Mortality Derivatives and the Option to Annuitize

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October 5, 2000

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Abstract

Most U.S.-based insurance companies offer holders of their tax-sheltered savings plans (VAs), the long-term option to *annuitize* their policy at a pre-determined rate over a pre-specified period of time. Currently, there is approximately $680 billion dollars invested in such policies, with guaranteed annuitization rates. The insurance company has essentially granted the policyholder an option on *two* underlying stochastic variables; future interest rates and future mortality rates. Although the (put) option on interest rates is obvious, the (put) option on mortality rates is not.

Motivated by this product, this paper attempts to value (options on) mortality-contingent claims, by stochastically modeling the future *hazard-plus-interest* rate. Heuristically, we treat the underlying life annuity as a defaultable coupon-bearing bond, where the default occurs at the exogenous time of death. From an actuarial perspective, rather than considering the force of mortality (hazard rate) at time t for a person now age x, as a number $\mu_x(t)$, we view it as a random variable forward rate $\widetilde{\mu}_x(t)$, whose expectation is the force of mortality in the classical sense. ($\mu_x(t) = E[\widetilde{\mu}_x(t)]$.)

Our main qualitative observation is that <u>both</u> mortality and interest rate risk can be hedged, and the option to annuitize can be priced by locating a replicating portfolio involving insurance, annuities and default-free bonds. We provide both a discrete and continuous-time pricing framework.

*...If life spans increase during the accumulation period, the guaranteed purchase rates may produce larger monthly payments than would the current or immediate annuity rates when the policyholder annuitizes. On the other hand, if the rates being offered to individuals who purchase immediate annuities are more favorable than the guaranteed purchase rates, the policy holder can annuitize using the immediate annuity rates. In effect, the policyholder is in a win/win situation....*Source: National Association for Variable Annuities, 1997.

1 Introduction and Motivation.

Most U.S.-based insurance companies offer holders of their tax-sheltered savings plans – also known as variable annuity contracts – the long-term option to annuitize their policy at a pre-determined rate over a pre-specified period of time. If annuity rates at the time of annuitization are more favorable than the contractually specified value, the policyholder can demand current rates or go elsewhere. Conceptually, this call option on annuity purchase rates can be viewed as the right, but not the obligation, to purchase a fixed immediate life annuity, for a time varying strike price during the life of the contract.

The company has essentially granted the policyholder an option on two underlying stochastic variables; future interest rates and future mortality rates. Interest rates may decrease - relative to the guaranteed return – thus putting the insurance company at investment risk. As well, mortality patterns can improve leaving the insurance company exposed to unanticipated longevity risk. We believe this risk is not trivial since at least \$680 billion in retirement savings are invested in these products, according to Morningstar Inc.[1]

The option to annuitize – and the implicit risk exposure – should be contrasted with a spot transaction of purchasing a single premium immediate annuity (SPIA) or a forward transaction of purchasing a single premium deferred annuity (SPDA); both of these immediately lock-in irreversible long-term rates. Prudent insurance companies tend to protect themselves by guaranteeing low interest rates combined with 'aggressive' mortality improvement projections for the annuitant population.

In this paper we attempt to value mortality-contingent claims, by stochastically modeling the *hazard-plus-interest* rate. Heuristically, we treat the underlying life annuity as a defaultable coupon-bearing bond, where the default occurs at the exogenous time of death. In practice, the option to annuitize is an American-style contingent claim on a corporate bond. From an actuarial perspective, rather than considering the force of mortality (hazard rate) at time t for a person now age x, as a number $\mu_x(t)$, we view it as a random vari-

[1] See Milevsky and Posner (2000) for a description of some other options which are embedded in Variable Annuity products.

${}_{(t-55)}p_{55}$	**1971** IAM		**1983** IAM		**1996** IAM	
t	F.	M.	F.	M.	F.	M.
55	1.00	1.00	1.00	1.00	1.00	1.00
60	.976	.952	.982	.966	.985	.974
65	.938	.886	.956	.919	.962	.937
70	.889	.799	.914	.848	.926	.880
75	.812	.682	.849	.742	.899	.791
80	.689	.530	.745	.596	.775	.663
85	.504	.353	.586	.415	.628	.496
90	.281	.181	.379	.234	.427	.313
95	.103	.056	.181	.100	.221	.154
100	.026	.007	.059	.028	.082	.055

Table 1: Annuitant Mortality Tables

able forward rate $\widetilde{\mu}_x(t)$, whose expectation is the force of mortality in the classical sense. ($\mu_x(t) = E[\widetilde{\mu}_x(t)]$.) In our full-fledged model, we will assume that the forward term structure of interest-plus-mortality is modeled in continuous time by using a Gompertz specification for the *expected* force of mortality.

Our main qualitative observation can be summarized as follows. Aside from the interest rate risk – which is well understood in the financial literature – there are two distinct categories of mortality risk that an insurance company faces when selling an option to annuitize. The first category we label as 'small sample' risk. It reflects the chance that any particular option holder is healthier than average. When faced with such a client, the insurance company is confronted with a payment stream that is longer than originally expected based on annuitant mortality rates. However, actuarial theory has long established that this particular risk can essentially be eliminated – and therefore should not be priced – by selling enough identical policies and taking advantage of the law of large numbers. If enough options are sold, the realization will converge to the expected.

The second risk is more subtle. It is the risk that the insurance company overestimated the population's force of mortality. Table 1 indicates the change in *Society of Actuaries* annuitant mortality tables over the last three decades. The symbol ${}_{t-55}p_{55}$ denotes the probability that a 55 year-old will survive to age $(t - 55)$. Clearly, mortality has been improving, and at an unpredictable rate.

This longevity risk can not be hedged by selling more annuities or options to annuitize. On the other hand, in all likelihood, the insurance company prices immediate annuities with an expected (dynamically projected) improvement rate. And thus, aggregate mortality may indeed be worse or better than expected. We therefore argue that this second type of risk can be hedged by selling more life insurance policies. The risks offset each other. If mortality rates improve, then the life insurance book will show unexpected profits. Clearly, in practice, the insurance company may not be able to sell insurance to the same group that is buying annuities – the young tend to buy life insurance, while the old purchase annuities – but this is a mere question of implementation. In complete markets, the value of the option is the price of the residual risk that can not be eliminated by selling more endowments and life insurance policies.

We will shortly present a binomial model that should help explain this concept.

1.0.1 Agenda.

The remainder of this paper is organized as follows. Section 2 presents a discrete time model which should provide the basic intuition for the hedging argument. Section 3 provides a continuous-time model of the hazard-plus-interest rate, which is based on the analytically tractable Cox-Ingersoll-Ross (1985) specification. Section 4 concludes the paper. In the appendix we provide an alternative formulation for the hazard-plus-interest rate, which is more realistic, but computationally quite cumbersome.

Finally, although we have motivated this investigation by alluding to the *options to annuitize* that are embedded in VA contracts, we must make absolutely clear that this paper is exclusively concerned with European-style options on a mortality-contingent claim that pays one lump sum upon surviving a pre-specified period. This is also known as a pure endowment policy – in contrast to a traditional life annuity – and is quite similar to a zero-coupon bond. The options contained in VA policies are (i) American style, (ii) on coupon-bearing bonds, (iii) with a variable notional principal, all of which complicate the modeling substantially.

2 Discrete Time.

In this section, we illustrate the basic ideas by means of some simple discrete time examples covering a small number of periods. Our main objective is to illustrate that *mortality improvement risk* can be hedged, and therefore the option to purchase a mortality contingent claim has a unique and quantifiable price.

We assume throughout the section that all activity takes place on a yearly basis, and insurance benefits will be paid at the end of the year of death. We clearly ignore expenses, profits and other administrative charges and thus assume that everything is presented on a net basis. We assume in addition that mortality rates used for life insurance are the same as those used for annuities. Of course in practice this is not true. We will discuss the more realistic case in a later section.

There is another simplification that we will adopt initially. In order that our theory should parallel the well developed results for bond options, we will concentrate at first on the case of options on pure endowments. These are contracts that provide a single payment at a specified time, if the purchaser is then alive. A life annuity can of course be viewed as a basket of several pure endowments. There is a complication however, in that the sum of option prices for all these pure endowments will in general be a strict upper bound to the option price for the annuity. In the latter case, the option holder must exercise either all or none of the pure endowment options. It is possible however that if interest and mortality move in opposite directions, (relative to their effect on annuity prices) some pure endowment options would be in the money at expiry, and some would not. We will elaborate, with examples, in a later section.

2.0.2 General Notation.

We begin with set Ω, representing all possible states of nature, equipped with a filtration to denote the information flow. This is an increasing sequence $\mathcal{F}_n$ of σ - algebras, where $\mathcal{F}_0$ is the trivial algebra $\{\Omega, \emptyset\}$. We are given an interest rate term structure of default-free discount bonds. That is, for each discrete $k \leq n$ we have random variables $D(k,n)$ which are $\mathcal{F}_k$ measurable, together with a probability measure Q on Ω such that:

$$D(k,m) = D(k,n)E_Q[D(n,m)|\mathcal{F}_k] \qquad \forall \quad k \leq n \leq m \tag{1}$$

The term $D(k,n)$ represents the price, at time k, of a \$1-unit zero coupon bond maturing at time n. Throughout the paper we assume that the risk neutral measure, Q, is used for pricing.

In addition to the default free market, we need a 'term structure' for mortality rates. This is most easily expressed in terms of survival probabilities which are analogous to the bond prices as given above. For each $k \leq n$ we have random variables $p_x(k,n)$ which are $\mathcal{F}_k$ measurable, and such that for the same probability measure Q as above:

$$p_x(k,m) = p_x(k,n)E_Q[p_x(n,m)|\mathcal{F}_k] \qquad \forall \quad k \leq n \leq m \tag{2}$$

The qualitative interpretation is that $p_x(k,n)$ is the probability that an individual in our cohort – who is currently aged x – will survive to time n conditional, upon surviving to time k. To contrast with the traditional actuarial approach, the random variable $p_x(k,n)$ would be denoted by ${}_{n-k}p_{x+k}$ and would be considered a constant. Indeed, the expectation could be removed from the above formula which reduces to a standard identity. We further postulate, quite naturally, that interest rates and mortality rates are independent. That is, for all $k \leq n, r \leq s$, the variable $D(k,n)$ and $p_x(r,s)$ are independent.

Finally, for all $k \leq n$ we let

$$\Lambda_x(k,n) = D(k,n)p_x(k,n), \tag{3}$$

which is $\mathcal{F}_k$ measurable, and denotes our pure endowment contract. In other words, $\Lambda_x(k,n)$ is the (random variable) price at time k that would be paid by an individual (currently age x) surviving to that time, for a contract paying a \$1 unit at time n, if alive. Our ultimate objective is to price a call option to acquire a pure endowment at time k, that pays \$1 at time n, if alive. Our notation is $C_x(k,n|\Lambda)$, where Λ is the strike price of the call option.

Of course, the ultimate goal is to price an option to purchase a life annuity,

2.0.3 Example 1

At time 0 an individual, now age 60, would like to receive a single payment of 1000 units at age 62 if they are then alive. See Figure #1 for details. This is a two period pure endowment contract. Suppose they purchase this from an insurance company which is using a mortality function which shows $p_{60}(0,1) = 1 - q_{60} = 1 - 0.20 = 0.8$ and $E_Q[p_{60}(1,2)] = 1 - 0.25 = 0.75$. Implicitly, we assume that the expected probability of death takes into account projected improvement over the next year. That is, it is the anticipated rate at age 61 which will apply in one years time. For simplicity let us assume a zero interest rate. Then the single premium cost to the individual for the desired pure endowment contract, as per equation (3), would be $\Lambda_{60}(0,2) = 1000(0.8)(0.75) = 600$ units. We emphasize, once again, that the rate of $1 - E_Q[p_{60}(1,2)] = 0.25$ is an estimate of the one period probability of death that will be applicable at age 61. The *actual* rate applicable to a person age 61 in one years time might well change, depending on random events which occur over the next year. Now, suppose that two possible states of natures could materialize over the following year. In the first case, a hoped for medical breakthrough materializes and the actual value of $p_{60}^{(1)}(1,2) = 1 - 0.20 = 0.8$. In the second, the medical breakthrough does not materialize and the value of $p_{60}^{(2)}(1,2) = 1 - 0.30 = 0.7$. (The superscripts are used to 'count' the possible realizations.) The insurer is effectively basing their rate by assuming that there is an equal chance of either possibility, $\{p_{60}^{(1)}(1,2), p_{60}^{(2)}(1,2)\}$, which leads to $1 - E_Q[p_{60}(1,2)] = 0.25$.

Now, an alternative for the individual would be to wait a year and purchase the pure endowment contract if they are then alive. Why buy it now if they forfeit nothing for death during the year? Rather than pay 600 however, they will pay either $\Lambda_{60}^{(2)}(1,2) = 1000(1 - 0.30) = 700$ or $\Lambda_{60}^{(1)}(1,2) = 1000(1 - 0.20) = 800$, assuming they are alive and depending on which state of nature occurs. Suppose that as another alternative the insurer offers them the option to purchase this contract at age 61 at a guaranteed price of 750. That is, if the improvement occurs, they will only have to pay 750 rather than 800. If the improvement does not occur, they will not exercise the option and pay the prevailing price of 700. Our intention is to 'price' the option to purchase the pure endowment contract. Using our previous notation, we are looking for $C_{60}(1,2|750)$

Our main claim is that regardless of the individual's assessment of the probability of either$\{p^{(1)}_{60}(1,2), p^{(2)}_{60}(1,2)\}$, the price of the option must be $C_{60}(1,2|750) = 20$ units. The reason, of course, is that the individual can replicate the option for a cost of 20. The basic strategy in this replication is to borrow money, and use it to purchase a combination of insurance and pure endowment contracts. Suppose they borrow 350 now, on a one year loan, for a total cash flow of 370, together with the 20 they begin with. They purchase a pure endowment contract paying 500 at age 62 if then alive, for a cost of 300. The other 70 is used to purchase an insurance policy which will pay 350 at age 61, and thereby pay off the loan, if they die during the year. At age 61, if the mortality improvement occurs, they can pay off the loan for 350, buy the remaining 500 of income for 400 spending a total of 750. If the improvement does not occur, they pay off the loan for 350, and buy the remaining 500 of income for 350, spending a total of 700. They have therefore replicated the option. Naturally, extensions to longer periods are similarly handled but with more complication.

Formally, one can think of the hedge portfolio in the classical Cox-Ross-Rubinstein one-period binomial framework. At time zero, we must decide how many units of the underlying security should be held as a hedge ($H > 0$), and how many bonds should be sold short ($B < 0$) to finance the position. At time 1, we have a system of two equations and two unknowns:

$$\begin{aligned} H\Lambda^{(1)}_{60}(1,2) + B &= \Lambda^{(1)}_{60}(1,2) - \Lambda \\ H\Lambda^{(2)}_{60}(1,2) + B &= 0, \end{aligned} \tag{4}$$

which leads to:

$$H = \frac{\Lambda^{(1)}_{60}(1,2) - \Lambda}{\Lambda^{(1)}_{60}(1,2) - \Lambda^{(2)}_{60}(1,2)}, \qquad B = \frac{-\Lambda^{(2)}_{60}(1,2)\left(\Lambda^{(1)}_{60}(1,2) - \Lambda\right)}{\Lambda^{(1)}_{60}(1,2) - \Lambda^{(2)}_{60}(1,2)}. \tag{5}$$

The price of the option is:

$$\begin{aligned} C_{60}(1,2|\Lambda) &= H\Lambda_{60}(0,2) + B - (1 - p_{60}(0,1))B \\ &= H\Lambda_{60}(0,2) + p_{60}(0,1)B. \end{aligned} \tag{6}$$

The extra term in equation (6), accounts from the cost of insuring the loan – which is due in all states of nature – against death. In our example, recall, $\Lambda_{60}(0,2) = 600$, $H = 0.5$,

$B = -350$ and $p_{60}(0,1) = 0.8$, which according to equation (6) recovers: $C_{60}(1,2|750) = 20$. Finally, combining equations (5) and (6), we can represent the price of the call option as:

$$\begin{aligned} C_{60}(1,2|\Lambda) &= \frac{\Lambda_{60}(0,2)\left(\Lambda_{60}^{(1)}(1,2)-\Lambda\right) - p_{60}(0,1)\Lambda_{60}^{(2)}(1,2)\left(\Lambda_{60}^{(1)}(1,2)-\Lambda\right)}{\Lambda_{60}^{(1)}(1,2)-\Lambda_{60}^{(2)}(1,2)} \qquad (7) \\ &= \left(\frac{\Lambda_{60}(0,2) - p_{60}(0,1)\Lambda_{60}^{(2)}(1,2)}{\Lambda_{60}^{(1)}(1,2)-\Lambda_{60}^{(2)}(1,2)}\right)\left(\Lambda_{60}^{(1)}(1,2)-\Lambda\right) \\ &= \left(\frac{E_Q[p_{60}(1,2)] - \Lambda_{60}^{(2)}(1,2)}{\Lambda_{60}^{(1)}(1,2)-\Lambda_{60}^{(2)}(1,2)}\right) p_{60}(0,1)\left(\Lambda_{60}^{(1)}(1,2)-\Lambda\right), \end{aligned}$$

We interpret this as follows. The first factor is the insurer's implied assessment of the probability of the mortality improvement occurring, as evidenced by their choice of the expected value of $p_{60}(1,2)$. This is then multiplied by the expected payoff from the call option, and discounted back to time 0 with interest (which is this example is zero) and survivorship.

2.0.4 Example 2

Assume again that the interest rate is 0. Suppose mortality rates at age 60 and 61 are as above. Suppose further, that if the value of q_{61} at age 61 is 0.20, then the value of q_{62} applicable at time two for an individual now age 60 is either 0.25 or 0.35. If the value of q_{61} at age 62 is 0.30, the corresponding value of q_{62} is either 0.35 or 0.45. See Figure #2 for details. Suppose as well that the price charged today to an individual age 60 for a contract paying 1000 at age 63, if alive, is 392 and the price which will be charged one year from now for such a contract is 560 if the improvement occurs and 420 if not. It is not difficult to check that this is an arbitrage free model for annuity prices, consistent with an assignment of equal probabilities for each of the two possible occurrences at each stage. Consider a contract that provides an option to purchase at age 62, for a price of 600, an annuity which pays 1000 at age 63 if then alive. The option price can be computed by taking an expectation using the risk neutral probabilities as given.

At time 62 the value of the pure endowment will be either 750, 650, 650, or 550, each with equal probability. The value of the option is computed as in the last line of equation (7), except we must now discount back on four paths rather than two. This value will be

$$\frac{1}{4}\left[(0.8)(0.8)150 + (0.8)(0.8)50 + (0.8)(0.7)50 + (0.8)(0.7)0\right] = 39$$

We will now illustrate how the individual can replicate this option for the 39 price, by a strategy similar to above. It is convenient to combine the borrowing and insurance into a single security of an insured loan, whereby the borrower receives the amount less the cost of insurance.

At time 0, (60) starts with 39, borrows 420 on an insured loan for 1 year year. The insurance cost will be 84, so they receive 336. With the 375 they buy income of 625 payable at age 62 if they are then alive.

Suppose the individual is alive at age 61 and value of $q_{61} = 0.2$ They borrow 1225 for one year, receiving 980 after insuring. The first year loan of 420 is repaid and the remaining 560 purchases 1000 payable at age 63 if then alive. If alive at age 62 they receive 625, and can pay an additional 600, to repay the loan, thereby replicating the option.

Suppose the individual is alive at age 61 and the value of $q_{61} = 0.3$. They now borrow 900 on an insured loan for 1 year, receiving proceeds of 630. They repay the 420 loan and use the remaining 210 to purchase income of 500 at age 63. If the value of $q_{62} = 0.35$, they pay an additional 600, using the 1225 to repay the 900 loan and to purchase an additional 500 of income age 63. If the value of $q_{62} = 0.45$, the cost of this additional income will be only 275, so they need pay only an additional 550. Again the option has been replicated.

Although we have assumed zero interest rates in the above, the situation is very similar for any deterministic interest rate assumptions. Additional complications set in when we deal with the more realistic model of stochastic interest along with the stochastic mortality. The replication strategy still involves borrowing and purchasing annuity contracts, but this must be done by using both fixed rate and floating rate loans.

2.0.5 Example 3

We redo Example 1, except instead of zero interest,we postulate interest rates by the following prices for zero coupon bonds paying 1000 at maturity. The price of a one year bond is 800 The price of a two year bond is 560. One year from now the price of the bond maturing at time two is either 800 or 600. We assume further that at time 0 price of the contract paying 1000 at age 62 if the purchaser is alive, is 336 and the price of the contract paying 1000 at age 61 if the purchaser is alive is 640. See Figure #3 for details. These prices are then consistent with risk neutral probabilities of 1/2 each, for the up or down movement in both interest or mortality. Consider a contract which gives the individual the option to purchase a one year pure endowment for 1000 at time 1 for a strike price of 525. The prices for the pure endowment at time 1 will be either 640, 560, 480, or 420. The one year discount factor at time 0 for interest and mortality is $(0.8)(0.8) = 0.64$, giving an option value of $\frac{1}{4}(0.64)\left[115 + 35 + 0 + 0\right] = 24$.

To replicate this option we must consider the two types of loans.

Type 1: (Floating rate). The loan bears interest at the prevailing rates. The borrower will purchase indexed insurance to repay the loan in the event of death in the first two years.

Type 2: (Fixed rate). The loan is made now at a fixed rate of interest, due at the end of two years That is, the individual just takes a short bond position. The individual will also purchase a nontypical insurance, which will provide for repayment of the loan in the event of death in the first year. We can think of this insurance as providing a death benefit at time 2 for death between time 1 and time 2, or equivalently, we can think of this insurance as providing at time 1, a bond maturing at time 2, should death occur between time 0 and 1.

These two securities, together with a one year and two year pure endowments are enough to replicate this option as follow .

The person borrows 1920 on the floating loan, and must pay 0.4(1920) to insure it, leaving proceeds of 1152 . Note that the amount due at time 2 is either 1920/.64 = 3000 or 1920/.48 = 4000 depending on the interest rate movement. They also short two- year bonds with a maturity of 2625 receiving, 2625(0.56) = 1470. From this they purchase the special insurance. Since the price of a one unit , one year bond at time 1 is either 0.8 or 0.6 with equal probability, the cost of this insurance will be 2625(0.2)(.8)(0.7) = 294. This leaves a net total of 1176. Adding the option premium, the total of 24 + 1152 + 1176 = 2352 at time 0 will be used to buy a 4000 unit pure endowment at age 62, for a cost of 1344 and a 1575 unit pure endowment at age 61 for a cost of 1008.

If death occurs in the first year, the special insurance covers the short position in bonds. and there are no further exchanges. Suppose that the individual lives to time 1. If the bond price is 0.8 The one year pure endowment pays 1575. Adding an amount of 525 gives 2100, which is just sufficient to cover the short bond position. At time 2, if alive , the individual will receive 4000 from the annuities keeping 1000 and use the rest to repay the floating rate loan. If the bond price is 0.6, The proceeds from the one year pure endowment will exactly cover the short bond position. The existing pure endowment will repay the floating loan. One can then just pay the regular price for the pure endowment which is now under the strike price.

We can also look at matters from the point of view of the insurer who writes the option, and show how they can in each case hedge the option. The insurer of course has a built in way to hedge annuity options, namely by selling life insurance. If mortality improvement above that expected occurs, the gain on the insurance makes up for the loss on the annuity options. They can not hedge all of the risk however, since if the mortality worsens, nobody will exercise the option and they lose on the insurance. The price of the options makes up for the portion of the risk which cannot be hedged by insurance.

2.0.6 Example 4

We look at Example 1 again from the insurer's viewpoint. Suppose for example that the insurer sells 100 such contracts, collecting 2000. They then sell 50 , one year deferred ,one year term insurance policies on people age 60 with a face amount of 1000. These are contracts which pay a death benefit at age 62 provided the person dies between the ages of 61 and 62. The total premium for this will be 50,000(0.8)(0.25) =10,000, leaving a total premium income of 12,000 Assuming that their estimate that 20% of people age 60 will die within the year, materializes, they can expect 80 of the option buyers to be living at the end of the year. If the improvement occurs, we can assume that all 80 of these will exercise the option, each paying 750 for a total of 60,000. Out of these they can expect to pay out 1000 to each of 64 survivors at the end of the year. Their net loss on the annuity contracts is 4000. In addition they expect to pay out death claims of 8000 = 50,000(.8)(.2). If on the other hand the disease occurs,nobody will exercise the option. They can expect to pay out 50,000(.8)(.3) = 12,000 in death claims from the insurance. In either case, the premiums received from the options and the insurance in the first year, pay for the total payouts in the second year.

2.0.7 Example 5

We now look at Example 2 from the insurer's viewpoint. The strategy can be done as in Example 3 working backwards one period at a time. Suppose the improvement in year 1 occurs. The insurer is now facing a one period situation with the two possible annuity prices the following year being 750 or 650. With a strike price of 600, they will need 80 = 0.8(150+50)/2 units per surviving option holder at time 1 to cover this. Similarly if the improvement doesn't occur they will need 17.50 =0.7(50+0)/2 to cover. To achieve this, for every 200 options they sell at time 1, they will also sell 125 one year - one year deferred insurances with 1000 face amount. They collect 200(39) + 125(200) = 32,800 total income. If the improvement occurs, they will need 20,000 to cover the insurance costs of the 100 surviving policies which will leave 80 each for the 160 surviving option holders. If the improvement does not occur, they will need 30,000 to cover death benefits and have the required 2800 for the surviving option holders.

Next we consider insurer hedging with stochastic interest.

2.0.8 Example 6

We look at Example 3 from the insurer's viewpoint. This will involve selling a new type of insurance policy. It is sold at time 0 and provided the insured dies between time 1 and time 2 it pays at time 2, 1000 if the bond price per unit is 0.8 in the second year and 0 if it is 0.6. While this may not be directly achievable, it can be manufactured by selling a regular deferred policy for 4000 and then reinsuring with an indexed policy which pays 3000 if the bond price is 0.8 and 4000 if the bond price is 0.6. The premium received for the regular policy is 448 and the premium paid for the reinsurance is 384, so the effective premium received for the special insurance is 64.

For any typical group of 100 options granted the insurer will sell 100 special policies receiving a total of 2400 for the options and 6400 for the special policies. The company will borrow 52,800 at the prevailing rate for one year agreeing to pay back 66,000 at the end of the year. This will used to buy 110 bonds at 560 If a the bond price is 0.8 at the end of the year it can be assumed that the option will be exercised by all of the expected 80 survivors. They each will pay 525. The total together with the sale of 30 bonds will provide the funds to repay the loan. The total expected benefits payments at the end of the year including both annuity and insurance payments is 80000 regardless of the mortality. and this is provided by the proceeds of the remaining 80 bonds. If the bond price is 0.6 at the end of the year, the company sells all 110 bonds at 600 and repays the loan.

2.1 General Formula for the Option in Discrete Time

We can now give a general formula for the option to purchase at time k a contract paying 1 unit at time n if the individual is alive at that time, for a strike price of Λ. The option price will just the expectation under Q of the difference between the market price and the strike price at time of exercise (if positive), discounted with both interest and mortality back to time 0. It is given by:

$$C_x(k, n|\Lambda) = E_Q\left[\prod_{i=1}^{k} \Lambda_x(i-1, i) \max[\Lambda_x(k, n) - \Lambda, 0]\right]$$

2.2 Annuity Options vs. Pure endowment Options

In this section we provide examples, as indicated above, to show that we can not in general consider an option on an annuity as equivalent to a basket of pure endowment options

Consider such a basket of options, one for each integer n, $k+1 \leq n \leq N$, (where N is the last duration at which members of the particular cohort will be living). The n-th contract gives the purchaser the option to purchase at time k, a 1-unit pure endowment maturing at time n for a strike price of K_n. Let $K = \sum_{n=k+1}^{N} K_n$ Now the n-th option will be exercised if and only if $K_n \leq \Lambda(k, n)$ while the annuity option with strike price K will be exercised iff $K \leq \sum_{n=k+1}^{N} \Lambda(k, n)$. In practice we would normally be given the strike price K for the annuity. The question is then, can we find as sequence K_n summing to K so that all or none of the individual pure endowments will be exercised. If we can find such a sequence then we could indeed reduce the pricing of annuity options to that of pricing pure endowment options. In the simplified lattice model where there are only finally many outcomes at each time, it is easy to derive conditions for this to hold. Fix k and consider the option to buy at time k, a 1-unit life annuity beginning at time $k+1$ Consider the possible outcomes, numbered $1, 2, \ldots s$ for interest and mortality at time k Let $\Lambda^i(k,n)$ be the price at time k of a one unit pure endowment payable at time n assuming outcome i. Let

$$a^i(k) = \sum_{n=k+1}^{N} \Lambda^i(k,n) \tag{8}$$

the price for a one unit life annuity at time k assuming outcome i at time k. We can partially order outcomes at time k by declaring that

$$i \leq j \text{ if } \Lambda^i(k,n) \leq \Lambda^j(k,n) \tag{9}$$

for all $n = k+1, \ldots N$.

THEOREM *Annuity options with exercise date k can be priced by pure endowment options iff the above ordering is linear.*

PROOF. Suppose the ordering is linear. Choose the minimal i such that $a^i(k) \geq K$, and then choose K_n so that $K_n \leq \Lambda^i(k,n)$ and $\sum_{k=1}^{N} K_n = K$. If an outcome j, with $a^j(k) \geq K$ occurs. Then $j \geq i$ and the annuity option will be exercised as will all the pure endowment options. If an outcome j with $a^j(k) < K$ occurs then necessarily $j < i$ and none of the pure endowment options will be exercised.

Conversely, suppose the ordering is not linear. Then we can find indices i, j, m, n such that

$$\Lambda^i(k,n) > \Lambda^j(k,n), \quad \Lambda^j(k,m) > \Lambda^i(k,m) \tag{10}$$

Suppose that $a^i(k) \leq a^j(k)$. and let $K = a^i(k)$ Consider a basket of pure endowment options with strike prices (K_n) summing to K. For a strike price of K the annuity option will be exercised if outcome i or j occurs, but we claim that not all the pure endowment options will be exercised for both outcomes i and j. Note first that in order for both the options payable at time m and n be exercised for both outcomes we would need

$$K_m \leq \Lambda^i(k,m) \tag{11}$$

and

$$K_n \leq \Lambda^j(k,n) < \Lambda^i(k,n) \tag{12}$$

This in turn implies that

$$\sum_{r \neq n,m} K_r > \sum_{r \neq n,m} \Lambda^i(k,r) \tag{13}$$

and therefore that at least one pure endowment option for times other than m, n will not be exercised in outcome i k.

EXAMPLE.

Suppose that the bond prices D(0,1), D(1,2) D(2,3) are given by

		800
	800	
800		600
	600	
		400

with equal probabilities of 1/2 on the up and down movements.

and the lattice of q's is given by

		0.7
	0.5	
0.2		0.6
	0.3	
		0.5

with equal probabilities of 1/2 on the up and down movements. Suppose that the option is given to purchase at time 1, a 1000 unit - two year annuity with the first payment at time 2 for a strike price of 497. At time 1 there are four possible events depending on whether bond prices are high and low, and whether the $q's$ are high or low. We need only consider two of these, namely outcome 1. high bond prices and high values of q , and outcome 2, low bond prices and low values of q. We have

$$\Lambda^1(1,2) = 400 < \Lambda^2(1,2) = 420 \tag{14}$$

but

$$\Lambda^2(1,3) = 98 > \Lambda^2(1,3) = 94.50 \tag{15}$$

so the condition is not met. In the example above, note that the changes in mortality have a monotone property. If we have at any time, a value of q that is higher than another, that all possible future values arising from rate will be higher than those arising from the other. If interest were deterministic we would achieve the linear ordering, but the problem is that interest rates have the same behaviour. If we modify this however by asserting a mean reverting nature to interest behaviour then it becomes easier to achieve the linear ordering. Suppose for example that the bond prices have the same values as above, but for each of the two outcomes at time 1, the probability that the prices will be 600 at time 2 is 3/4 rather than 1/2. Now we get the same values for outcome 1 and also

$$91 = \Lambda^1(1,3) < \Lambda^2(1,3) = 103.95 \tag{16}$$

and we do have linear ordering (it is clear that the other two outcomes will give sequences larger than outcome 2 and smaller than outcome 1).

3 Continuous-Time Model.

In this section we develop a continuous-time model of the annuity price curve, which is driven by two independent 'short rates'; the instantaneous interest rate and hazard rate. We derive annuity-bond (pure endowment contract) prices and then value options using a two factor Gompertz-CIR model. We start with some basic notation and terminology, which is well established in the bond pricing literature. We then extend these ideas to the mortality arena.

3.1 General Framework, Notation and Terminology

Following the approach taken by Duffie (1996) for the pricing of defaultable bonds, we model a hazard-plus-interest rate, which is denoted by:

$$\xi_t = r_t + h_t, \tag{17}$$

where r_t is the instantaneous risk-free rate of interest, and h_t is the hazard rate. In contrast to the corporate bond pricing literature, and quite naturally, we assume that r_t is independent of h_t. We also ignore recovery rates since, in the personal insurance context, default is complete.

3.1.1 Default-Free Bonds

In the absence of mortality (default) risk, the time$-t$ price of a zero-coupon bond, which matures at $T > t$, is denoted by, and equal to:

$$D_t(T) := E_t^* \left[e^{-\int_t^T r_u du} \right]. \tag{18}$$

This pricing axiom is the widely cited fundamental theorem of bond pricing. The time$-t$ bond price is equal to risk neutral expected value of the bond price in the next infinitesimal period, discounted at the current short rate. The risk neutral expectation $E_t^*[.]$, should be contrasted with the real world expectation, which is denoted by $E_t[x]$. See Musiela and Rutkowski (1997, ch. 12) for details on the relationship between the two possible expectations.

Clearly, in the trivial (deterministic, constant) case that $r_t = r$, equation (18) collapses to $D_t(T) = \exp\{-r(T-t)\}$. On the other hand, when r_t can be represented by a continuous-time stochastic process, equation (18) reduces to evaluating the Laplace transform of the random variable $\int_t^T r_t dt$. Quite commonly, the expression $D_t(T)$ is also written as:

$$D_t(T) = e^{-y_t(T)(T-t)}, \tag{19}$$

where $y_t(T)$ denotes the time$-t$ yield to maturity of a zero-coupon bond. The family $\{y_t(T); \forall T \geq t\}$ is referred to as the yield curve, or term structure of interest rates. See ibid. or Hull (2000, ch. 17) for an in depth discussion.

With some slight abuse of notation, we use $D_t(s,T)$ to denote the time$-t$ forward price of a zero-coupon bond maturing at time $T \geq s$. (A commitment is made at time t to purchase a zero-coupon bond at time $s \geq t$.) Applying the fundamental pricing equation, or a simple cost of carry argument, we have that:

$$D_t(s,T) = E_t^* \left[e^{-\int_s^T r_u du} \right] = E_t^* \left[e^{\int_t^s r_u du} \, e^{-\int_t^T r_u du} \right] = \frac{D_t(T)}{D_t(s)}, \qquad \forall \quad T \geq s \geq t. \tag{20}$$

Finally, the instantaneous forward rate at time s, is defined equal to:

$$\begin{aligned} f_t(s) &= \lim_{T \downarrow s} \frac{\ln[D_t(s)] - \ln[D_t(T)]}{T-s} \\ &= \lim_{T \downarrow s} \frac{y_t(T)(T-t) - y_t(s)(s-t)}{T-s}. \end{aligned} \tag{21}$$

This leads to the well-known forward-based pricing equation for a zero-coupon bond price:

$$D_t(T) = e^{-\int_t^T f_t(u) du},$$

which we will generalize to the annuity context.

3.1.2 Mortality Functions

Using the same framework, the probability of survival to time T – conditional on being alive at time t – is denoted and equal to:

$$p_t(T) := E_t \left[e^{-\int_t^T h_u \, du} \right]. \tag{22}$$

Equation (22) implicitly assumes we are focusing on one cohort group, all born in a particular year. In the event we want to distinguish between different cohorts, we will attached the birth year y, to the probability $p_t(T|y)$.

To actuaries, equation (22) may seem like an odd way to represent survival functions. But, in fact, this generalization is consistent with, and has embedded within it, traditional practice. For example, in the trivial case that the hazard (failure) rate is constant, $h_t = \lambda$, the probability of survival is given by: $p_t(T) = \exp\{-\lambda(T-t)\}$. This constant fore of mortality assumption is synonymous with a future lifetime that is exponentially distributed.

In the more general case that the hazard rate itself is stochastic – which is the essence of our paper – the probability of survival can be represented via the Laplace transform of the integral of the hazard rate.

Actuaries might be more familiar with the expectation of the hazard rate, $\mu_s = E[h_s]$, as the underlying pricing factor. Using their approach, $p_t(T) = \exp\{-\int_t^T \mu_u du\}$, where μ_u is the instantaneous force of mortality. However, as we argued in the introduction to the paper, we prefer to define the force of mortality, analogous to an interest rate, where

$$\lambda_t(s) = \lim_{T \downarrow s} \frac{\ln[p_t(s)] - \ln[p_t(T)]}{T - s}, \qquad \forall \quad T \geq s \geq t. \tag{23}$$

In this framework, the instantaneous force of mortality is defined directly from the probability of survival, which, in turn, is driven by the stochastic hazard rate. In fact, a simple application of Jensen's inequality reveals that:

$$p_t(T) = E\left[e^{-\int_t^T h_u du}\right] = e^{-\int_t^T \lambda_t(u) du} \geq e^{-\int_t^T E[h_u] du} = e^{-\int_t^T \mu_u du}, \tag{24}$$

which, in turn, implies that $\lambda_t(u) \leq \mu_u$. Practically speaking, we are arguing that the traditional force of mortality, used to price insurance and annuity claims, is less than the expected value of the hazard rate. Only when the uncertainty in the hazard rate is set to zero, do the two expressions coincide.

The idea of a stochastic hazard rate does not seem to have previously appeared in the actuarial literature. However, related concepts have been employed by population biologists in studying human aging. See Woodbury and Manton (1977) or Yashin, Manton, and Vaupel (1985).

3.1.3 Mortality Contingent-Claim Curve

As in the discrete time model, we start with pure endowment contracts, which can then be joined (summed) to form a life annuity. This is analogous to creating coupon bearing bonds using a mixture of zero coupon contracts. In continuous time, the price of pure endowment contract, with maturity T, is denoted and equal to:

$$\begin{aligned}\Lambda_t(T) &= E_t^*\left[e^{-\int_t^T h_u\, du}\right] E_t^*\left[e^{-\int_t^T r_u\, du}\right] \\ &= E_t^*\left[e^{-\int_t^T (h_u + r_u)\, du}\right] = E_t^*\left[e^{-\int_t^T \xi_u\, du}\right].\end{aligned} \tag{25}$$

Intuitively, the pure endowment price is equal to the product of the default-free bond price and the probability of survival. Naturally, $\Lambda_t(T) \to 0$, when $T \to \infty$. Also, as in the case of survival probability, the notation $\Lambda_t(T|y)$ will be used when a particular cohort is being identified.

As one can see from the superscript, the expectations in equation (25) are taken with respect to a unique risk neutral probability measure. In the context of bond pricing this is quite natural, and to be expected. However, a risk neutral mortality expectation may seem redundant, or even superfluous. Our notation is meant to draw attention to the fact that 'complete market' pricing for annuity-bonds, implies the ability to entirely eliminate mortality risk. In other words, each individuals subjective probability of survival is irrelevant – just as subjective expectations are irrelevant for bond pricing – all that matters is the risk neutral expectation of the insurance company, which is the aggregate survival function. As we argued in the previous section, the mortality risk can be eliminated by suitably constructing a portfolio of insurance and life annuities.

Analogous with the (default free) zero-coupon bond, we define $\Lambda_t(s,T)$ to represent the time$-t$ forward price of a pure endowment maturing at time $T \geq s$. In the context of a life annuity, a commitment is made at time t to purchase a pure endowment at time $s \geq t$. We emphasize that $\Lambda_t(s,T)$ is not exactly a deferred annuity (in the classical actuarial sense), since premium payment is paid at time $s \geq t$, contingent on survival. In any event, applying the fundamental pricing equation, we have that:

$$\Lambda_t(s,T) = E_t^*\left[e^{-\int_s^T \xi_u du}\right] = \frac{\Lambda_t(T)}{\Lambda_t(s)}, \qquad \forall \quad T \geq s \geq t. \tag{26}$$

Following the same line of reasoning, the time$-t$ price of a call option to acquire a pure endowment with maturity T, at time $s \leq T$, for a fixed price of Λ, is denoted by $c_t(s,T,\Lambda)$. The payoff at maturity is:

$$C_s(s,T,\Lambda) = \max\left[\Lambda_s(T) - \Lambda, 0\right] \tag{27}$$

The complete-market (or No Arbitrage) price of this call option is:

$$\begin{aligned} C_t(s,T,\Lambda) &= E_t^*\left[e^{-\int_t^s \xi_u \, du} \max\left[\Lambda_s(T) - \Lambda, 0\right]\right] \\ &= E_t^*\left[e^{-\int_t^s \xi_u \, du} \max\left[E_s^*\left[e^{-\int_s^T \xi_u \, du}\right] - \Lambda, 0\right]\right]. \end{aligned} \tag{28}$$

As one would expect intuitively, as $\Lambda \to 0$, $c_t(s,T,\Lambda) \to \Lambda_t(T)$, We now proceed to derive explicit expressions for $\Lambda_t(T)$ and $C_t(s,T,\Lambda)$, assuming a particular parametrization of the process ξ_t.

3.2 Gompertz-CIR Assumption

3.2.1 The Underlying Process.

In this section assume that r_t obeys the Cox-Ingersoll-Ross (1985) specification and the h_t obeys a Gompertz expectation with square root volatility. The CIR assumption is one of the most popular short-term models for interest rates. In contrast, the Gompertz assumption may not be very familiar to financial economists, but it ubiquitous in the actuarial literature. Our only innovation is to assume that the expected value of the stochastic hazard rate obeys a Gompertz specification. The actual hazard rate process itself follows a square root law. More specifically:

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r \sqrt{r_t} dB_t^r, \tag{29}$$

$$dh_t = m h_t dt + \sigma_h \sqrt{h_t} dB_t^h. \tag{30}$$

The two factors (Brownian motions) driving h_t and r_t are independent. The three interest rate parameters, κ, θ and σ_r, are risk-neutralized, as are the mortality parameters m and σ_h.

Equation (29) and (30) imply that both $r_s|r_t$ and $h_s|h_t$, properly scaled, are non-central chi-square distributed.[2] This fact can be traced to original results by Feller (1951), but is widely attributed to Cox-Ingersoll-Ross (1985). In this particular formulation of the hazard rate, the process does not display any mean reversion. The expect value and variance of $r_s|r_t$ is:

$$E\left[r_s|r_t\right] = r_t e^{-\kappa(s-t)} + \theta(1 - e^{-\kappa(s-t)}) \tag{31}$$

$$\text{var}\left[r_s|r_t\right] = r_t \left(\frac{\sigma_r^2}{\kappa}\right) \left(e^{-\kappa(s-t)} - e^{-2\kappa(s-t)}\right) + \theta \left(\frac{\sigma_r^2}{2\kappa}\right) \left(1 - e^{-\kappa(s-t)}\right)^2, \tag{32}$$

while the expect value and variance of $h_s|h_t$ is:

$$E\left[h_s|h_t\right] = h_t e^{m(s-t)} \tag{33}$$

$$\text{var}\left[h_s|h_t\right] = h_t \left(\frac{\sigma_h^2}{m}\right) \left(e^{2m(s-t)} - e^{m(s-t)}\right), \tag{34}$$

The properties of the CIR short rate process are well known in the term structure literature, so we avoid deriving them here. Taking the Laplace transform in equation (18), the bond price is equal to:

$$D_t(T) = C_1(t,T) e^{-r_t C_2(t,T)}, \tag{35}$$

where

$$C_1(t,T) = \left[\frac{2\gamma e^{(\kappa+\gamma)(T-t)/2}}{(\gamma+\kappa)(e^{\gamma(T-t)} - 1) + 2\gamma}\right]^{2\kappa\theta/\sigma_r^2}, \tag{36}$$

$$C_2(t,T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma+\kappa)(e^{\gamma(T-t)} - 1) + 2\gamma}, \tag{37}$$

[2] Technically speaking, the 'degrees of freedom' variable in our model for the hazard rate, is zero. Therefore, to be precise, we must define all relevant quantities for the non-central chi-square distribution, as the appropriate limit. See Siegel (1979) for a discussion of the properties of the non-central chi-square distribution with zero degrees of freedom.

and where $\gamma = \sqrt{\kappa^2 + 2\sigma_r^2}$.

As for the hazard rate process, our particular specification may at first seem arbitrary. However, in selecting our model, we desired a (i) an expected hazard rate that corresponds with the Gompertz specification, and (ii) a variance that is proportional to the level of the hazard rate, and (ii) analytic tractability. Indeed, equation (30) allows for all three. An unfortunate by-product is that a Gompertz-CIR specification allows for the possibility of h_t reaching zero, and does not incorporate mean reversion. In the appendix, we discuss the implications of extending our model to an alternative hazard rate process.

In any event, according to equation (22), the probability of survival is given by:

$$p_t(T) = e^{-h_t C_3(t,T)}, \tag{38}$$

where

$$C_3(t,T) = \frac{2(e^{\delta(T-t)} - 1)}{(\delta - m)(e^{\delta(T-t)} - 1) + 2\delta}, \tag{39}$$

and where $\delta = \sqrt{m^2 + 2\sigma_h^2}$. This follows from 'plugging' into the standard CIR bond price with $\kappa = -m$ and $\theta = 0$. The conditional probability of survival, as specified by equation (38), depends on the current hazard rate h_t, the growth rate of the expected hazard rate m, and the 'volatility' of the growth rate, σ_h. The higher the volatility factor, the greater is the probability of survival. For example, if we let $t = 0$, $T = 20$, $h_0 = 0.03$, and $m = 0.1$, then $p_0(20) = 0.2684$ when $\sigma_h = 0.05$, but $p_0(20) = 0.9069$ when $\sigma_h = 0.5$.

Indeed, when $\sigma_h \to 0$, equation (38) leads to, $\exp\{-h_t(\exp\{m(T-t)\} - 1)/m\}$, which is an alternative representation of the classical Gompertz specification. Also, following our definition of the force-of-mortality, (namely a forward rate), as defined by equation (23), we get that:

$$\lambda_t(s) = h_t \frac{4\delta^2 e^{\delta(s-t)}}{\left(m(1 - e^{\delta(s-t)}) + \delta(1 + e^{\delta(s-t)})\right)^2}, \tag{40}$$

which, in the limit, when $\sigma_h \to 0$, (and therefore $\delta \to m$) collapses to $\lambda_t(s) = h_t \exp\{m(s - t)\}$, which is the familiar Gompertz specification. We emphasize, once again, that the expected value of the hazard rate process, as specified by equation 30, is not equivalent to the classical definition of a hazard rate or force of mortality. The analogous quantity, which generates the probability of survival, is defined by equation (40).

3.2.2 Pure Endowment Pricing

Finally, we obtain closed-form expression for $\Lambda_t(T)$, by combining equations (36), (37) and (39), to yield:

$$\Lambda_t(T) = p_t(T) D_t(T) = C_1(t,T) e^{-r_t C_2(t,T) - h_t C_3(t,T)}. \tag{41}$$

Equation (41) represents the price of an annuity-bond (pure endowment contract), and is the mortality-contingent analogue of a zero-coupon bond. Equation (41) contains five free parameters, which are $\kappa, \theta, \sigma_r, m, \sigma_h$. The main distinction between equation (41) and the well-known CIR bond pricing formula, is that we allow for default, in a way that is consistent with actuarial theory. On a slightly more technical level, equation (41) represents a bond price under a two factor CIR model, where the second factor exhibits zero mean reversion. The pricing of option in a two factor CIR models has been analyzed by Chen and Scott (1992), and we shall therefore use some of their techniques for pricing options on life annuities.

3.2.3 Numerical Examples for Endowment Prices

For example, when the current short rate is $r_0 = 0.06$ while $t = 0, s = 20$ and $\kappa = 0.15, \theta = 0.08, \sigma_r = 0.02$, we get that a generic (default) free zero coupon bond is worth $D_0(20) = \$0.23$, as per equation (35). However, when we incorporate a hazard rate function, with parameters $h_0 = 0.02$, $m = 0.1$ and $\sigma_h = 0.05$, we get that $\Lambda_0(20) = \$0.096$, as per equation (41). Once again, the importance of the volatility is evident. If we let $\sigma_h = 0.5$, we obtain $\Lambda_0(20) = \$0.216$. The very high price, compared to the certainty of $D_0(20)$, should probably be attributed to the undesirable probability of the hazard rate hitting zero. However, the volatility one would use in practice, should be much smaller than $\sigma_h = 0.5$, so that the probability of hitting zero is minuscule.

3.2.4 Option Values

In the two factor general case of stochastic interest and hazard rates, one must resort to the formula provided by Chen and Scott (1992) for the option price. In the specific case of constant (zero) interest rates, the appropriate formula comes from the original Cox-Ingersoll-Ross (1985) formula (pg. 396), suitably modified to incorporate the parameters for our hazard rate.

$$\begin{aligned} C_t(s,T,\Lambda) &= \Lambda_t(s)\chi^2\left(2h^*(\delta_1+\delta_2+C_3(T,s)); 0; \frac{2\delta^2 h_t e^{\delta(T-t)}}{\delta_1+\delta_2+C_3(T,s)}\right) \\ &\quad - \Lambda\Lambda_t(T)\chi^2\left(2h^*(\delta_1+\delta_2); 0; \frac{2\delta^2 h_t e^{\delta(T-t)}}{\delta_1+\delta_2}\right), \end{aligned} \tag{42}$$

where $\chi^2(.)$ is the non-central chi-square distribution, δ is defined as before, Λ is the strike price, and:

$$\delta_1 = \frac{2\delta}{\sigma^2\left(e^{\delta(T-t)}-1\right)}, \quad \delta_2 = \frac{\delta - m}{\sigma^2}, \quad h^* = \frac{\ln[\Lambda^{-1}]}{C_3(T,s)}. \tag{43}$$

4 Conclusion.

In this paper we have proposed a model for pricing options on future mortality (and interest) rates. These option currently exist in the market place and very little has been written on how they should be priced or reserved against. These options 'pay off' if life annuity (pure endowment) prices are higher than some pre-specified strike price, at the time the contract was issued. We have presented both a discrete and continuous time model, and have demonstrated how to hedge these options using pure endowments, default free bonds and life insurance contracts.

Our main conceptual contribution is in treating the actuarial hazard rate itself as a stochastic variable, as opposed to a deterministic force of mortality. Indeed, in classical actuarial models, the option to annuitize – struck at the forward – should have zero value since actuaries do not allow for uncertainty in the mortality factors that will be applicable upon annuitization. In contrast, we treat the probability of survival as an expectation over the path of the hazard rate over time. Pure endowment prices can then be viewed as generating a mortality term structure. We interpret the classical force of mortality as a forward curve, off which all claims are priced – together with the default free curve – in a No Arbitrage framework. This mortality forward curve aggregates market expectations about futures mortality rates, but at the same time can change over time in response to new information. This approach is similar in spirit to recent work by Mullin and Philipson (1997), where they use market prices for term life insurance policies to imply expectations of future mortality rates. However, we take this process one step further by assuming the market has a cohort specific expectation that actually changes over time. The stochasticity in the future survival probabilities for any particular cohort is what gives value to the option. Of course, the problem then becomes to model the evolution of this mortality forward curve in a parsimonious and consistent manner. We have adopted a hazard-plus-interest rate model based on the Cox-Ingersoll-Ross (1985) process, with an expectation equal to the Gompertz specification. Option prices can be obtained using a two-factor procedure, as described by Chen and Scott (1992), for example. In the appendix we provide an alternative specification. Future research will attempt to 'fit' market annuity (endowment) prices to our formulas, in an attempt to calibrate the parameters and to determine the appropriate hazard rate models in practice.

5 Appendix

One of the unsatisfying properties of the hazard rate process described in the body of the paper, is the (small) possibility that h_t reaches zero. (Although h_t will never become negative under the Gompertz-CIR specification.) An additional drawback is the lack of mean reversion. Of course, this is the price that one has to pay for analytic tractability. Nevertheless, in this appendix we present an alternative model for the hazard rate which corrects both shortcomings. Specifically, we bound the process away from zero and allow for mean reversion. To focus attention on the hazard rate, and our alternative specification, we will limit our analysis to computing endowment (bond) prices (i.e. survival probabilities) with constant interest rates. The techniques and models used in the body of the paper for the interest rate uncertainty, can be applied without any modifications as a result of the independence assumption.

Our process is a generalization of geometric Brownian motion, but with a variance that is lower. We label it the mean reverting Gompertz-GBM specification, for which the hazard rate process obeys the diffusion:

$$h_t = h_0 e^{gt+\sigma Y_t}, \qquad g, \sigma, h_0 > 0 \tag{44}$$

$$dY_t = -bY_t dt + dB_t, \qquad Y_0 = 0, b \geq 0. \tag{45}$$

Clearly, when $b = 0$, the process is a geometric Brownian motion, since Y_t collapses to B_t. However, as b increases, Y_t – which is known as an Ornstein Uhlenbeck process – displays stronger mean reversion. The stochastic differential equation for Y_t can be solved explicitly to yield:

$$Y_t = \int_0^t e^{-b(t-u)} dB_u, \tag{46}$$

with a mean value (stochastic integral) of zero, and a variance (by Ito's isometry) of:

$$\sigma_t = E[Y_t^2] = \int_0^t \left(e^{-b(t-u)}\right)^2 dB_u = \frac{1-e^{-2bt}}{2b}, \tag{47}$$

which goes to t as $b \to 0$, and is always smaller than t, for $b > 0$. In other words, the process has a smaller variance compared to a standard Brownian motion. The log-hazard rate $\ln[h_t]$, is therefore *normally* distributed with a mean value of $\ln[gt]+\ln[h_0]$ and a variance of $\sigma^2(1-\exp(-2bt))/2b$.

To gain further insight into the structure of h_t, if we let: $F(t, y) := h_0 \exp\{gt + \sigma y)$,then by Ito's lemma, we have:

$$dF(t,y) = \frac{\partial F(t,y)}{\partial t} dt + \frac{\partial F(t,y)}{\partial y} dy + \frac{1}{2}\frac{\partial^2 F(t,y)}{\partial y^2} d\langle Y \rangle_t. \tag{48}$$

This, in turn, implies that:

$$dh_t = gh_t dt + \sigma h_t dY_t + \frac{\sigma^2}{2} h_t dt \tag{49}$$

$$= gh_t dt + \sigma h_t \left(-bY_t dt + dB_t\right) + \frac{\sigma^2}{2} h_t dt. \tag{50}$$

Finally, since $Y_t = (\ln[h_t/h_0] - gt)/\sigma$, as per the definition in equation (44), arrive at:

$$dh_t = gh_t dt + \sigma h_t \left(\frac{-b(\ln[h_t/h_0] - gt)}{\sigma} dt + dB_t \right) + \frac{\sigma^2}{2} h_t dt \tag{51}$$

$$= \left(g + b\ln[h_0] + \frac{\sigma^2}{2} - b\ln[h_t] + bgt \right) h_t dt + \sigma h_t dB_t, \tag{52}$$

which, interestingly enough, is the Black-Derman-Toy (1991) model for the short-rate. See Black and Karasinsky (1990) for more details.

As we mentioned earlier, this particular specification does not allow us to obtain a closed-form expression for the bond/option price, since the relevant expectation involves the sum of lognormal variables, which does not have a known analytic density function. We will, however, employ moment-matching techniques, similar to recent work by Hansen and Jorgensen (1999), but applied to the Reciprocal Gamma distribution. Our ultimate objective – as in the body of the paper – is to obtain an expression for the probability of survival $p_t(T)$. To this end, we must compute the (higher) moments of h_t, Y_t and $\int_0^t h_s ds$. The moments of Y_t are given by

$$E[Y_t^{2n-1}] = 0, \ n = 1, 2, \ldots, \tag{53}$$

$$E[Y_t^{2n}] = E[(\frac{Y_t}{\sigma_t})^{2n}] \ \sigma_t^{2n} \tag{54}$$

$$= \frac{(2n)!}{2^n (n!)} \left[\frac{(1 - e^{-2bt})}{2b} \right]^n$$

$$= \frac{(2n)!}{(n!)} \left[\frac{(1 - e^{-2bt})}{4b} \right]^n, \ n = 1, 2, \ldots .$$

The (non-central) moments of h_t are given by

$$E[h_t^n] = E\left[(h_0 e^{gt + \sigma Y_t})^n\right] \tag{55}$$

$$= h_0^n \ e^{ngt} \ E\left[e^{n\sigma Y_t}\right]$$

so

$$E[h_t^n] = h_0^n \ e^{ngt} \ \exp\left[\frac{(n\sigma)^2}{2} \left(\frac{1 - e^{-2bt}}{2b} \right) \right], \ n = 1, 2, \ldots . \tag{56}$$

For example, when $n = 1$, we have the expected hazard rate of:

$$E[h_t] = h_0 \exp\left(gt + \frac{\sigma^2}{2} \left(\frac{1 - e^{-2bt}}{2b} \right) \right) \tag{57}$$

The (non-central) moments of $\int_0^T h_u \, du$ are given by

$$E\left[\left(\int_0^T h_u \, du\right)^n\right] = E\left[\prod_{i=1}^n \left(\int_0^T h_{u_i} \, du_i\right)\right] \tag{58}$$

$$= E\left[\int_0^T du_1 \int_0^T du_2 \cdots \int_0^T du_n \prod_{i=1}^n h_{u_i}\right]$$

$$= \int_0^T du_1 \int_0^T du_2 \cdots \int_0^T du_n \; E\left[\prod_{i=1}^l h_{u_i}\right] \; , \; n = 1, 2, \ldots$$

Since $h_{u_i} = h_0 e^{g u_i \, + \, \sigma Y_{u_i}}$, then

$$\prod_{i=1}^n h_{u_i} = h_0^l \; \exp\left[g \sum_{i=1}^n u_i\right] \exp\left[\sigma \sum_{i=1}^n Y_{u_i}\right] \; , \tag{59}$$

so

$$E\left[\prod_{i=1}^n h_{u_i}\right] = h_0^n \; \exp\left[g \sum_{i=1}^n u_i\right] \; E\left[\exp\left(\sigma \sum_{i=1}^n Y_{u_i}\right)\right] \; . \tag{60}$$

The random variable $\sum_{i=1}^n Y_{u_i}$ is normally distributed with mean zero, and if we denote its standard deviation by $\widehat{\sigma}$, then

$$E\left[\exp\left(\sigma \sum_{i=1}^n Y_{u_i}\right)\right] = E\left[\exp\left(\sigma \widehat{\sigma}\left[\left\{\sum_{i=1}^n Y_{u_i}\right\} \; / \; \widehat{\sigma}\right]\right)\right] \tag{61}$$

$$= e^{(\sigma \widehat{\sigma})^2/2} \; ,$$

so

$$E\left[\prod_{i=1}^n h_{u_i}\right] = h_0^n \; \exp\left[g \sum_{i=1}^n u_i\right] \exp\left[\frac{\sigma^2}{2} Var\left(\sum_{i=1}^n Y_{u_i}\right)\right] \; . \tag{62}$$

But

$$Var\left(\sum_{i=1}^n Y_{u_i}\right) = E\left[\left(\sum_{i=1}^n Y_{u_i}\right)^2\right] \tag{63}$$

$$= E\left[\sum_{i,j=1}^n Y_{u_i} Y_{u_j}\right]$$

$$= \sum_{i,j=1}^n E\left[Y_{u_i} Y_{u_j}\right] \; .$$

Suppose $u_i \leq u_j$. Then,

$$Y_{u_j} = \int_0^{u_j} e^{-b(u_j - v)} dB_v \tag{64}$$

$$= \int_0^{u_i} e^{-b(u_j - v)} dB_v + \int_{u_i}^{u_j} e^{-b(u_j - v)} dB_v$$

$$= e^{-b(u_j - u_i)} \int_0^{u_i} e^{-b(u_i - v)} dB_v + \int_{u_i}^{u_j} e^{-b(u_j - v)} dB_v$$

$$= e^{-b(u_j - u_i)} Y_{u_i} + \int_{u_i}^{u_j} e^{-b(u_j - v)} dB_v \; .$$

Now, the random variable $\int_{u_i}^{u_j} e^{-b(u_j-v)}dB_v$ is independent of Y_{ui}

$$E\left[Y_{u_i}Y_{u_j}\right] = E\left[Y_{u_i}\left(e^{-b(u_j-u_i)}Y_{u_i} + \int_{u_i}^{u_j} e^{-b(u_j-v)}dB_v\right)\right] = e^{-b(u_j-u_i)}E\left[Y_{u_i}^2\right] \ . \tag{65}$$

Therefore, for example:

$$E\left[\int_0^T h_u\ du\right] = h_0 \int_0^T du\ e^{gu} \exp\left[\frac{\sigma^2}{2}\left(\frac{1-e^{-2bu}}{2b}\right)\right] = \frac{h_0}{2b}\left(\frac{4b}{\sigma^2}\right)^{-\frac{g}{2b}} \exp\left(\frac{\sigma^2}{4b}\right)\ \left[\Gamma\left(-\frac{g}{2b}, \frac{\sigma^2}{4b}\ e^{-2bT}\right) - \Gamma\left(-\frac{g}{2b}, \frac{\sigma^2}{4b}\ \right)\right]\ , \tag{66}$$

where

$$\Gamma(a,x) = \int_x^\infty dv\ e^{-v}\ v^{a-1} \tag{67}$$

denotes the incomplete Gamma function. For $n=2$, we have:

$$\begin{aligned} E\left[\left(\int_0^T h_u\ du\right)^2\right] &= 2\ h_0^2 \int_0^T du_1 \int_0^{u_1} du_2\ \exp\left[g\left(u_1+u_2\right)\right] \\ &\exp\left[\frac{\sigma^2}{2}\left\{\left(\frac{1-e^{-2bu_1}}{2b}\right) + \left(\frac{1-e^{-2bu_2}}{2b}\right)\right\} + \sigma^2 e^{-b(u_1-u_2)}\left(\frac{1-e^{-2bu_2}}{2b}\right)\right] \\ &= 2\ h_0^2 \int_0^T du_1\ e^{gu_1} \exp\left[\frac{\sigma^2}{2}\left(\frac{1-e^{-2bu_1}}{2b}\right)\right] \int_0^{u_1} du_2\ e^{gu_2} \\ &\exp\left[\frac{\sigma^2}{2}\left(\frac{1-e^{-2bu_2}}{2b}\right) + \sigma^2 e^{-b(u_1-u_2)}\left(\frac{1-e^{-2bu_2}}{2b}\right)\right] \\ &= 2\ h_0^2 \exp\left(\frac{\sigma^2}{2b}\right) \int_0^T du_1\ e^{gu_1} \exp\left[-\frac{\sigma^2}{4b}e^{-2bu_1}\right] \int_0^{u_1} du_2\ e^{gu_2} \\ &\exp\left[-\frac{\sigma^2}{4b}e^{-2bu_2} + \frac{\sigma^2}{2b}\ e^{-b(u_1-u_2)}\left(1-e^{-2bu_2}\right)\right]\ . \end{aligned} \tag{68}$$

Unfortunately, the double integral can not be evaluated analytically, which forces us to rely on numerical methods for all moments of order two and higher.

Nevertheless, at this point, we have computed the first two moments of the $\int_0^T h_u du$, as opposed to the actual density function of $\int_0^T h_u du$. At this point we employ our moment-matching approximation. Specifically, we let M_1 and M_2 denote the first two moments of the random variable $X = \int_0^T \xi_u$, and then *assume* that X is Reciprocal Gamma distributed, based on the approximation developed in Milevsky and Posner (1998). This approximation is motivated by a well-known problem in Asian option pricing theory, and the fact that when $T \to \infty$, (and $b \to 0$), the random variable X *is*, in fact, Reciprocal Gamma distributed.

Distribution of $X = \int_0^T h_u du$ assumed Reciprocal Gamma						
(M_1)	**b=0**	**b=0.10**	**b=0.20**	**b=0.30**	**b=0.50**	**b=1.00**
T=5	0.0628638	0.0626348	0.0624912	0.0623966	0.0622843	0.0621698
T=10	0.1606358	0.1585660	0.1576991	0.1572621	0.1568449	0.1564948
T=15	0.3127002	0.3046010	0.3022050	0.3011796	0.3002843	0.2995761
T=20	0.5492056	0.5264107	0.5214231	0.5194889	0.5178672	0.5166157
T=25	0.9170418	0.8630347	0.8539579	0.8506416	0.8479179	0.8458423
T=30	1.4891365	1.3737489	1.3583797	1.3529662	1.3485708	1.3452452
T=35	2.3789136	2.1485001	2.1235357	2.1149406	2.1080096	2.1027877
T=40	3.7627816	3.3237457	3.2841983	3.2707772	3.2599998	3.2519016

Table 2: First Moment

Distribution of $X = \int_0^T h_u du$ assumed Reciprocal Gamma						
(M_2)	**b=0**	**b=0.10**	**b=0.20**	**b=0.30**	**b=0.50**	**b=1.00**
T=5	0.0040262	0.0039737	0.0039410	0.0039196	0.0038947	0.0038707
T=10	0.0268929	0.0256456	0.0251403	0.0248954	0.0246759	0.0245130
T=15	0.1046830	0.0949567	0.0922912	0.0912321	0.0903895	0.0898066
T=20	0.3330483	0.2839895	0.2745414	0.2712336	0.2687354	0.2670408
T=25	0.9612846	0.7635606	0.7358701	0.7268988	0.7202736	0.7158015
T=30	2.6327636	1.9343188	1.8609859	1.8382973	1.8217003	1.8105045
T=35	6.9982115	4.7297418	4.5463032	4.4910448	4.4507842	4.4236122
T=40	18.2781626	11.3153679	10.8714415	10.7397344	10.6439348	10.5792360

Table 3: Second Moment

$p_0(T)$	**b=0**	**b=0.10**	**b=0.20**	**b=0.30**	**b=0.50**	**b=1.00**
T=5	0.9391062	0.9393102	0.9394381	0.9395225	0.9396230	0.9397257
T=10	0.8520617	0.8535802	0.8542224	0.8545502	0.8548694	0.8551455
T=15	0.7339211	0.7382121	0.7395409	0.7401381	0.7406886	0.7411510
T=20	0.5858562	0.5927221	0.5944604	0.5952299	0.5959541	0.5965856
T=25	0.4205090	0.4257366	0.4271263	0.4278445	0.4285870	0.4292835
T=30	0.2622260	0.2589251	0.2590829	0.2594710	0.2600126	0.2606124
T=35	0.1369223	0.1230235	0.1217845	0.1217168	0.1219123	0.1221786
T=40	0.0575090	0.0406604	0.0390505	0.0387691	0.0387047	0.0387938

Table 4: Survival probability assuming a mean-reverting stochastic hazard rate and Reciprocal Gamma approximation.

In our context, we define two new parameters: $\alpha = (2M_2 - M_1^2)/M_2 - M_1^2$ and $\beta = (M_2 - M_1^2)/M_2 M_1$, which then leads to:

$$P[\int_0^T h_u \leq x] = P[X \leq x] \approx \int_0^x \frac{\beta^{-\alpha} u^{1-\alpha}}{\Gamma(\alpha) e^{u\beta} u^2} du \tag{69}$$

The (probability of) survival function is:

$$p_0(T) = E_0\left[e^{-\int_0^T h_u du}\right] = E_0\left[e^{-X}\right], \tag{70}$$

which is the Laplace transform of the RG distribution, (evaluated at minus one):

$$p_0(T) = \frac{2\beta^{(-\alpha/2)}\,\mathrm{BesselK}(\alpha, 2/\sqrt{\beta})}{\Gamma(\alpha)} \tag{71}$$

In summary, although we do not have a precise density function for the quantity $\int_0^t h_u du$, we approximate the integral by a Reciprocal Gamma distribution, which then implies that survival probabilities (endowment prices) are obtained via the Laplace transform.

To get a sense of the impact of mean reversion of hazard rate, on the survival probabilities, the following tables provides some numerical values for $p_0(T)$. We assume that $g = 0.083$ and $\sigma = 0.10$, and $h_0 = 0.01$.

As one can see from the table – and one would expect intuitively – the mean reversion does not have a significant impact on the survival probability (or the endowment bond prices) unless we are looking at relatively long time horizons. By the same argument, it is doubtful that one can statistically detect mean reversion in hazard rates given market quotes for annuity prices.

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Figure # 1

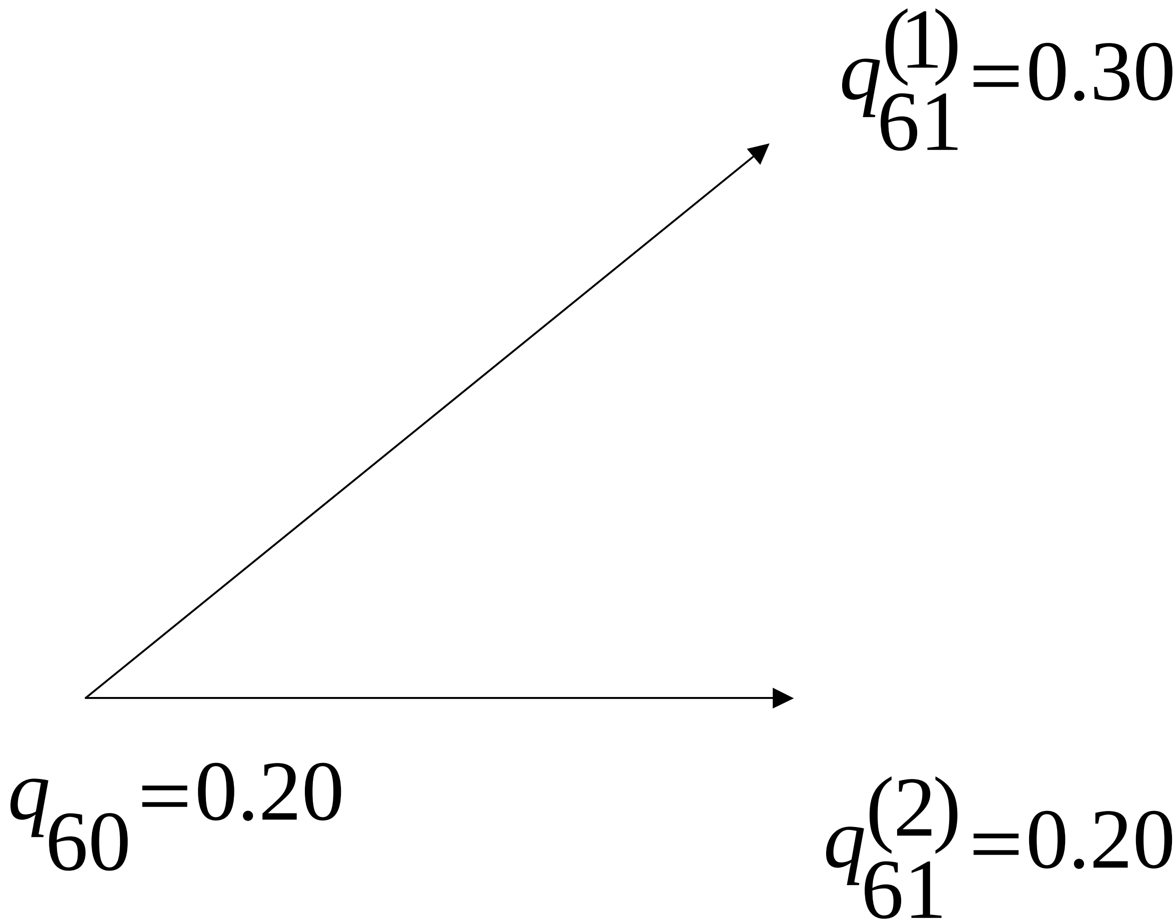

Figure # 2

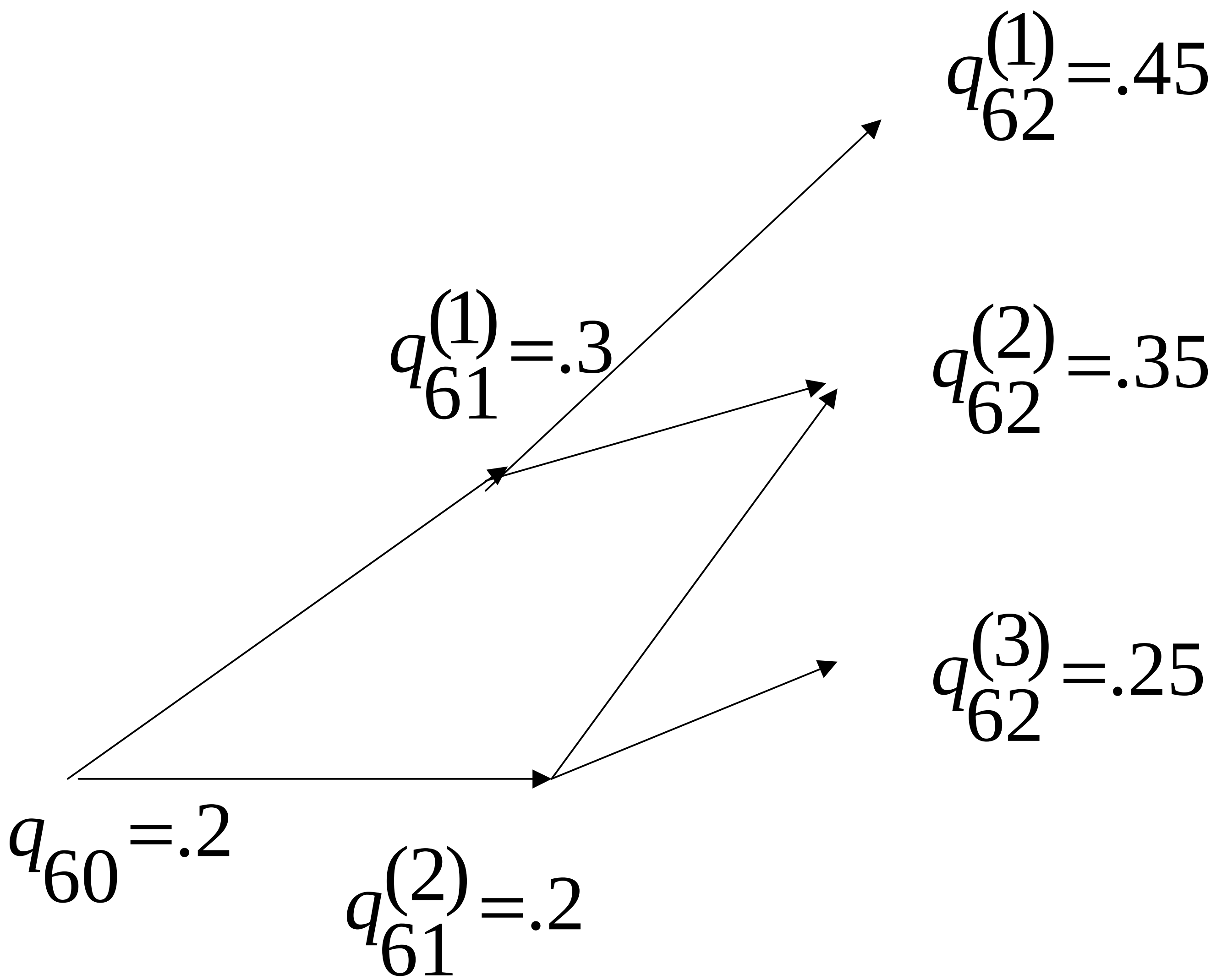

Figure # 3