{\sigma_k}\right)$ *for the cash-flows at time* T_k*, when the underlying driving uncertainty* S_t *is a BM, and* $S_0 = 0$*, is given by*

$$\varphi_k(s) = \frac{\sigma_k}{\sqrt{T_k}} s + \mu_k^F.$$

Value of the Cash-Flows for Normally Distributed Estimates

Theorem

Value of the Cash-Flows. *For a given set of cash-flow estimates, normally distributed with mean μ_k and standard deviation σ_k, given at times T_k, where $k = 1, 2, ..., n$, the value of these cash-flows at time $t < T_1$ is given by*

$$V_t(S_t) = \sum_{k=1}^{n} e^{-r(T_k - t)} \left(\frac{\sigma_k}{\sqrt{T_k}} (S_t + \widehat{\nu}(T_k - t)) + \mu_k \right),$$

and for the case where $t = 0$,

$$V_0 = \sum_{k=1}^{n} e^{-rT_k} \left(\widehat{\nu} \sigma_k \sqrt{T_k} + \mu_k \right).$$

Option Pricing for Normally Distributed Estimates

Theorem

Real Option Value of Risky Cash-Flows Estimates. *For a given set of cash-flow estimates, normally distributed with mean μ_k and standard deviation σ_k, given at times T_k, where $k = 1, 2, ..., n$, the value of the option at time $t < T_0$ to invest the amount K at time $T_0 < T_k$ to receive these cash flows is given by*

$$RO_t(S_t) = e^{-r(T_0-t)} \left[(\xi_1(S_t) - K)\, \Phi\left(\frac{\xi_1(S_t) - K}{\xi_2}\right) + \xi_2\, \phi\left(\frac{\xi_1(S_t) - K}{\xi_2}\right) \right]$$

where $\Phi(\bullet)$ and $\phi(\bullet)$ are the standard normal distribution and density functions, respectively, and

$$\xi_1(S_t) = \sum_{k=1}^{n} e^{-r(T_k-T_0)} \left(\frac{\sigma_k}{\sqrt{T_k}} (S_t + \widehat{\nu}(T_k - t)) + \mu_k \right),$$

$$\xi_2 = \sqrt{T_0 - t} \sum_{k=1}^{n} e^{-r(T_k-T_0)} \frac{\sigma_k}{\sqrt{T_k}}.$$

Utility Maximization

- Assume exponential utility

$$u(x) = -\frac{e^{-\gamma x}}{\gamma}$$

- $\gamma \geq 0$ represents managerial risk aversion
- Manager has two options: 1) invest in the market, or 2) invest in the real option
- Goal is to maximize the terminal utility in each of the two options and determine the indifference price

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Optimal Investment in the Traded Index (Merton Model)

- Invest in market only, with π_t invested in the risky asset

$$dX_t = (rX_t + \pi_t(\mu - r))dt + \pi_t \sigma dW_t$$

- And maximize expected terminal utility

$$V(t,x) = \sup_{\pi_t} \mathbb{E}\left[u(X_T) \middle| X_t = x \right]$$

- Applying standard arguments leads to the PDE

$$\partial_t V - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{(\partial_x V)^2}{\partial_{xx} V} + rx\partial_x V = 0$$

- with $V(T,x) = u(x)$, and the solution is given by

$$V(t,x) = -\frac{1}{\gamma} e^{-\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 (T-t) - \gamma e^{r(T-t)} x}.$$

Optimal Investment in the Real Option Project

- Wealth dynamics are given as

$$\left.\begin{array}{rcl} dX_t &=& (rX_t + \pi_t(\mu - r))\,dt + \pi_t \sigma dW_t,\;\; t \notin [T_0, T_1, ..., T_n] \\ X_{T_0} &=& X_{T_0^-} - K\mathbf{1}_{\mathcal{A}} \\ X_{T_j} &=& X_{T_j^-} + \varphi(S_j)\mathbf{1}_{\mathcal{A}},\;\; j \in [1, 2, ..., n] \end{array}\right\}$$

  where $\mathbf{1}_{\mathcal{A}}$ represents the indicator function equal to 1 if the real option is exercised
- The manager seeks to maximize his expected terminal utility as

$$U(t, x, s) = \sup_{\pi_t} \mathbb{E}\left[u(X_T) \middle| X_t = x, S_t = s \right]$$

- Applying standard arguments, it can be shown that the solution to $U(t, x, s)$ can be achieved by solving the following PDE

$$\partial_t U + rx\partial_x U + \nu s \partial_s U + \tfrac{1}{2}\partial_{ss} U \eta^2 s^2 - \frac{1}{2}\frac{((\mu - r)\partial_x U + \rho\sigma\eta s \partial_{sx} U)^2}{\sigma^2 \partial_{xx} U} = 0$$

Optimal Investment in the Real Option Project (con't)

- Boundary conditions

$$U(T_j, x, s) = U(T_j^+, x, s)e^{-\gamma\varphi(s)}, \text{ for } j = 1, \ldots, n-1$$
$$U(T_n, x, s) = u(x + \varphi_n(s))$$

- Using the substitution $U(t, x, s) = V(t, x)(H(t, s))^{\frac{1}{1-\rho^2}}$ results in the simplified PDE

$$\partial_t H + \hat{\nu} s \partial_s H + \tfrac{1}{2}\eta^2 s^2 \partial_{ss} H = 0$$

with $H(T_n, s) = e^{-\gamma(1-\rho^2)\varphi_n(S_{T_n})}$, and $t \in (T_{n-1}, T_n]$

- Apply dynamic programming, where at each $t = T_j$, $j = \{1, 2, \ldots, n-1\}$, set $H(T_j, s) = H(T_j^+, s)e^{-\gamma(1-\rho^2)\varphi_j(S_{T_j})}$

The Indifference Price

- At $t = T_0$, we should invest in the real option if

$$(H(T_0^+, s))^{\frac{1}{1-\rho^2}} e^{\gamma K e^{r(T_n - T_0^+)}} \leq 1$$

- Defining f as the *indifference price*, i.e. the value of the real option, and setting $U(t, x - f, s) = V(t, x)$ leads to

$$f(t, s) = -\frac{1}{\gamma(1-\rho^2)} \ln H(t, s) e^{-r(T_n - t)}$$

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The Indifference Price for Normally Distributed Estimates

Theorem

Real Option Value of Risky Cash-Flows Accounting for Risk Aversion. *For a given set of cash-flow estimates, normally distributed with mean* μ_k *and standard deviation* σ_k*, given at times* T_k*, where* $k = 1, 2, ..., n$*, the value of the option at time* $t < T_0$ *to invest the amount* K *at time* $T_0 < T_k$ *to receive these cash flows accounting for risk aversion, where the utility of the investor is given by* $u(x) = -\frac{e^{\gamma x}}{\gamma}$*, is given by*

$$f(t, s) = -\frac{1}{\gamma(1-\rho^2)} \ln H(t, s) e^{-r(T_n - t)}$$

where

$$H(t, s) = \Phi(\widehat{B}(t, s)) + e^{\frac{\xi_t^2}{2}} \widehat{C}(t, s) \Phi(\xi_t - \widehat{B}(t, s)).$$

The Indifference Price for Normally Distributed Estimates

$$\xi_t = -\gamma(1-\rho^2)\widehat{a}_1\sqrt{T_0 - t}$$

$$\widehat{a}_j = \sum_{k=j}^{n} \frac{\sigma_k}{\sqrt{T_k}} e^{r(T_n - T_k)}, \quad \widehat{b}_j = \sum_{k=j}^{n} \mu_k e^{r(T_n - T_k)}$$

$$A_j = \widehat{a}_j \left(\frac{\gamma \widehat{a}_j}{2}(1-\rho^2) - \widehat{\nu} \right), \quad A_0 = \sum_{j=1}^{n} A_j (T_j - T_{j-1})$$

$$\widehat{B}(t,s) = \frac{\frac{A_0 - \widehat{b}_1 + K e^{r(T_n - T_0)}}{\widehat{a}_1} - s - \widehat{\nu}(T_0 - t)}{\sqrt{T_0 - t}}$$

$$\widehat{C}(t,s) = e^{\gamma(1-\rho^2)(A_0 - \widehat{a}_1(s + \widehat{\nu}(T_0 - t)) - \widehat{b}_1 + K e^{r(T_n - T_0)})}$$

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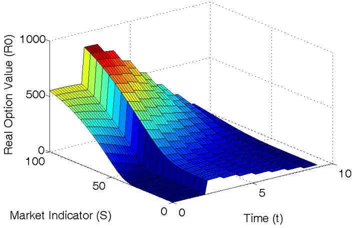
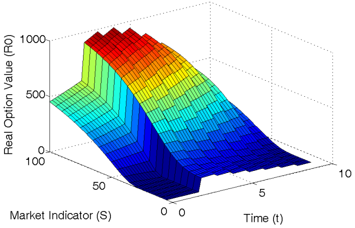
Real Option Value (MMM)



(a) Market Stochastic Driver as GBM

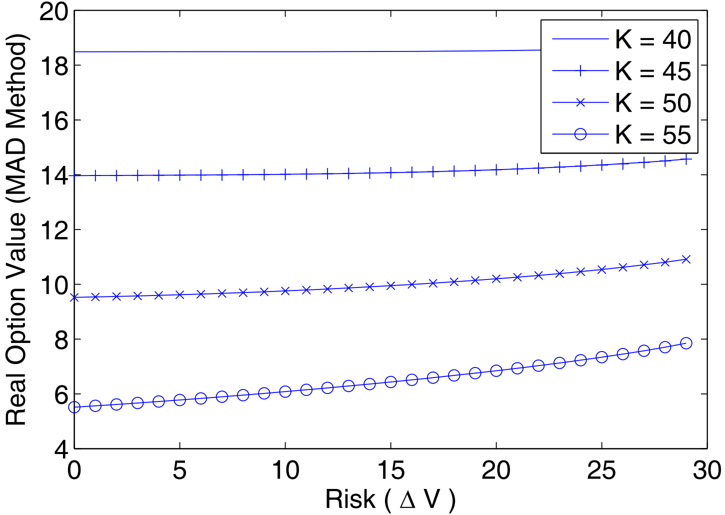


(b) Market Stochastic Driver as GMR

Project value and real option value of the UAV project for varying correlation (note that they are independent of S_0, ν and η)

Correlation (ρ)	0.0	0.2	0.4	0.6	0.8	1.0
Project Value (V_0)	493.69	467.31	441.49	416.35	392.00	368.54
Option Value (RO_0)	199.82	173.83	148.71	124.82	102.45	82.02

Sensitivity to Risk - Standard Approach

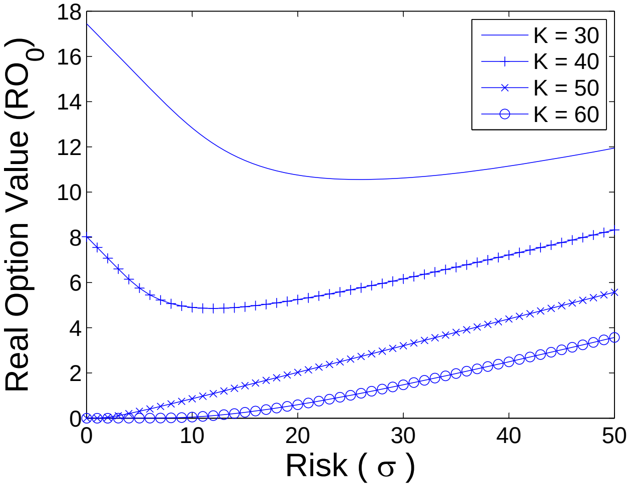


Sensitivity to Risk - MMM

Assumptions:

- Single cash-flow at $T_1 = 3$
- Expected value of the cash-flow: $\mu_1 = 50$
- Correlation to traded index: $\rho = 0.5$
- Investment time: $T_0 = 2$

Sensitivity to Risk - MMM ($\rho = 0.5$)



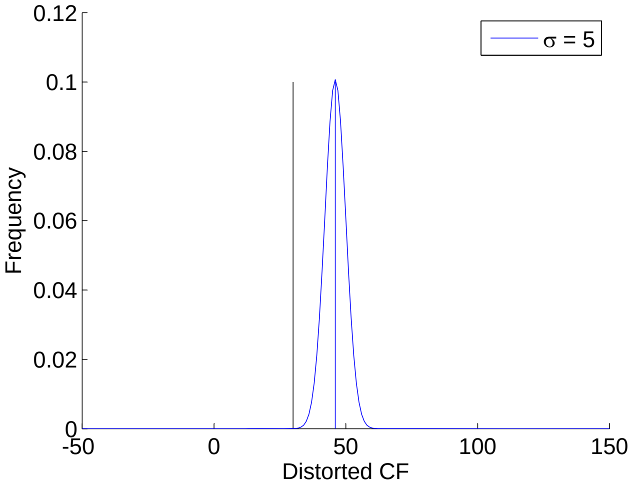
Sensitivity to Risk - MMM

For a single cash-flow, the real option value is given as

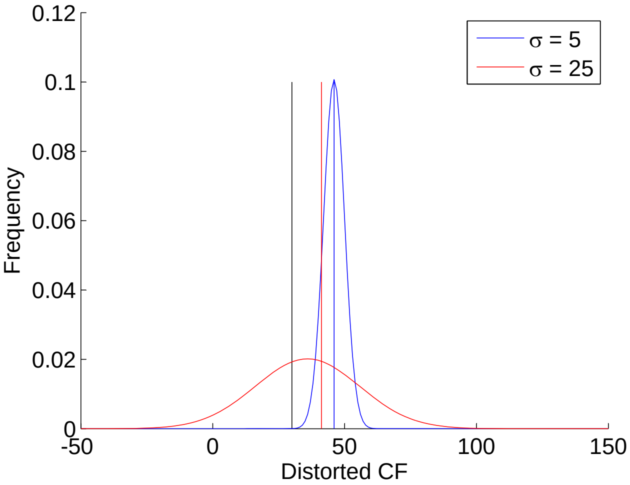
$$RO_0 = e^{-rT_0}\,\mathbb{E}^Q\left[\left(\underbrace{e^{-r(T_1-T_0)}\left(\mu_1+\widehat{\nu}\sigma_1\sqrt{T_1}\right)}_{\text{Distorted Mean}}+\underbrace{e^{-r(T_1-T_0)}\sqrt{\frac{T_0}{T_1}}\sigma_1 Z}_{\text{Standard Deviation}}-K\right)_+\right]$$

Recall $\widehat{\nu} = -\rho\frac{\mu-r}{\sigma}$

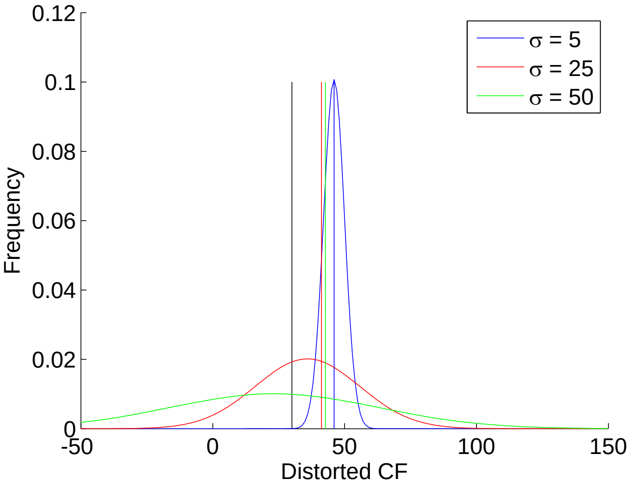
Sensitivity to Risk - MMM



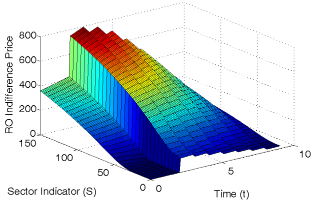
Sensitivity to Risk - MMM



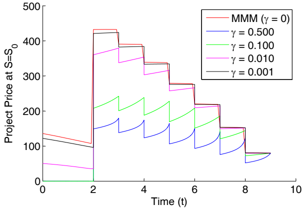
Sensitivity to Risk - MMM



Real Option Value - Indifference Price



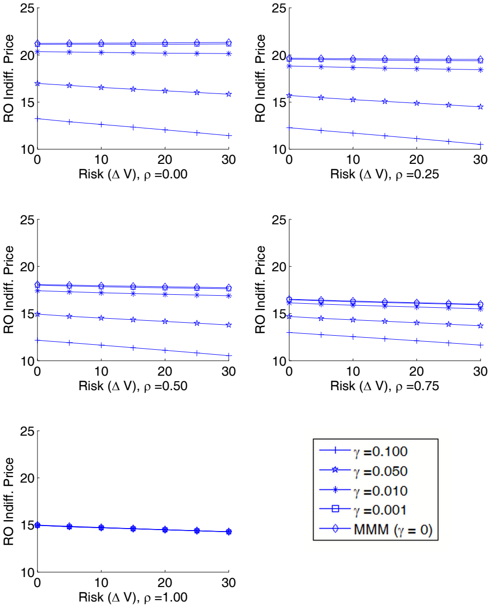
(c) Real option indifference price as a function of S_t and t at $\gamma = 0.01$.



(d) Real option indifference price at $S_t = S_0$ for varying levels of risk aversion.

Sensitivity to Risk - Indifference Price

Sensitivity to Risk – Indifference Price



Practical Implementation of the Matching Method

- Assume managers supply revenue and GM% estimates

Scenario	End of Year Sales / Margin						
	3	4	5	6	7	8	9
Optimistic	80	116	153	177	223	268	314
	(50%)	(60%)	(65%)	(60%)	(60%)	(55%)	(55%)
Most Likely	52	62	74	77	89	104	122
	(30%)	(40%)	(40%)	(40%)	(35%)	(35%)	(35%)
Pessimistic	20	23	24	18	20	20	22
	(20%)	(20%)	(20%)	(20%)	(15%)	(10%)	(10%)
SG&A*	10%	5%	5%	5%	5%	5%	5%
Fixed Costs	30	25	20	20	20	20	20

* Sales / General and Administrative Costs

Sales and GM% Stochastic Drivers

- Traded index

$$dI_t = \mu I_t dt + \sigma I_t dW_t$$

- Sales stochastic driver to drive revenues

$$dX_t = \rho_{SI} dW_t + \sqrt{1 - \rho_{SI}^2} dW_t^S$$

- GM% stochastic driver to drive GM%

$$dY_t = \rho_{SM} dX_t + \sqrt{1 - \rho_{SM}^2} dW_t^M$$

- Cash flow

$$V_k(t) = (1 - \kappa_k)\varphi_k^S(X_t)\varphi_k^M(Y_t) - \alpha_k$$

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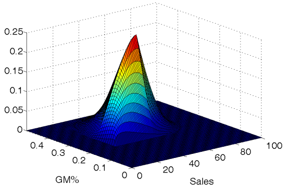
Bivariate Density of Sales and GM%

Theorem

The Bivariate Density of Sales and GM%. *The bivariate probability density function between sales and GM% is given by*

$$u(s,m) = \phi_{\Omega_{\rho_{SM}}}\left(\Phi^{-1}\left(F^*(s)\right), \Phi^{-1}\left(G^*(m)\right)\right) \bullet \frac{f^*(s)}{\phi\left(\Phi^{-1}\left(F^*(s)\right)\right)} \frac{g^*(m)}{\phi\left(\Phi^{-1}\left(G^*(m)\right)\right)}$$

where ϕ_{Ω_ρ} represents the standard bivariate normal PDF with correlation ρ, and ϕ is the standard normal PDF.



Project and Real Option Value

- Project value

$$V_{T_0}(X_{T_0}, Y_{T_0}) = \sum_{k=1}^{n} e^{-r(T_k - T_0)}\, \mathbb{E}^{\mathbb{Q}} \left[v_k(X_{T_k}, Y_{T_k}) \mid X_{T_0}, Y_{T_0} \right]$$

- Real option value

$$RO_t(X_t, Y_t) = e^{-r(T_0 - t)}\, \mathbb{E}^{\mathbb{Q}} \left[(V_{T_0}(X_{T_0}, Y_{T_0}) - K)_+ \middle| X_t,\ Y_t \right]$$

- Risk-neutral measure $\left(\widehat{\nu} = -\rho_{SI}\, \frac{\mu - r}{\sigma} \text{ and } \widehat{\gamma} = -\rho_{SI}\rho_{SM}\, \frac{\mu - r}{\sigma}\right)$

$$\frac{dI_t}{I_t} = r\, dt + \sigma\, d\widehat{W}_t,$$

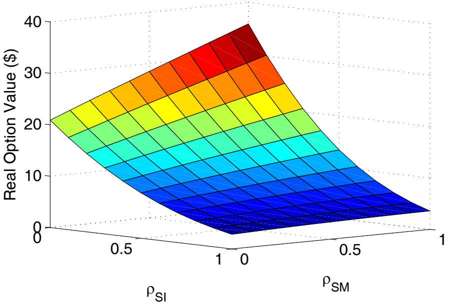
$$dX_t = \widehat{\nu}\, dt + \rho_{SI}\, d\widehat{W}_t + \sqrt{1 - \rho_{SI}^2}\, d\widehat{W}_t^S,$$

$$dY_t = \widehat{\gamma}\, dt + \rho_{SI}\rho_{SM} d\widehat{W}_t + \rho_{SM}\sqrt{1 - \rho_{SI}^2}\, d\widehat{W}_t^S + \sqrt{1 - \rho_{SM}^2}\, d\widehat{W}_t^M$$

Computing the Real Option

- Resulting PDE

$$rH = \frac{\partial H}{\partial t} + \widehat{\nu}\frac{\partial H}{\partial x} + \widehat{\gamma}\frac{\partial H}{\partial y} + \frac{1}{2}\frac{\partial^2 H}{\partial x^2} + \frac{1}{2}\frac{\partial^2 H}{\partial y^2} + \rho_{SM}\frac{\partial^2 H}{\partial x \partial y}$$



Value of the real option for varying ρ_{SI} and ρ_{SM}

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Matching Method Conclusions

- The approach utilizes managerial cash-flow estimates
- The approach is theoretically consistent
  - Provides a mechanism to account for systematic versus idiosyncratic risk
  - Provides a mechanism to properly correlate cash-flows from period to period
- The approach requires little subjectivity with respect to parameter estimation
- The approach provides a **missing link** between practical estimation and theoretical frame-works

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References

Berk, J., R. Green, and V. Naik (2004).
Valuation and return dynamics of new ventures.
The Review of Financial Studies 17(1), 1–35.

Block, S. (2007).
Are real options actually used in the real world?
Engineering Economist 52(3), 255–267.

Borison, A. (2005).
Real options analysis: Where are the emperor's clothes?
Journal of Applied Corporate Finance 17(2), 17–31.

Henderson, V. (2007).
Valuing the option to invest in an incomplete market.
Mathematics and Financial Economics 1, 103–128.

Miao, J. and N. Wang (2007, December).
Investment, consumption, and hedging under incomplete markets.
Journal of Financial Economics 86(3), 608–42.

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