The Valuation of Basket Credit Derivatives: A Copula Function Approach

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Outline

- The Construction of Credit Curves
- On Default Correlation: The Joy of Copula Functions
- The Valuation of Credit Default Swaps
- The Valuation of Basket Credit Derivatives
  - First default / first loss
  - CBOs/CLOs

Credit Markets Are Being Transformed

- Shrinking Loan Profit Margin
- Low interest rate environment
- Huge amount of investment money
- Changing regulatory environment
- Theoretical and analytical advancements
- Technology

Credit Derivative Products

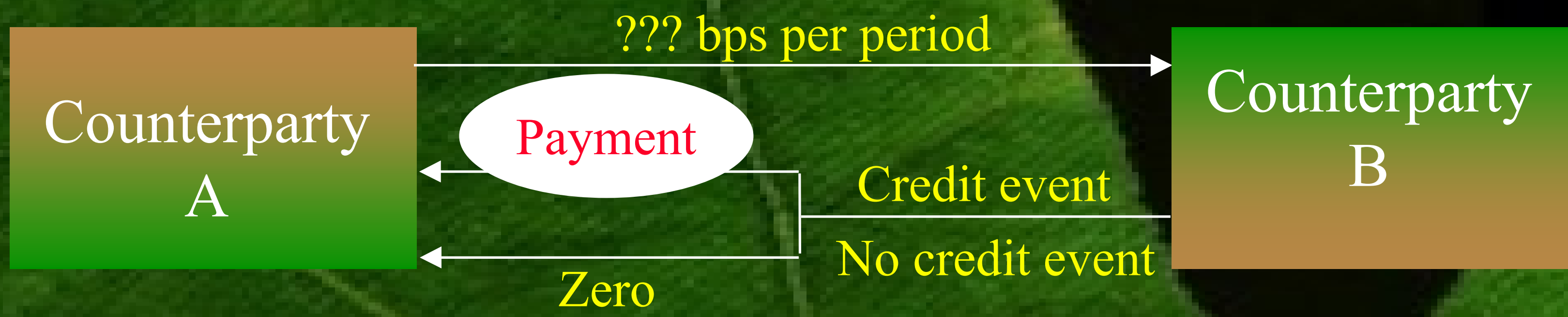
Structures

- Total return swap
- Default contingent forward
- Credit swap
- Credit linked note
- Spread forward
- Spread option

Underlying Assets

- Corporate loans
- Corporate bonds
- Sovereign bonds/loans
- Specified loans or bonds
- Portfolio of loans or bonds

Credit Swap Pricing: Illustration



Reference Credit:	Company X
Swap Tenor:	3 Years
Event Payment:	Par - Post Default Market Value

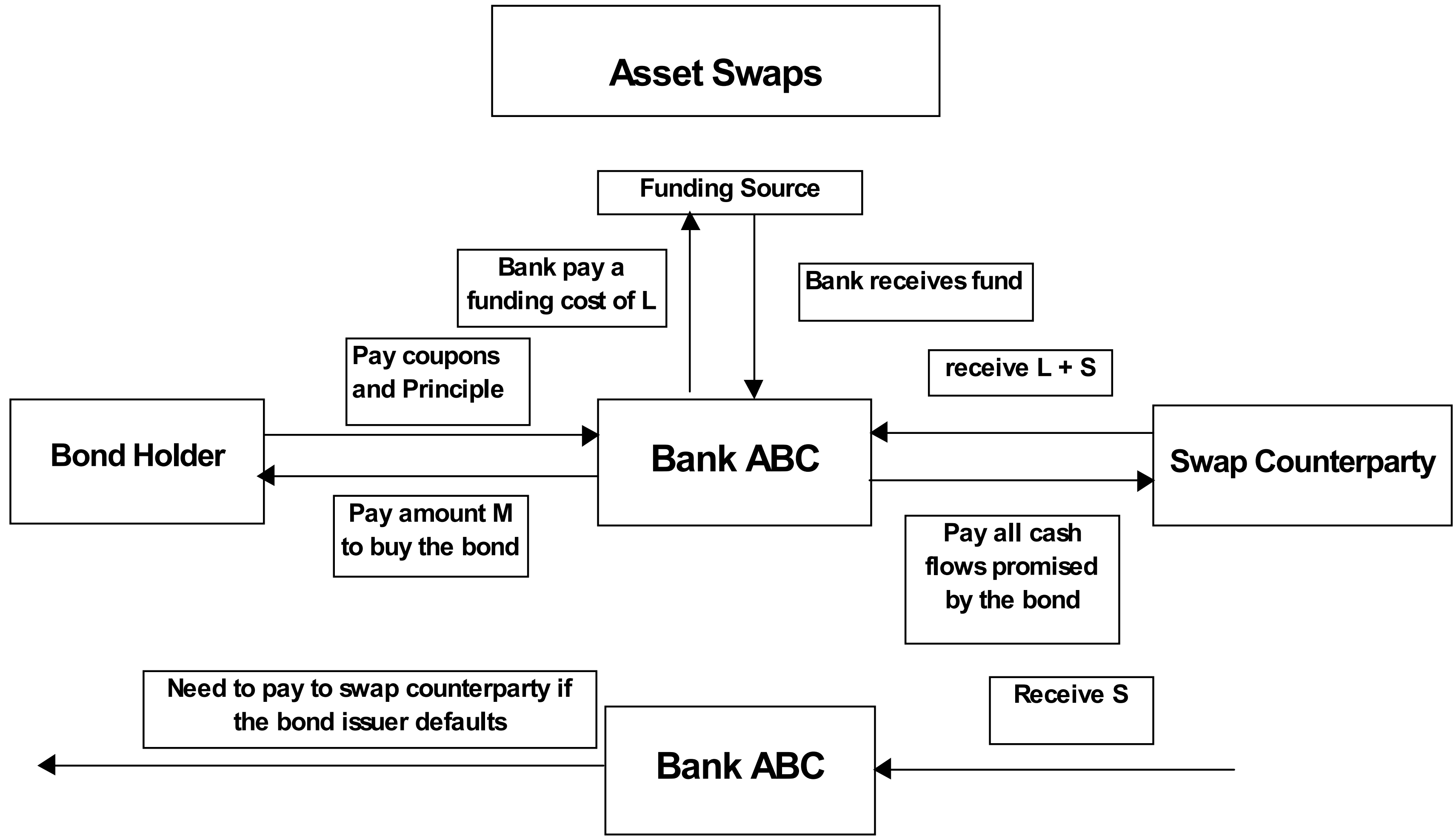
Bond Insurance v.s. Credit Default Swaps

Bond insurance	Credit Default Swaps
• Player	
Insurance company	Banking
• which side of credit risk	
long	long and short
• How to price	
Actuarial approach based on historical data	Relative pricing based asset swap spread

Default Probabilities

- From Historical Data
  - Moody’s and S&P publish historical data
- From Merton’s Option Framework
  - data
  - method to address term structure of default rates
- From Market Observed Credit Spread or Asset Spread

Asset Swaps of Bonds



Credit Swap Pricing:

A *credit curve* gives instantaneous default probabilities of a credit at any time in the future conditional on the survival at that time

- Construct a discount curve, such as LIBOR
- Construct a credit curve for the reference credit
- Construct a credit curve for the counterparty
- Calculate the NSP of the protection
- Amortize the NSP into a number of years

The Characterization of Default

- Define a random variable called the *time-until-default* to denote the survival time   Pr[T < t] = F(t)
- Use survival function or hazard rate function to describe this survival time

$$S(t) = 1 - F(t)$$

$$h(t) = \frac{f(t)}{1 - F(t)} = -\frac{S'(t)}{S(t)}$$

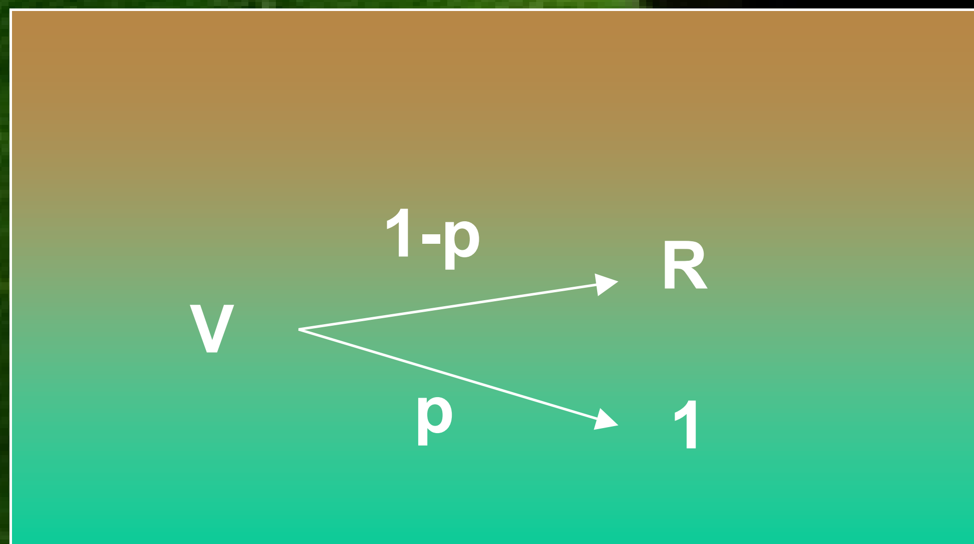
$$S(t) = e^{-\int_0^t h(s)\,ds}$$

$${}_t q_x = \Pr[\,T - t \le t \mid T > x\,]$$

$${}_t p_x = 1 - {}_t q_x$$

Constructing a Credit Curve

- Valuation of Risky Bond -- Duffie and Singleton Approach
- Default Treatment: Recover a fixed % R of the value just before default
- One period

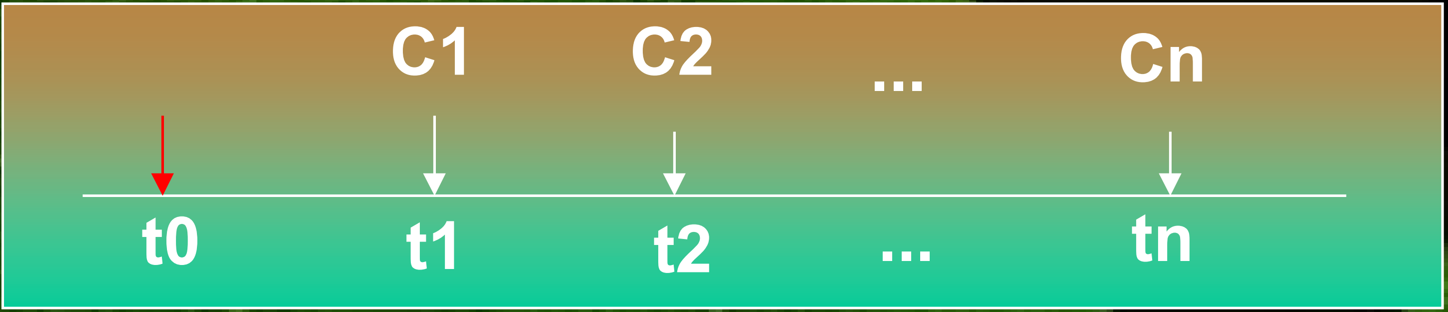


$$V = [p + (1 - p)R]e^{-r\Delta t}$$

$$\cong e^{-\int_0^{\Delta t}[r(s) + (1 - R(s))h(s)]ds}$$

Multiperiod

- General Case



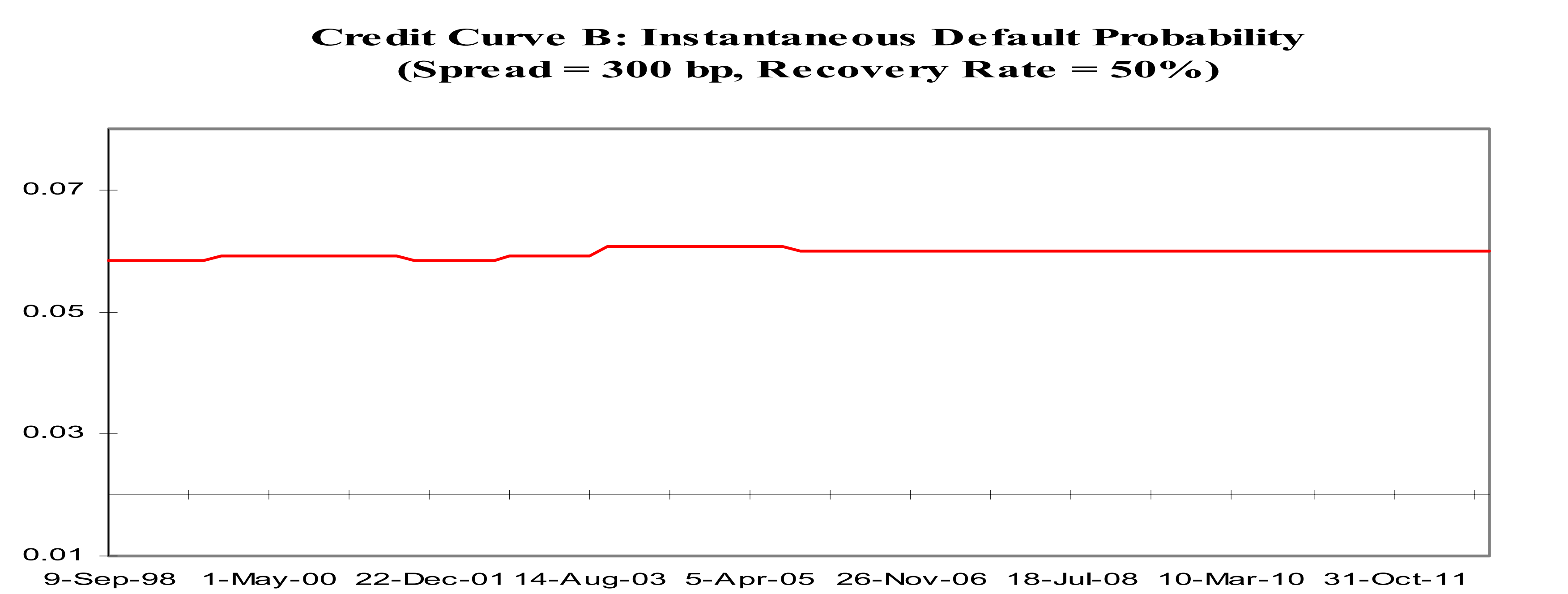
$$V(t_0) = \sum_i^n C_i \cdot e^{-\int_{t_0}^{t_i} [r(s) + (1 - R(s))h(s)]ds}$$

Asset Swap Spreads

Maturity Year	LIBOR Yield	Asset Swap Spread
1	5.89%	200 bp
2	6.13%	200
3	6.30%	200
4	6.40%	200
5	6.48%	200
7	6.62%	200
10	6.78%	200

An Example

Maturity	Coupon	Spread	Price
1 year	7.89%	200	100.00
2 year	8.13%	200	100.00
3 year	8.30%	200	100.00
4 year	8.40%	200	100.00
5 year	8.48%	200	100.00
7 year	8.62%	200	100.00
10 year	8.78%	200	100.00



Default Correlation

- What is the default correlation?
- Traditional Correlation defined in the current finance literature

$$Corr(A,B) = \frac{\Pr[A \cap B] - P[A] \cdot P[B]}{\sqrt{P(A)[1 - P(A)]P(B)[1 - P(B)]}}$$

Problems with This Approach

- One year is an arbitrary choice, useful information about the term structure of default rates could be lost
- Default correlation is a time dependent variable
- Need correlation over a number of years instead of only one year
- Estimation of default correlation has its problem

Lucas Approach

Default Correlation: The Joy of Copulas

- We first know the marginal distribution of survival time for each credit
- We need to construct a joint distribution with given marginals and a correlation structures
- Copula function in multivariate statistics can be used
- The correlation parameters used in copula function can be interpreted as the asset correlation between two credits used in CreditMetrics

What is a Copula Function?

- Function that join or couple multivariate distribution functions to their one-dimensional marginal distribution functions
- For m uniform r. v., U1, U2, …., Um

$$C(u_1, u_2, \cdots, u_m) = \Pr[U_1 \le u_1, U_2 \le u_2, \cdots, U_m \le u_m]$$

- Suppose we have m marginal distributions with distribution function $F_i(x_i)$
- Then the following defines a multivariate distribution function

$$F(x_1, x_2, \cdots, x_m) = C(F_1(x_1), F_2(x_2), \cdots, F_m(x_m))$$

A Few Copula Functions

- **Normal Copula Function**

$$C(u.v) = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v), \rho)$$

- **Frank Copula Function**

$$C(u,v) = \frac{1}{\alpha} \ln\left[1 + \frac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{e^{\alpha} - 1}\right]$$

- **Mixture Copula Function**

$$C(u,v) = (1 - \rho)uv + \rho \min(u,v)$$

Credit Swap Pricing:

- Calculate the PV of Payment
  - 100 - Q(ti) if bond issuer defaults, but the seller does not
  - [100 - Q(ti)]Rc   if both the bond issuer and the default protection seller defaults

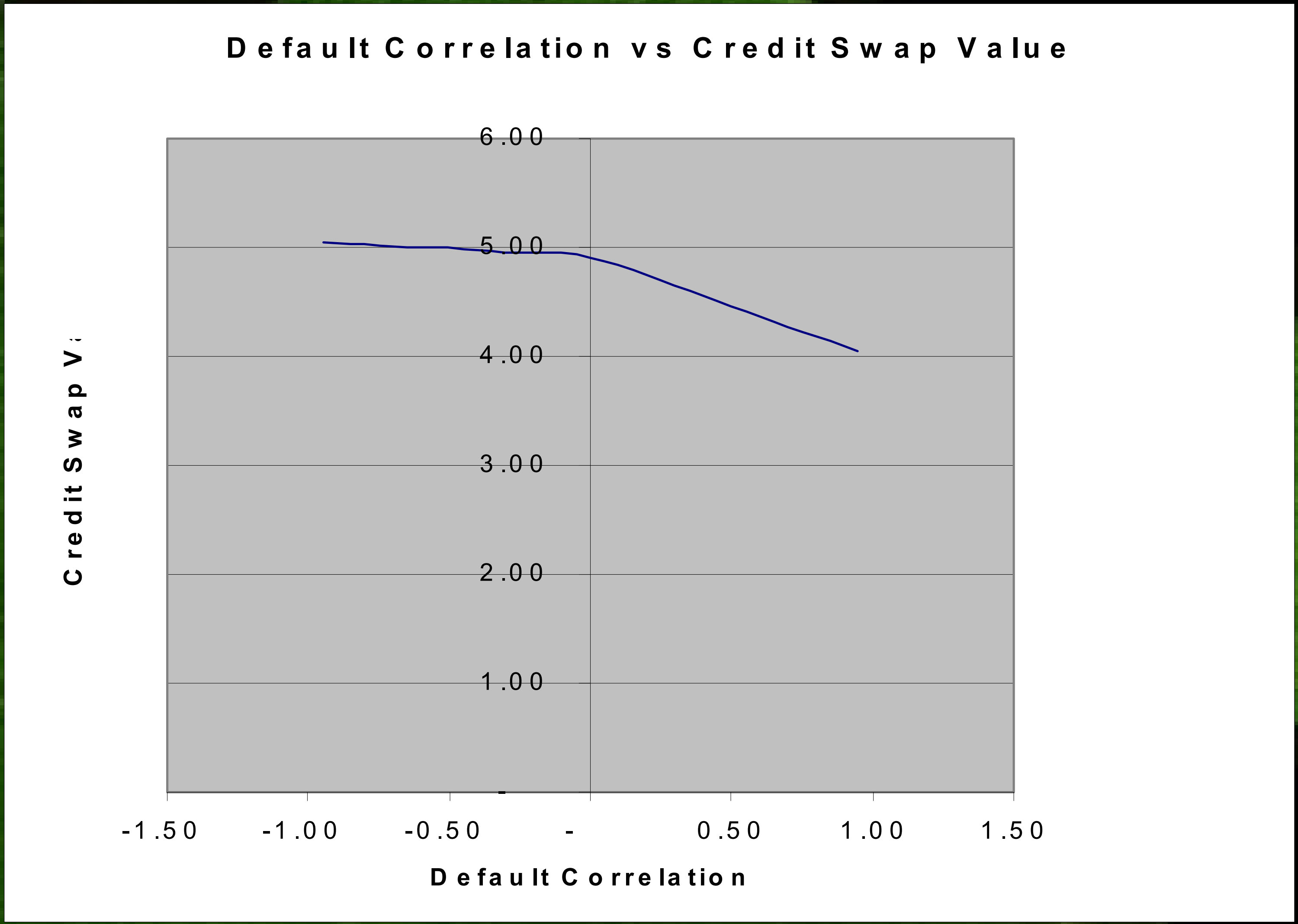
$$\sum_{i=1}^{n}\left(\begin{array}{l}[100-Q(t_i)]\Pr[t_{i-1}<\tau_B\le t_i,\tau_c>t_i]+\\ R_C[100-Q(t_i)]\Pr[t_{i-1}<\tau_B\le t_i,\tau_c\le t_i]\end{array}\right)\bullet D(t_i)$$

- Calculate the PV of Premium

$$X\sum_{i=0}^{n-1}\Pr[\tau_B>t_i,\tau_C>t_i]\cdot D(t_i)$$

- The Periodic or Level Premium X can be solved by equating the above two equations

Numerical Examples of Default Swap Pricing



How do we simulate the default time?

- Map obligors to countries and industries
- Calculate asset correlation based on the historical data of equity indices, use CreditManager
- Simulate y1, y2, …, yn from a multivariate normal distriution with the asset correlation matrix
- Transform the equity return to survival time by

$$T_i = F_i^{-1}(\Phi(Y_i))$$

Summary of the Simulation

Asset Correlation → Copula Function → Survival Time

- Use CreditMetrics Approach to Default Correlation
- Simulate correlated multivariate normal distribution with the asset correlation
- Translate the multivariate normal random variable into survival times by using marginal term structure of default rates

Details: CreditMetrics Monitor, May 1999

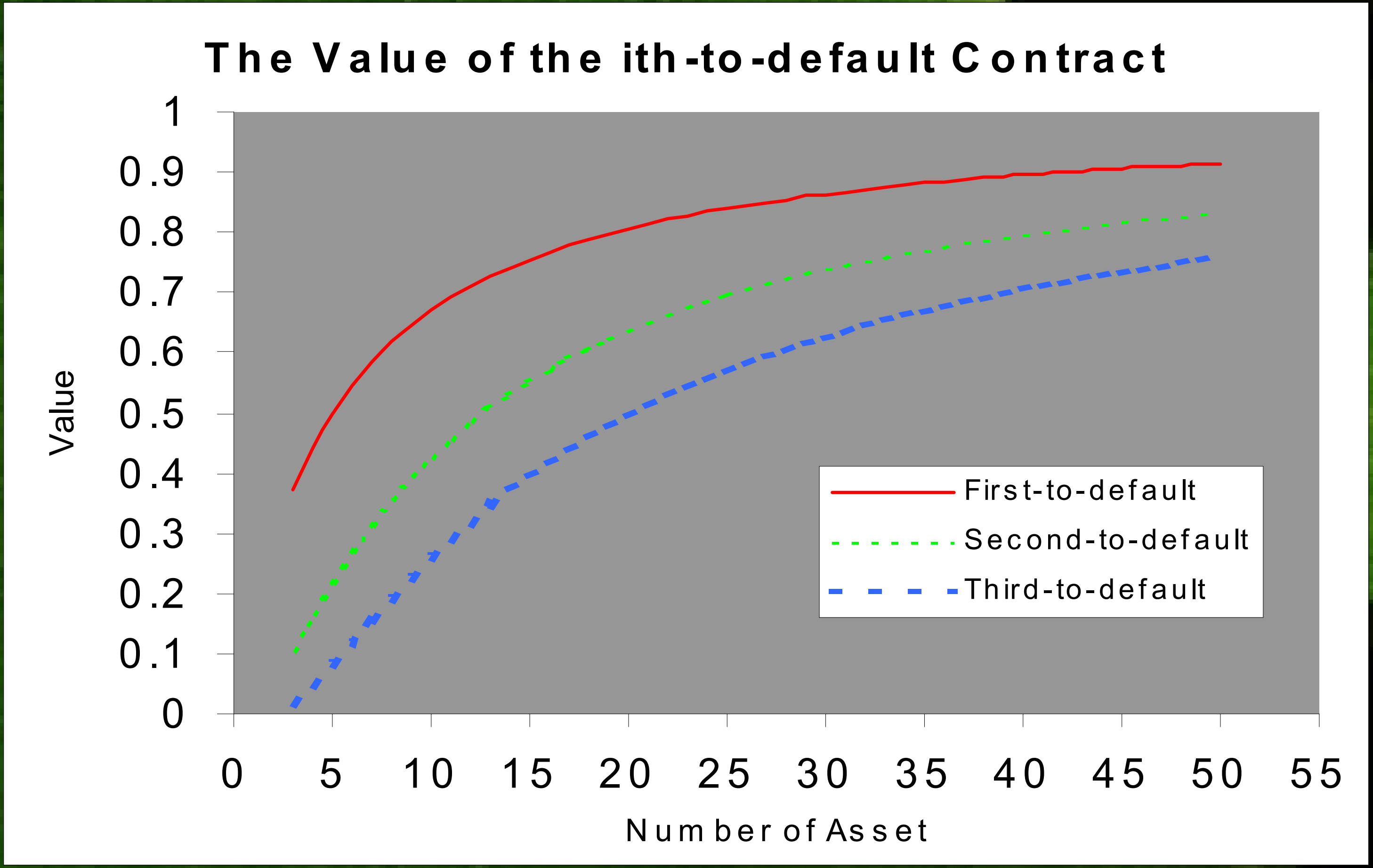
The Valuation of the First-to-Default

- An Example: The contract pays $1 if the first default of 5-credit portfolio occurs during the first 2 years
- We use the above approach to construct a credit curve for each credit
- Using asset correlation and normal copula function we can construct a joint distribution of survival times
- Then we can simulate the survival times for all 5 credits

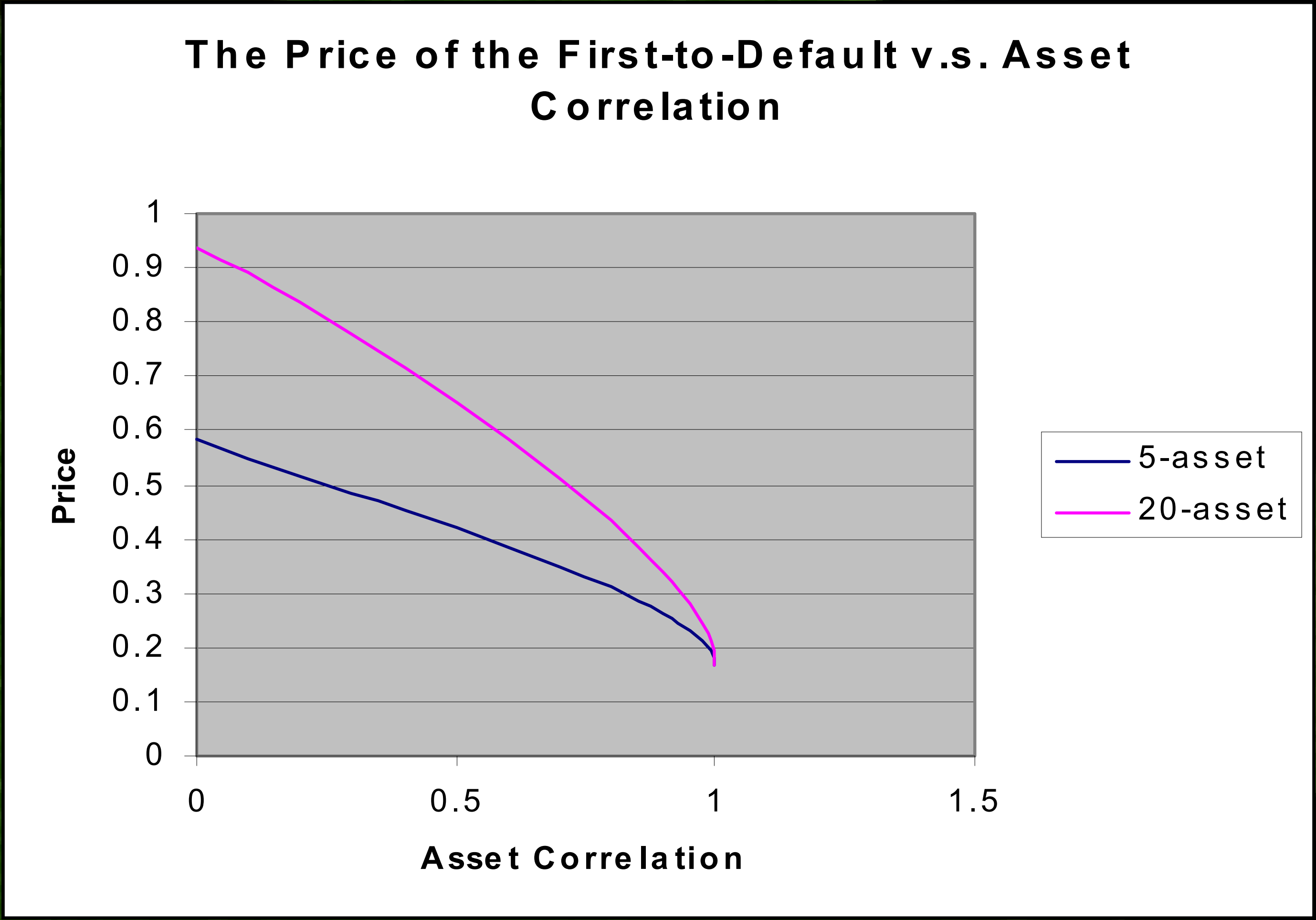
An Numerical Example

Input Parameter

hazard rate = 0.1,
Interest rate = 0.1
Asset Correlation = 0.25

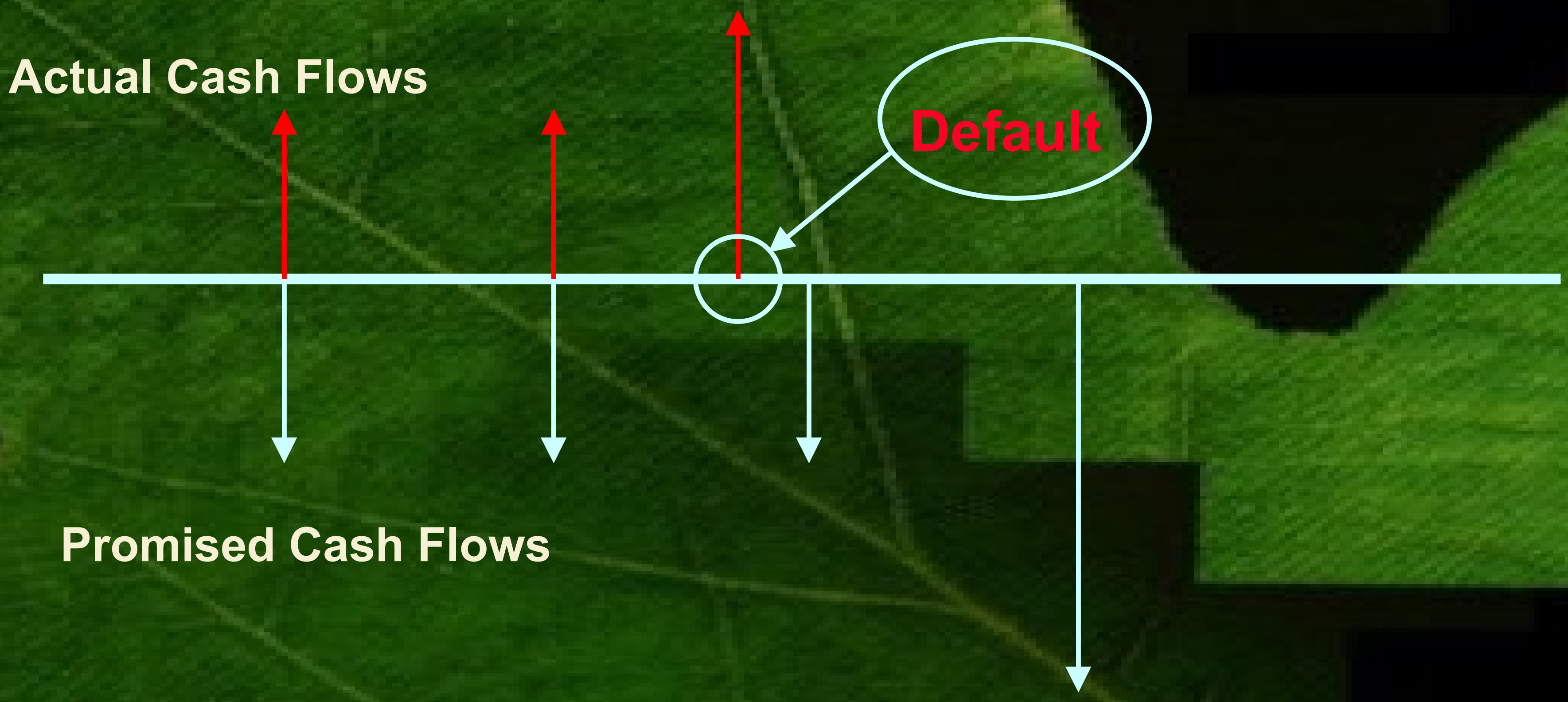


The Impact of Asset Correlation



CBO/CLO Models:
Extraction of Cash Flows from Simulation

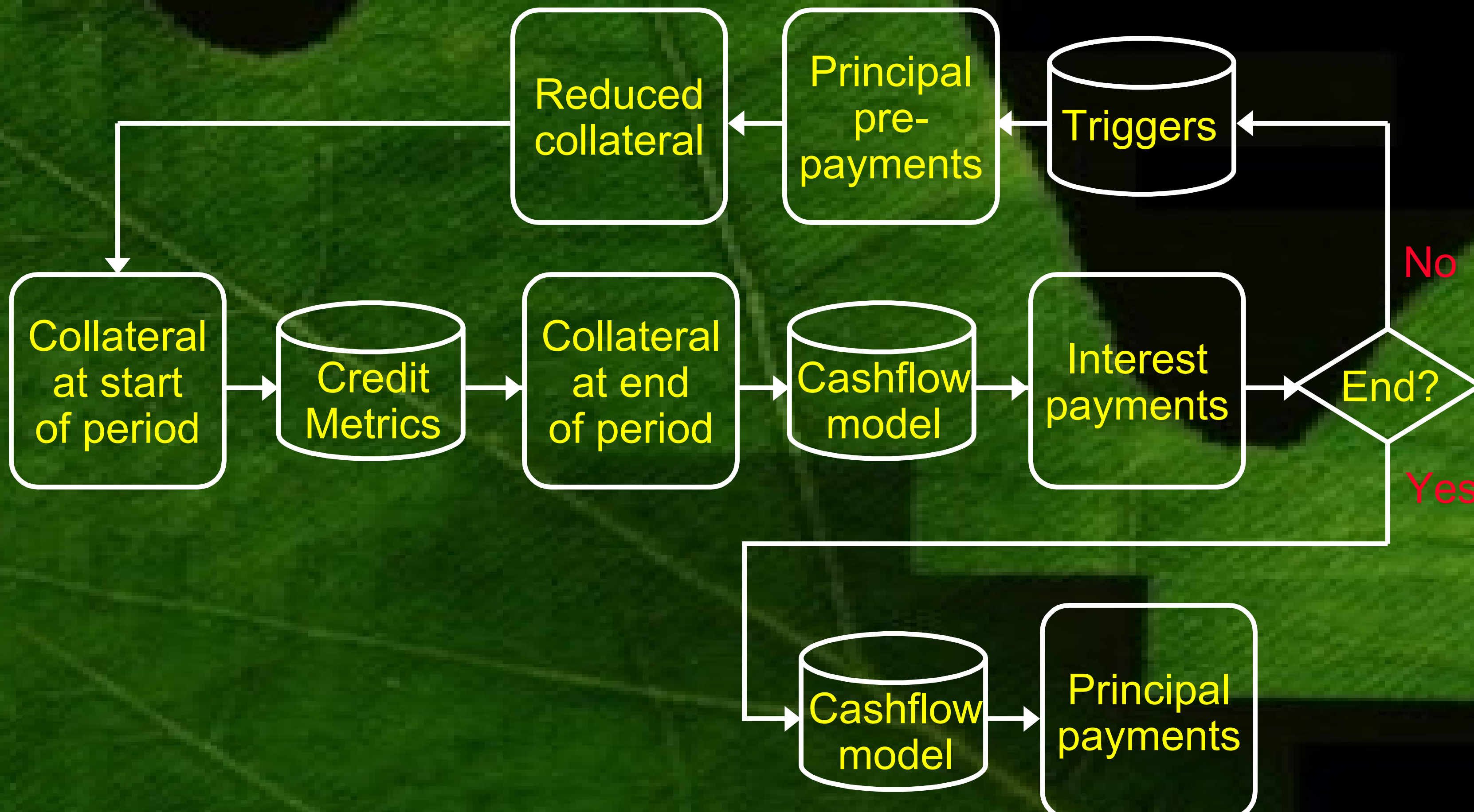
For a defaultable bond we can project the cash flow if we know when default occurs



Cash Flow Distribution

- Interest Proceeds
  - Pass OC and IC test - payment each tranche consecutively
  - Fail OC and IC test - Retire principal
- Principal Proceeds
  - Pass OC and IC test
    - During the reinvestment period - buy additional high yield
    - After the reinvestment period - retire principal from the top to bottom
  - Fail OC and IC test
    - During the reinvestment period - flow through each tranche until tests are passed, remaining one is used to buy additional collateral assets
    - After the reinvestment period - flow through each

Flow Chart of Cash Flow Distribution



Simple cashflow CBO

- Collateral pool -- total value of $100M
  - 80 identical assets, face value of $1.25M
  - one year maturity
  - annual coupon of L+180bp
  - in default, recover 40% of face value
- Securitization
  - Senior tranche -- $90M of one year notes paying L+80bp
  - Equity -- $10M held as loss reserve

What is the probability that the Senior notes pay their coupon?
What is the return for the equity investors?
How to characterize risk in general?

Collateral guidelines and ratio tests

- Overcollateralization
  - ratio of performing collateral to par value of Senior notes
  - here, 100/90= 1.11
- Interest coverage
  - ratio of collateral interest to interest on Senior notes
  - here, 100*(L+180bp)/90*(L+80bp)= 1.29 (assume L=5.5%)
- Other guidelines on average rating, maturity, diversification
- Typically, minimum ratio levels must be maintained throughout the life of the structure

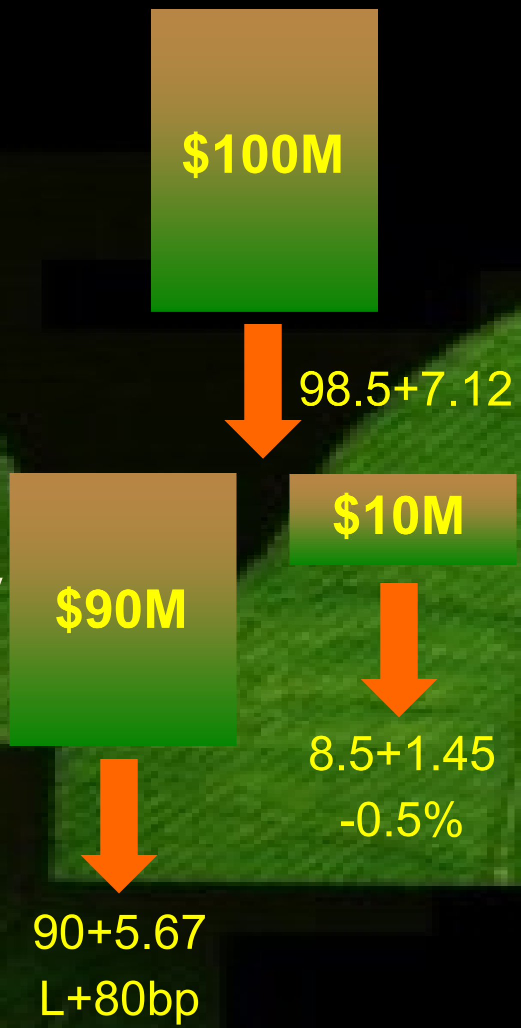
Ratios characterize risk generally,
but for more information, we must look at default scenarios.

The best scenario -- no defaults

- Assume LIBOR is 5.5%
- Interest
  - receive 80*$1.25M*(L+180bp)=$7.30M
  - pay to Senior $90M*(L+80bp)=$5.67M
  - pay remainder ($1.63M) to Equity
- Principal
  - receive 80*$1.25M=$100M
  - pay $90M to Senior notes, $10M to Equity
- Yield
  - Senior receives the contracted L+80bp
  - Equity appreciates by 16.3%, or L+1080bp

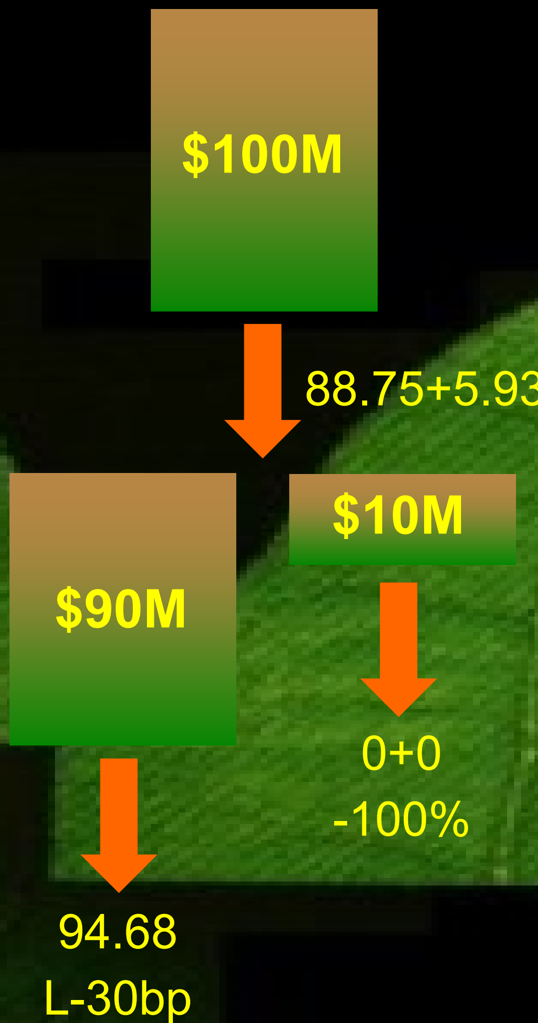
A moderate scenario -- two defaults

- Interest
  - receive 78*$1.25M*(L+180bp)=$7.12M
  - pay to Senior $90M*(L+80bp)=$5.67M
  - pay remainder ($1.45M) to Equity
- Principal
  - receive 78*$1.25M+2*40%*$1.25M=$98.5M
  - pay $90M to Senior notes, $8.5M to Equity
- Yield
  - Senior receives the contracted L+80bp
  - Equity depreciates by 0.5%



At fifteen defaults, Senior investors get hit

- Receive
  - Interest -- 65*$1.25M*(L+180bp)=$5.93M
  - Principal -- 65*$2M+15*40%*$2M=$88.75M
- Pay
  - All receipts ($94.68M) to Senior
  - Equity receives nothing
- Yield
  - Senior only appreciates 5.2%, or L-30bp
  - Equity is worthless



Use CreditMetrics to evaluate the likelihood of each scenario

- Individual default probabilities
  - 1.2% for each asset, consistent with Ba rating
- Correlations
  - assume a homogeneous portfolio; all pairs are the same
  - what level of correlation?

	Low	Med	Hi
Asset corr.	0%	20%	50%
Joint def. prob.	1.4bp	4.6bp	16.7bp

- Simulation gives probabilities for scenarios

# defaults	0	1	…	14	15
probability	57.0%	21.6%	…	3.8bp	2.6bp
cum prob	57.0%	78.6%	…	99.93%	99.95%

Putting the probabilities together with cashflows gives risk and return information

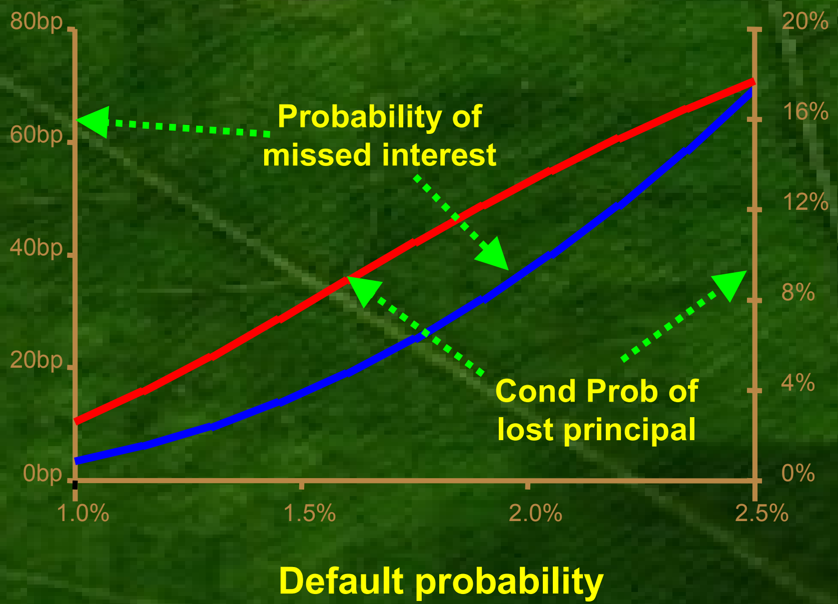
- Senior
  - probability that L+80bp is not paid -- 7.2bp
  - conditional probability that some principal is not repaid, *given that some interest is missed* -- 4.5%

- Equity
  - mean return -- L+280bp
  - standard deviation -- 14.0%
  - probability of positive (L+1080bp or L+239bp) return -- 78.6%
  - probability of losing more than 50% -- 78bp

Not very meaningful!

Can now examine losses under stressed default rates

- Senior notes



- Equity