Debt allocation: To fix or float?

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Abstract

We study an economic agent who has an exogenously determined initial amount of debt. The agent is equipped with a constant relative risk aversion utility function and a deterministic terminal wealth (before debt interest payments) and faces a debt allocation problem: The choice between fixed interest rate debt or floating interest rate debt. The problem is thus related to the seminal Merton (1969), Merton (1971) asset allocation problem. In order to model fixed and floating interest rates we use a version of the Hull and White (1990) term structure model, essentially the Vasicek (1977) model fitted to the initial term structure.

First, the static case is considered, where no rebalancing of debt is allowed after the initial point in time. Next, the dynamic case is treated where the debt portfolio can be rebalanced continuously at no cost. We find a surprisingly low increase in welfare, measured by expected utility, in the dynamic case compared to the static case. The optimal debt portfolio in the dynamic case is sensitive to the initial shape of the initial forward rates and therefore may or may not resemble the static case.

1 Introduction

The last decades financial institutions have developed new products in several areas. This rapid and innovative development leaves customers with more choices so more tailor-made financial solutions can be constructed, hopefully in better accordance with individuals' needs. In this paper we focus on the choice of fixed or floating rate loans.

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	floating		fixed		
	$> \frac{1}{2}$m	$< \frac{1}{2}$m	3 y	5 y	10 y
$< 60\%$	8.1%	8.45%	7.45%	7.40%	7.40%
$< 80\%$	8.85%	9.15%	7.9%	7.85%	7.85%

Table 1: Interest rate conditions of Postbanken Oct 30, 2002, found at `www.postbanken.no`.

As an example consider the Norwegian State Education Loan Fund, a government run organization under the Ministry of Education, which is the most important source of student financial aid in Norway. Earlier the loan interest rate was politically determined. Later years the loan interest rate has been determined by interest rates in the financial market and since 1999 customers have been given the choice of floating rates or 3 year fixed rates (in 2002 a third option of 5 year fixed rate was introduced).

Most Norwegian banks offer customers a menu of choices for mortgage financing. The interest rate conditions of the major Norwegian bank Postbanken are presented in table (1). The conditions depend on whether the loan amount is within 60% or 80% of the value of the house and whether the amount of loan is above or below 500 000 NOK (roughly 70 000 USD). The customer has to choose between floating, fixed for 3, 5, or, 10 years interest rates, and also the allocation of debt between these 4 alternatives.

To get some insight into this debt allocation problem we choose a simplified and idealized approach, which excludes important aspects as income, inflation or consumption choices. Many of these aspects are included in the article by Cambell and Cocco (2002), who use a different modelling approach to analyze the optimal mortgage choice.

We study an economic agent who borrows an exogenously determined amount of money. Two types of loan alternatives are present: Loan with a fixed interest rate through the loan horizon and loans with floating interest rates. Both fixed and floating rate interest rates are determined from the prevailing term structure of interest rates derived from a financial market.

The investor must determine his initial distribution between fixed rate and floating rate loan. Two extreme cases are treated with respect to intermediate rebalancing of the loan portfolio: No rebalancing and continuous rebalancing.

The problem is in many ways related to the classical Merton (1969), Merton (1971) problem. Merton studies the investment decision or asset allocation where capital may be invested either in a risky security or to the riskfree interest rate. Whereas Merton deals with allocation of assets, we focus on the liability side of the balance sheet and study debt allocation. In our set-up fixed rate debt corresponds to a 'risky' investment in the sense that the intermediate market value of the fixed-rate debt fluctuates randomly. Although we allow for stochastic interest rates, floating rate debt has similar dynamics as a bank or money market account which in Merton's model corresponds to the 'riskfree' investment.

A term structure model including random interest rates is essential to our problem. In order to keep things simple we use a one factor Gaussian spot interest rate process with mean reversion. The same process was first used by Vasicek (1977). By assuming a special structure of the market price of interest rate risk process the Vasicek model is consistent with the initial observable term structure. This extension is credited Hull and White (1990). By calibrating the model to the initial observable term structure one does not need to make ad-hoc assumption with respect to the market price of interest rate risk (like the typical example of a constant market price of interest rate risk). At the same time forward interest rates, which are observable, in our model as well as in real world financial markets, enter the model in a natural way.

Our set-up is similar to the recent literature on asset allocation in models with stochastic interest rates (Sørensen, 1999; Brennan and Xia, 2000; Bajeux-Besnainou, Jordan, and Portait, 2001; Munk and Sørensen, 2001). Brennan and Xia (2000) and Bajeux-Besnainou et al. (2001) suggest a solution to the apparent asset allocation puzzle established by Canner, Mankiw, and Weil (1997), However, most of these papers assume a constant market price of risk. This difference may play a crucial role when it comes to optimal allocations.

Section 2 of the article contains a description of one version of the Hull and White (1990) term structure model.

In section 3 we present results for a static model without the possibility to rebalance the loan portfolio before expiration. We derive a lower bound for the fixed rate which is interpreted as follows: The agent will not borrow at the floating rate (though he may wish to *lend* money at the floating rate) if the fixed rate is below this lower bound. This lower bound depends only on, in a specific sense, expected future interest rates. We also derive an upper bound for the fixed rate interpretable as follows: The agent will not borrow at the fixed rate (also for this case he may wish to lend money to the fixed rate) if the fixed interest rate is above this upper bound. In addition to expectations about the future interest rates this upper bound depends on characteristics of the agent related to his financial wealth and preferences. For values of the fixed rate between the lower and upper bounds it is optimal to keep positive fractions of both floating rate debt and fixed rate debt.

In section 4 the agent is allowed to rebalance his loan portfolio continuously at no cost. For this problem we derive closed form expressions both for optimal expected utility and optimal fractions of floating rate debt. We present some numerical comparisons of the dynamic case with the static case both in terms of welfare measured by optimal expected utility and in terms of initial fractions of floating rate debt. Our numerical examples indicate, perhaps surprisingly, only a marginal increase in optimal expected utility. It turns out that the interplay between the market price of risk and the initial forward rates plays an important role, especially for the optimal debt fractions. As a consequence the optimal initial fractions of debt may be substantially different from the static case. Several examples are included to illustrate this point.

Finally, section 5 concludes the article.

2 A term structure model

This section contains a description of a version of the Hull and White (1990) term structure model. A time horizon T is fixed and we denote by s the initial time point. Uncertainty is given by a fixed probability space $(\Omega, \mathcal{F}, P)$ together with the filtration $\{\mathcal{F}_t, s \leq t \leq T\}$ where $\mathcal{F}_T = \mathcal{F}$. A financial market with continuous trading opportunities consists only of unit discount bonds from which information about the term structure is derived. In this model there exists a unique equivalent martingale measure Q which may be applied for pricing purposes.

2.1 Spot interest rate process

We denote the spot interest rate process by r_t and assume it is given by the following stochastic differential equation under the original probability measure P

$$dr_t = q(m - r_t)dt + vdB_t, \tag{1}$$

where the initial value r_s is a given constant. This is the well known Ornstein-Uhlenbeck process, first used in financial economics by Vasicek (1977). The parameters m, q and v are interpreted as the long-run mean to which the process tend to revert, the speed of reversion and the volatility of the process, respectively.

We denote by $f_s(t)$ the instantaneous time t forward rate observable at time s. The connection between market prices of default free unit discount bonds and the instantaneous forward rates is given by

$$P_{s,\tau} = e^{-\int_s^\tau f_s(t)dt},$$

where $P_{s,\tau}$ denotes the market price at time s of a default free unit discount bond with maturity at time τ.

We assume that the market price of (interest rate) risk at time s as a function of time t is

$$\lambda_s(t) = \frac{qm}{v} - \frac{1}{v}\left[qf_s(t) + \frac{\partial}{\partial t}f_s(t)\right] - \frac{v}{2q}(1 - e^{-2q(t-s)}). \tag{2}$$

Notice that for fixed s the market price of risk is a deterministic process of time which depends on the time s forward rates $f_s(t)$ as well as the derivative of the time s forward rate $\frac{\partial}{\partial t}f_s(t)$ and 3 parameters (q, m, v) of the spot interest rate process.

By this choice of market price of risk the dynamics of the spot interest rate process under the equivalent martingale measure Q can be written as

$$dr_t = q\left(\theta_t - r_t\right)dt + vd\hat{B}_t$$

where $\hat{B}_t$ is a Brownian motion under the equivalent martingale measure Q, r_s is a given constant, and

$$\theta_t = \frac{1}{q}\frac{\partial}{\partial t}f_s(t) + f_s(t) + \frac{v^2}{2q^2}(1 - e^{-2q(t-s)}).$$

This model[1] of spot interest rates under the equivalent martingale measure is known as a version of the Hull and White (1990) one-factor model.

The solution of (1) is

$$r_t = m + (r_s - m)e^{-q(t-s)} + \int_s^t ve^{-q(t-u)}dB_u. \tag{3}$$

For future use we define

$$R_{s,T} = \int_s^T r_t dt$$

and calculate $R_{s,T}$ as a function of r_s the spot rate at time s as

$$R_{s,T} = m(T-s) + (r_s - m)\frac{1-e^{-q(T-s)}}{q} + \int_s^T \frac{v}{q}(1-e^{-q(T-u)})dB_u. \tag{4}$$

Observe that $R_{s,T}$ is Gaussian and calculate the expectation and variance of $R_{s,T}$ as

$$\mu_{s,T} = m(T-s) + \frac{1}{q}(r_s - m)(1-e^{-q(T-s)}) \tag{5}$$

and

$$\sigma^2_{s,T} = \frac{v^2}{2q^3}\left(2q(T-s) - 3 + 4e^{-q(T-s)} - e^{-2q(T-s)}\right). \tag{6}$$

2.2 Bond price dynamics

The dynamics of market prices of default free unit discount bonds under the original probability measure P in this model are

$$P_{t,\tau} = P_{s,\tau} + \int_s^t [r_u + b(u,\tau)]P_{u,\tau}du + \int_s^t a(u,\tau)P_{u,\tau}dB_u, \tag{7}$$

where

$$a(t,\tau) = \frac{v}{q}(e^{-q(\tau-t)} - 1) \tag{8}$$

and

$$b(t,\tau) = \left(m - f_s(t) - \frac{1}{q}\frac{\partial}{\partial t}f_s(t) - \frac{v^2}{2q^2}(1-e^{-2q(t-s)})\right)(e^{-q(\tau-t)} - 1).$$

[1] Alternatively, our model can be expressed under the equivalent martingale measure as

$$r_t = f_s(t) + \int_s^t q(\hat{\theta}_u - r_u)du + \int_s^t vd\hat{B}_u,$$

where

$$\hat{\theta}_t = f_s(t) + \frac{v^2}{2q^2}(1-e^{-2q(t-s)}).$$

This latter equivalent formulation is in spirit with the more general Heath, Jarrow, and Morton (1992) term structure formulation in the sense that the initial value of the process is the instantaneous forward rate. It can also be shown that the current model is a special case of the Heath et al. (1992) model, see e.g., Miltersen and Persson (1999). The above equation (2) corrects an error in corresponding equation (for λ_t) at the bottom of page 310 in Miltersen and Persson (1999).

Observe that the relationship

$$\lambda_s(t) = \frac{b(t,\tau)}{a(t,\tau)}$$

holds for all $\tau \geq t$.

3 The static case: No intermediate rebalancing of debt

3.1 The agent's problem

We assume that utility is derived from final time T wealth only, and that the agent is equipped with a constant relative risk aversion (CRRA) utility function given by

$$u(x) = \frac{1}{1-\rho} x^{1-\rho}, \tag{9}$$

where ρ can be interpreted as the relative risk aversion coefficient ($\frac{-u''(x)}{u'(x)}x = \rho$). Here ρ is assumed positive and the special case $\rho = 1$ corresponds to the utility function $u(x) = \ln(x)$.

The floating rate loan is assumed to accrue interest according to the spot rate r_t given in expression (1), whereas the time t fixed rate for the period time (t, T) is determined at each point in time from the time t observable forward rates as

$$r_t^x = \frac{1}{T-t}\int_t^T f_t(u)du.$$

We denote the initial (time s) amount of debt by D_s and assume that the agent has a deterministic time T wealth $\bar{W}$, which can be interpreted as the *collateral* for the loan. All interest payments are assumed to take place at the horizon T.

In this section we assume that no rebalancing of the debt portfolio can take place after the investor has chosen the initial distribution between fixed and floating rate debt.

We denote the fraction of floating rate debt to total debt by α (the optimal value of α is denoted by α^*). The agent's terminal (time T) wealth may be written as

$$W_T = \bar{W} - \alpha D_s e^{\int_s^T r_t dt} - (1-\alpha) D_s e^{r_s^x (T-s)}.$$

Unless $\int_s^T r_t dt$ is bounded there may be a potential problem of negative terminal wealth for high values of α.

Values of $\alpha < 0$ represent *short* positions of floating rate debt, which means that the investor acts as *lender* instead of *borrower*. Similarly, values of $\alpha > 1$ imply *short* positions of fixed rate debt, which means that the investor acts as a bond investor instead of a bond issuer. We do not formally exclude values of α greater than 1 or lower than zero, but since we are primarily concerned with optimal debt allocation, we focus on the case $0 < \alpha < 1$ in numerical examples.

Observe that $\bar{W}e^{-r_s^x(T-s)} - D_s$ can be interpreted as the time s market value of the time T wealth. We denote the ratio between the time s market value of the time T wealth and the (market value of) the time s debt by L_s. Then

$$L_s = \frac{\bar{W}e^{-r_s^x(T-s)} - D_s}{D_s}.$$

Sometimes we refer to L_s as just the (time s) wealth to debt ratio.

Now rewrite[2] W_T in terms of L_s as

$$W_T = D_s e^{r_s^x(T-s)}[L_s + \alpha\left(1 - e^{\int_s^T (r_t - r_s^x)dt}\right)].$$

The agent's problem is stated as

$$\max_{\alpha} E\left[u(W_T)\right].$$

The first order condition of this problem is

$$D_s e^{r_s^x(T-s)} E\left[u'(W_T)(1 - e^{\int_s^T (r_t - r_s^x)dt})\right]. \tag{10}$$

3.2 A lower fixed rate bound for floating rate borrowing

In order to analyze this first order condition (10) further we apply the standard arguments used, e.g., in Huang and Litzenberger (1988). We set $\alpha = 0$. For this special case all debt is fixed rate debt and $W_T = \bar{W} - D_s \exp\left(r_s^x(T-s)\right)$ is deterministic. The first order condition may then be written as

$$D_s u'(W_T)\left(e^{r_s^x(T-s)} - E\left[e^{\int_s^T r_t dt}\right]\right).$$

The value of the first order condition may be interpreted as the marginal increase in expected utility of time T wealth from a marginal increase in α. If the value of the above expression is positive, we may conclude that the optimal value of α is positive. Using equations (5) and (6) we calculate

$$E\left[e^{\int_s^T r_t dt}\right] = E\left[e^{R_{s,T}}\right] = e^{\mu_{s,T} + \frac{1}{2}\sigma_{s,T}^2}.$$

Now, define

$$r_L = \frac{1}{T-s}(\mu_{s,T} + \frac{1}{2}\sigma_{s,T}^2). \tag{11}$$

These arguments lead to the following result:

[2] Observe that the following expression is on the form $W_T = K + \alpha\tilde{Y}$, where K is constant and $\tilde{Y}$ is a random variable. The problem has thus the same structure as both the classical asset allocation problem, see e.g. Huang and Litzenberger (1988) as well as optimal purchase of insurance problems, cf. Mossin (1968), as e.g., explained in the textbook by Eeckhoudt and Gollier (1995).

Proposition 1 *The optimal fraction of floating rate debt α^* is strictly positive if and only if the fixed rate r_s^x is strictly greater than r_L defined in expression (11).*

This result is interpreted as follows: If $r_s^x > r_L$, it is optimal to accept *some* floating interest rate loan. If the fixed rate is r_L or lower, at least 100% of the loan amount is financed by fixed rate debt. In the case where strictly more than 100% of the loan amount is financed by fixed rate debt, the agent 'shorts' floating rate debt, i.e., the agent *lends* instead of *borrows* to the floating rate.

Note that this result holds for any utility function with strictly positive marginal utility and does therefore not depend on our particular choice of utility function in expression (9).

3.3 An upper bound for fixed rate debt borrowing

An upper bound for *some* fixed rate debt may be derived in an analogous matter. Define

$$Z = \frac{\bar{W}}{D_s} - e^{\int_s^T r_t dt}.$$

We now study the situation with only floating rate loan, i.e., we let $\alpha = 1$. By using the CRRA utility function in expression (9) the first order condition (10) is proportional to

$$E[Z^{1-\rho}] + \left(e^{r_s^x (T-s)} - \frac{\bar{W}}{D_s} \right) E[Z^{-\rho}]$$

If this first order condition takes a negative value, it is optimal to decrease α, i.e., to accept *some* fixed rate loan. Define

$$r_U = \frac{1}{T-s} \left[\ln \left(\frac{\bar{W}}{D_s} - \frac{E[Z^{1-\rho}]}{E[Z^{-\rho}]} \right) \right]. \tag{12}$$

We now have the following result:

Proposition 2 *The optimal fraction of floating rate debt α^* is strictly less than 1 if and only if the fixed rate $r_s^x < r_U$ defined in expression (12).*

This result tells us that for fixed rates lower than r_U it is optimal to accept *some* fixed rate loan. As opposed to the lower bound the upper bound r_U depends on the agent specific factors, $\frac{\bar{W}}{D_s}$, ρ, and our specific choice of utility function (see expression (9). Numerical results require the calculation of two moments of the random variable Z defined above. The expression for r_U also holds for the case $\rho = 1$, i.e., for $u(x) = \ln(x)$.

From expression (12) the following result follows immediately for the special case of a risk neutral investor:

Proposition 3 *For a risk neutral investor, i.e., $\rho = 0$, the upper bound equals the lower bound $r_U = r_L$.*

Thus, a risk neutral investor chooses the debt alternative with the lowest expected interest rate, i.e., either fixed rate loan or floating rate loan, never a combination of both.

Base case parameters:	
Interest rate process (1)	$dr_t = q(m - r_t)dt + vdB_t$
Initial interest rate	$r_s = 0.05$
Speed of mean reversion	$q = 0.15$
Long term mean reversion level	$m = 0.045$
Volatility of interest rate	$v = 0.02$
Other parameters:	
Wealth to debt ratio	$L_s = 1$
Time horizon	$T - s = 3$
Fixed interest rate	$r_s^x = 0.05$

Table 2: Base case parameters

3.4 Numerical illustrations — static case

In order to do numerical calculations of the optimal α we set the first order condition (10) equal to zero for the CRRA utility function and obtain

$$E\left[\left(L_s + \alpha(1 - e^{\int_s^T (r_t - r_s^x)dt})\right)^{-\rho}\left(1 - e^{\int_s^T (r_t - r_s^x)dt}\right)\right] = 0. \tag{13}$$

From this expression it is clear that α^* only depends on the time s wealth to debt ratio (in addition to ρ, T, r_s^x and properties of $R_{s,T}$) and not, for example, on the levels of either $\bar{W}$ or D_s.

By inspection of equation (13) it is clear that the optimal α is proportional to the parameter L_s, i.e., if $\bar{\alpha}$ solves the equation for $\bar{L}_s$, $k\bar{\alpha}$ will solve the equation for $k\bar{L}_s$, for any constant k.

Numerical results are presented in the following tables. Table (2) presents the base case parameters, which are intended to be within reasonable ranges. In particular, the base case values of the mean reversion speed q and the volatility v are close to the values estimated for the Vasicek spot rate interest by Chan, Karolyi, Longstaff, and Sanders (1992). The chosen values of the initial interest rate r_s and the mean reversion level m are in the same range as used by Munk and Sørensen (2001). The chosen time horizon represents a typical option for consumers who want to fix their debt interest rate.

In table (3) we present the interval (r_L, r_U) where the debt will be divided into both fixed and floating rate debt for different levels of risk aversion ρ for some alternative parameter values.

3.5 Constant relative risk aversion? A reformulation

From the proportionality property discussed above It is clear that the fraction of floating rate debt increases with the wealth to debt ratio L_s. However, a well known property of CRRA utility (9) is that the total fraction of 'risky investments' is independent of wealth. In order to obtain results in this spirit we reformulate the problem as follows: We now express the amount of floating

	$r_L(\rho=0)$	$\rho=\frac{1}{2}$	$\rho=1$	$\rho=2$	$\rho=4$	$\rho=8$	$P(\text{neg})$
Base case	4.95%	5.01%	5.07%	5.19%	5.45%	6.00%	$.51 \cdot 10^{-26}$
$L=4$	4.95%	4.96%	4.98%	5.02%	5.09%	5.23%	$.14 \cdot 10^{-129}$
$v=4\%$	5.08%	5.34%	5.61%	6.20%	7.7%$*$	14% $*$	$.44 \cdot 10^{-7}$
$T=6$	4.96%	5.29%	5.72%	7.35%	$*$	$*$	0.006
$q=30\%$	4.86%	4.91%	4.95%	5.04%	5.23%	5.62%	$.55 \cdot 10^{-35}$
$m=5.5\%$	5.14%	5.20%	5.26%	5.39%	5.65%	6.21%	$.17 \cdot 10^{-25}$
$r_0=r_0^x=7\%$ $m=6.5\%$	6.95%	7.02%	7.09%	7.24%	7.54%	8.20%	$.82 \cdot 10^{-21}$

Table 3: Bounds for fixed rate debt r_L and r_U (equation (12)). Let $W_T^{\alpha=1} = \bar{W} - D_0 e^{\int_0^T r_s ds}$ be the terminal wealth for $\alpha = 1$. Then $P(\text{neg}) = P(W_T^{\alpha=1} < 0) = P(\int_0^T r_s ds > \ln(L))$. An asterix ($*$) indicates numerically unstable values.

	$\rho=\frac{1}{2}$	$\rho=1$	$\rho=2$	$\rho=4$	$\rho=8$
α^*	1.231	0.6187	0.3100	0.1551	0.07759
$E[U(W_T^*)]$	2.157	0.1505	-0.8605	-0.2125	-0.04997

Table 4: Optimal α (from equation (13)) and optimal expected utility $E[U(W_T^*)]$ for various levels relative risk aversion ρ for $L_s = 1$ and $D_s = 1$.

rate debt as a fraction β of the time s market value of the time T wealth as follows

$$W_T = (\bar{W} - D_s e^{r_s^x(T-s)}) \left(1 + \beta\left(1 - e^{\int_s^T (r_t - r_s^x)dt}\right)\right).$$

The *amount* of floating rate debt was αD_s by the previous formulation and is $\beta(\bar{W}e^{-r_s^x(T-s)} - D_s)$ by the current reformulation. These amounts are identical in both formulations, i.e.,

$$\alpha D_s = \beta(\bar{W}e^{-r_s^x(T-s)} - D_s).$$

The first order condition of this reformulation is

$$E\left[\left(1 + \beta(1 - e^{\int_s^T (r_t - r_s^x)ds})\right)^{-\rho}\left(1 - e^{\int_s^T (r_t - r_s^x)dt}\right)\right] = 0. \tag{14}$$

By inspection of the first order condition (14) it is clear that the optimal β (denoted by β^*) does neither depend on $\bar{W}$, D_s, nor L_s.

Based on a second order Taylor approximation, we have calculated the following approximation of β

$$\hat{\beta} = \frac{\sigma_Y^2\rho - 2\gamma^2 e^{2r_s^x(T-s)} - \sigma_Y e^{r_s^x(T-s)}\sqrt{\rho^2\sigma_Y^2 e^{-2r_s^x(T-s)} - 2\gamma^2(1+\rho)\rho}}{2\gamma^3 e^{2r_s^x(T-s)} + \sigma_Y^2\gamma\rho(\rho-1)}, \tag{15}$$

where

$$\sigma_Y^2 = e^{2\mu_{s,T} + \sigma_{s,T}^2}(e^{\sigma_{s,T}^2} - 1)$$

	$\rho = \frac{1}{2}$	$\rho = 1$	$\rho = 2$	$\rho = 4$	$\rho = 8$
β^*	1.231	0.6187	0.3100	0.1551	0.07759
$\hat{\beta}$	1.249	0.6236	0.3116	0.1557	0.07785

Table 5: Optimal β from equation (14) and approximate β from equation (15) for the base case parameters.

and

$$\gamma = 1 - e^{\mu_{s,T} + \frac{1}{2}\sigma^2_{s,T} - r^x_s(T-s)}.$$

As mentioned, the reformulation only involves a change of base for a fraction, so the total optimal *amounts* of floating rate debt are the same for the two formulations. Thus

$$D_s\alpha^* = (\bar{W}e^{-r^x_s(T-s)} - D_s)\beta^*.$$

This insight leads to a simpler way of calculating α^* for different wealth to debt ratios: First, calculate β^* which is independent of L_s. Then, calculate the corresponding α^* as

$$\alpha^* = L_s\beta^*,$$

i.e., the optimal α is given as the optimal β multiplied by the time s wealth to debt ratio.

In table (5) some numerical values of β^* are calculated together with the value of the approximated β. We are tempted to conlude that the approximation performs reasonably well.

From table (5) and the above relationship between β^* and α^*, the first line in table (4) may easily be reproduced[3].

4 The dynamic problem: Continuous rebalancing of debt

In this section we allow the investor to rebalance his debt portfolio at any point in time between the initial time 0 ant the time of expiration T. Moreover, rebalancing does not impose any cost for the agent.

[3] Another polar case is the situation with constant absolute risk aversion, i.e., the investor is equipped with a negative exponential utility function

$$u(x) = -e^{-\eta x}.$$

For comparison we present some results for this case as well. The first order condition, similar to expression (13) for the problem now becomes

$$E\left[e^{\eta\alpha D_s e^{\int_s^T r_t dt}}\left(1 - e^{\int_s^T (r_t - r^x_s)dt}\right)\right] = 0.$$

From this equation it is clear that the optimal α is independent of $\bar{W}$. Furthermore, it is clear that if that the optimal α is inverse proportional to η, i.e., if $\bar{\alpha}$ is the optimal value for a given value of η, say $\eta = \bar{\eta}$, then by e.g., doubling the risk aversion parameter to $\eta = 2\bar{\eta}$, the new optimal α is $\alpha^* = \frac{1}{2}\bar{\alpha}$, i.e., half the value of the given α.

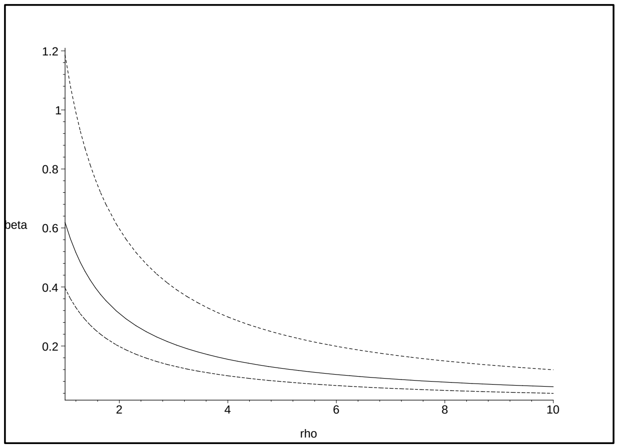

Figure 1: Optimal β as a function of the relativ risk aversion parameter ρ for 3 alternative parameter sets. The upper curve shows β^* for the case where the parameter $r_s^x = 0.0505$, the lower curve shows β^* for the parameter $m = 0.046$, the center curve depicts the base case parameters.

Our methodology is based on the martingale formulation by Pliska (1986) and Cox and Huang (1989) as recently extended by Sørensen (1999) and Munk and Sørensen (2001).

4.1 Intermediate market value of debt and debt dynamics

In the dynamic setting of this section we both need the market value at intermediate points in time as well as the stochastic dynamics of fixed and floating rate debt.

First we derive expressions for the market values of debt at time t, $s \leq t < T$. Let D_s^L be a time s amount of floating rate debt which has to be paid back including interest rates at time T. The market value of this debt at time $t > s$ is

$$\begin{aligned} D_t^L &= D_s^L E_t^Q \left[e^{-\int_t^T r_u du} e^{\int_s^T r_u du} \right] \\ &= D_s^L e^{\int_s^t r_u du}. \end{aligned} \tag{16}$$

Let D_s^X denote a time s amount of fixed rate debt which has to be paid back including interest rates at time T . The market value of this debt at time $t > s$

is

$$
\begin{aligned}
D_t^X &= D_s^X E_t^Q \left[e^{-\int_t^T r_u du} e^{\int_s^T r_s^x du} \right] \\
&= D_s^X e^{r_s^x (T-s)} E_t^Q \left[e^{-\int_t^T r_u du} \right] \\
&= D_s^X e^{r_s^x (T-s)} P_{t,T}. \qquad (17)
\end{aligned}
$$

From the previous expressions (16) and (17) we derive the fixed and floating rate dynamics below. The market value of fixed rate debt can be described by the following stochastic differential equation

$$
dD_t^X = (r_t + b_{t,T}) D_t^X dt + a_{t,T} D_t^X dB_t, \qquad (18)
$$

together with the given constant D_s^X. As one should expect, fixed rate debt has identical dynamics as a bond with the same expiration as the debt, see expression (7).

The corresponding stochastic differential equation for the market value of floating rate debt is

$$
dD_t^L = r_t D_t^L dt, \qquad (19)
$$

where the initial value is the given constant D_s^L. Floating rate debt has the same dynamics as a bank account (sometimes called a money market account) where interest accrues according to the spot rate r_t.

In addition to debt the agent's time T wealth consists only of the deterministic amount $\bar{W}$. The market value at time t, $s \leq t \leq T$ of the agent's time T wealth for given amounts of fixed and floating rate debts is therefore

$$
W_t = \bar{W} P_{t,T} - D_t^X - D_t^L.
$$

From the equations (7), (18), (19), and Itô's lemma the dynamics of the wealth process may be written as

$$
dW_t = \left((r_t + b_{t,T}) W_t + b_{t,T} D_t^L \right) dt + a_{t,T} (W_t + D_t^L) dB_t, \qquad (20)
$$

where the initial value W_s is a given constant. We now let D_t denote the market value of the total debt at time t, i.e., $D_t = D_t^X + D_t^L$, and let α_t denote the fraction of floating rate debt, i.e., $D_t^L = \alpha_t D_t$. By substituting in expression (20) we obtain

$$
dW_t = ((r_t + b_{t,T}) W_t + b_{t,T} \alpha_t D_t) dt + a_{t,T} (W_t + \alpha_t D_t) dB_t. \qquad (21)
$$

Alternatively, we may express the floating rate amount at time t as a fraction β_t of time t wealth, $D_t^L = \beta_t W_t$ (somewhat similar as we did in the previous section above expression (14)). By substituting in expression (20) we now obtain

$$
dW_t = (r_t + b_{t,T}(1 + \beta_t)) W_t) dt + a_{t,T} (1 + \beta_t) W_t dB_t. \qquad (22)
$$

The connection between α_t and β_t is

$$\begin{aligned}\alpha_t &= \frac{W_t}{D_t}\beta_t, \\ &= L_t\beta_t, \end{aligned} \tag{23}$$

where L_t is the wealth to debt ratio as previously defined. This is exactly the same relationship between α and β as in the static case.

4.2 The agent's problem

The agent's problem is similar as in the previous section. Also in this dynamic set-up utility is derived only from time T wealth. At time s the investors problem is:

$$J_s = \sup_{W_T} E_s \left[\frac{1}{1-\rho} (W_T)^{1-\rho} \right]$$

subject to

$$E_s \left[\xi_{s,T} W_T\right] \leq W_s,$$

where

$$\xi_{s,t} = \exp\left(-\int_s^t r_u du - \int_s^t \lambda_s(u) dB_u - \frac{1}{2}\int_s^t \lambda_s(u)^2 du \right) \tag{24}$$

is sometimes called the *state price deflator* and $\lambda_s(t)$ is given by expression (2). For the special case $\rho = 1$ we assume that $J_s = \sup_{W_T} E_s[\ln(W_T)]$.

For example the market price of a default free unit discount bond expiring at time T may be expressed by the state price deflator as

$$P_{s,T} = E_s[\xi_{s,T}].$$

Under the condition that $\ln(\xi_{s,T})$ is normally distributed, which always will be the case for our model, we can calculate $P_{s,T}$ as

$$P_{s,T} = \exp\left(-\mu_{s,T} - \frac{1}{2}\int_s^T \lambda_s(u)^2 du + \frac{1}{2} V_{s,T}^2 \right),$$

where

$$V_{s,T}^2 = \mathrm{Var}\left(\int_s^T r_u du + \int_s^T \lambda_s(u) dB_u | \mathcal{F}_s \right),$$

and $\mu_{s,T}$ and $\lambda_s(t)$ are given in expression (5) and (2), respectively.

4.3 Solution of the problem

The optimal indirect utility for this problem is given in the following proposition.

Proposition 4 *The optimal expected utility for this problem for $\rho \neq 1$ is*

$$J_s = \frac{1}{1-\rho}\left[\left(\frac{W_s}{P_{s,T}}\right)^{1-\rho} e^{\frac{1}{2}\frac{1-\rho}{\rho}V_{s,T}^2}\right]. \tag{25}$$

Optimal expected utility for logarithmic utility ($\rho = 1$) is

$$J_s = \ln\left(\frac{W_s}{P_{s,T}}\right) + \frac{1}{2}V_{s,T}^2. \tag{26}$$

Proof 1 *Consider first the case $\rho \neq 1$. From the first order condition of the corresponding Lagrangian we obtain*

$$W_T = \mathcal{L}^{-\frac{1}{\rho}}(\xi_{s,T})^{-\frac{1}{\rho}}, \tag{27}$$

where $\mathcal{L}$ denotes the Lagrangian multiplier. Inserting the expression (27) for W_T into the budget constraint we obtain

$$W_s = \mathcal{L}^{-\frac{1}{\rho}} E_s[(\xi_{s,T})^{\frac{\rho-1}{\rho}}],$$

from which we determine $\mathcal{L}$ as $\mathcal{L}^{-\frac{1}{\rho}} = \frac{W_t}{E_s[(\xi_{s,T})^{\frac{\rho-1}{\rho}}]}$. From equation (27) we write the optimal terminal wealth W_T^* as

$$W_T^* = \frac{W_s}{E_s[(\xi_{s,T})^{\frac{\rho-1}{\rho}}]}(\xi_{s,T})^{-\frac{1}{\rho}}. \tag{28}$$

Finally, we insert this expression into the objective function and obtain

$$J_s = E_s\left[\frac{1}{1-\rho}(W_T^*)^{1-\rho}\right] = \frac{1}{1-\rho}W_s^{1-\rho}E_s[(\xi_{s,T})^{\frac{\rho-1}{\rho}}]^{\rho}.$$

Equation (25) is obtained by calculating $E_s[(\xi_{s,T})^{\frac{\rho-1}{\rho}}] = (P_{s,T})^{\frac{\rho-1}{\rho}} e^{\frac{1}{2}\frac{1-\rho}{\rho^2}V_{s,T}^2}$.

Equation (27) also holds for the case $\rho = 1$. The expression corresponding to equation (28) is $W_T^* = W_s\frac{1}{\xi_{s,T}}$. Equation (26) follows by inserting this expression into the objective function.

4.4 Optimal debt positions

In the following propositions we present the optimal fractions of floating rate debt and, thus, implicitly, the optimal fixed rate debt.

Proposition 5 *The optimal time $t \geq s$ fraction of floating rate debt, expressed as a fraction of total debt, is*

$$\begin{aligned}\alpha_t &= \frac{1}{\rho}\left(\frac{\lambda_s(t)}{a_{t,T}} - 1\right)L_t \\ &= \frac{1}{\rho}\left(\frac{\lambda_s(t)}{a_{t,T}} - 1\right)\left(\frac{\bar{W}P_{t,T}}{D_t} - 1\right).\end{aligned}$$

Proposition 6 *The optimal time $t \geq s$ fraction of floating rate debt, expressed as a fraction of the market value of total time t wealth, is*

$$\beta_t = \frac{1}{\rho}\left(\frac{\lambda_s(t)}{a_{t,T}} - 1\right).$$

Proof 2 *The proof consists of deriving the dynamics of the optimal wealth process (28). By equating this process with the wealth processes derived earlier in equations (22) and (21) the optimal β and α, respectively, are determined.*

We start by defining the process Y_t for $t \geq s$ as

$$Y_t = \frac{W_s}{Q_{s,t}} \left(\xi_{s,t}\right)^{-\frac{1}{\rho}}, \tag{29}$$

where

$$Q_{s,t} = E_s\left[\left(\xi_{s,t}\right)^{\frac{\rho-1}{\rho}}\right] = \left(P_{s,t}\right)^{\frac{\rho-1}{\rho}} e^{\frac{1}{2}\frac{1-\rho}{\rho^2}V_{s,t}^2}.$$

Observe that Y_t can be interpreted as the optimal wealth process for the given time horizon t, in particular $Y_T = W_T^*$ from equation (28). By applying Itô's lemma to the above equation $Q_{s,t}$ for $t \geq s$ may be expressed as

$$Q_{s,t} = 1 + \int_s^t (\cdot)\, dv + \int_s^t \frac{\rho-1}{\rho} a_{v,t} Q_{s,v} dB_v, \tag{30}$$

where the drift term is left unspecified. Furthermore, we obtain from equation (24)

$$\xi_{s,t} = 1 - \int_s^t r_v \xi_{s,v} dv - \int_s^t \lambda_s(v) \xi_{s,v} dB_v. \tag{31}$$

We now apply Itô's lemma to equation (29) to find the dynamics of Y_t and evaluate this expression for $t = T$:

$$W_T^* = W_s + \int_s^T (\cdot)\, dv + \int_s^T \left[\frac{\rho-1}{\rho} a_{v,T} + \frac{\lambda_s(v)}{\rho}\right] W_v dB_v$$

The time t instantaneous dB_t term of this equation is $\left[\frac{\rho-1}{\rho}a_{t,T} + \frac{\lambda_s(t)}{\rho}\right] W_t$. By equating this term of with the similar term $a_{t,T}(1+\beta_t)W_t$ of expression (22) the expression for β in the proposition is obtained.

Proposition (5) then follows from the general connection between α_t and β_t in expression (23). Alternatively, it can be derived by equating the dB_t term of the above equation with the dB_t term of equation (21).

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4.5 Numerical illustrations — dynamic case

We present some numerical results to compare the dynamic case with the static case both in terms of welfare measured by expected utility and initial fractions of floating rate debt.

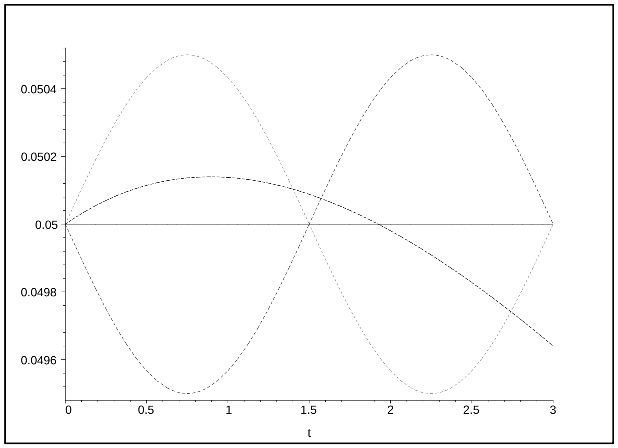

Figure 2: The 4 different initial term structures

From proposition 5 the optimal fraction of floating rate debt depends on the market price of risk, which again (see equation (2)) depends on the initial (time s) forward rates. We will therefore consider the following 4 cases (see Figure 2):

- Case 1. As Brennan and Xia (2000) and Bajeux-Besnainou et al. (2001) among others, we first assume that $\lambda_s(t) = \bar{\lambda}$, a constant. This assumption implies (from equation (2)) that the initial forward rates are of the form:

$$f_s^{(1)}(t) = r_s e^{-q(t-s)} + (m - \frac{v\bar{\lambda}}{q})(1 - e^{-q(t-s)}) - \frac{v^2}{2q^2}(1 - e^{-q(t-s)})^2.$$

The derivative of of the initial forward rate is

$$\frac{\partial}{\partial t} f_s^{(1)}(t) = q e^{-q(t-s)} \left(m - \frac{v\bar{\lambda}}{q} - r_s - \frac{v^2}{q^2}(1 - e^{-q(t-s)}) \right).$$

For our choice of parameter values $f_s^{(1)}(t)$ will be humped, i.e., increasing for small t values and decreasing for larger t values.

By definition $P_{s,T} = e^{r_s^x (T-s)}$. Also, $P_{s,T}$ can be calculated as

$$P_{s,T} = E_s^Q[e^{-\int_s^T r_t dt}] = e^{-\hat{\mu}_{s,T} + \frac{1}{2}\sigma_{s,T}^2} = e^{-\mu_{s,T} + \frac{1}{2}\sigma_{s,T}^2 + \bar{\lambda}\frac{v}{q}[T - s - \frac{1}{q}(1 - e^{-q(T-s)})]},$$

where $\hat{\mu}_{s,T}$ denotes the expectation of $\int_s^T r_t dt$ under the equivalent martingale measure. By equating these two expressions the market price of risk at time s is determined as

$$\bar{\lambda} = \frac{q}{v} \frac{\mu_{s,T} - \frac{1}{2}\sigma^2_{s,T} - r^x_s(T-s)}{(T - s - \frac{1}{q}(1 - e^{-q(T-s)}))}.$$

Here $V^2_{s,T}$ is given by equation (38).

- Case 2. The initial forward rates are constant, i.e., $f^{(2)}_s(t) = r_s$ for all t. Then $\frac{\partial}{\partial t} f^{(2)}_s(t) = 0$ and

$$\lambda^{(2)}_s(t) = \frac{q}{v}(m - r_s) - \frac{v}{2q}(1 - e^{-2q(t-s)}).$$

  Here $V^2_{s,T}$ is given by equation (34).

- Case 3. The initial forward rates are initially increasing, given by the function

$$f^{(3)}_s(t) = r_s + \sin(\frac{2\pi(t-s)}{T-s})\frac{1}{2000}.$$

  The derivative of of the initial forward rate is

$$\frac{\partial}{\partial t} f^{(3)}_s(t) = \cos(\frac{2\pi(t-s)}{T-s})\frac{\pi}{1000(T-s)}$$

  and

$$\begin{aligned}\lambda^{(3)}_s(t) = \frac{q}{v}(m - r_s) &- \frac{q}{2000v}\sin(\frac{2\pi(t-s)}{T-s}) \\ &- \frac{\pi}{1000(T-s)v}\cos(\frac{2\pi(t-s)}{T-s}) - \frac{v}{2q}(1 - e^{-2q(t-s)}).\end{aligned}$$

  In this case $V^2_{s,T}$ is given by equation (34).

- Case 4. The forward rates are initially decreasing. In particular the initial forward rates are given by the function

$$f^{(4)}_s(t) = r_s + \sin(\frac{2\pi(t-s)}{T-s} + \pi)\frac{1}{2000}.$$

  The derivative of of the initial forward rate is

$$\frac{\partial}{\partial t} f^{(4)}_s(t) = \cos(\frac{2\pi(t-s)}{T-s} + \pi)\frac{\pi}{1000(T-s)}$$

  and

$$\begin{aligned}\lambda^{(4)}_s(t) = \frac{q}{v}(m - r_s) &- \frac{q}{2000v}\sin(\frac{2\pi(t-s)}{T-s} + \pi) \\ &- \frac{\pi}{1000(T-s)v}\cos(\frac{2\pi(t-s)}{T-s} + \pi) - \frac{v}{2q}(1 - e^{-2q(t-s)}).\end{aligned}$$

  Here $V^2_{s,T}$ is given by equation (34).

J_s	$\rho=\frac{1}{2}$	$\rho=1$	$\rho=2$	$\rho=4$	$\rho=8$
static case	2.157	0.1505	-0.8605	-0.2125	-0.04997
constant λ	2.162	0.1515	-0.8594	-0.2121	-0.04986
constant $f_s(t)$	2.167	0.1527	-0.8584	-0.2117	-0.04975
increasing $f_s(t)$	2.176	0.1546	-0.8568	-0.2111	-0.04959
decreasing $f_s(t)$	2.177	0.1550	-0.8564	-0.2110	-0.04956

Table 6: Optimal initial utility levels J_s calculated from equation (25) and compared with the results of the previous static model in table (4) for the base case parameters.

ΔCE in %	$\rho=\frac{1}{2}$	$\rho=1$	$\rho=2$	$\rho=4$	$\rho=8$
constant λ	0.46	0.10	0.13	0.06	0.03
constant $f_s(t)$	0.93	0.22	0.24	0.13	0.06
increasing $f_s(t)$	1.77	0.41	0.43	0.22	0.11
decreasing $f_s(t)$	1.86	0.45	0.48	0.24	0.12

Table 7: Percentage increase in certainty equivalent wealth (ΔCE) compared with static case for the four dynamic cases. Let $\bar{u}$ denote the optimal utility level from table (6). The certainty equivalent wealth is then calculated as $(\bar{u}(1-\rho))^{\frac{1}{1-\rho}}$ for $\rho \neq 1$ and as $e^{\bar{u}}$ for $\rho = 1$.

These choices of initial forward rates all produce the same fixed rate $r_s^x = 5\%$. To make the choice between fixed and floating rate less obvious we set the initial spot rate equal to the fixed rate in the numerical examples. Given this restriction the sinus function is a natural choice as a model of the initial forward rates in case 3 and case 4.

In table (6) some values of J_s are calculated for the four cases for the base case parameters. In order to interpret these results the increase in *certainty equivalent wealth* from the static case to each of the dynamic case is presented in table (7). The overall conclusion is that the increase in optimal expected utility from the static to the dynamic case is low, less that 2%. The increase in certainty equivalent wealth seems to be decreasing in the risk aversion parameter ρ (with one exception), so that the welfare increase in a dynamic setting is largest for investors with low levels of risk aversion.

In the table (8) the optimal values for the initial fractions of floating rate debt β_s are presented. The main conclusion from these tables is that the initial positions of floating rate are sensitive to the initial term structure.

5 Concluding remarks and further research

Our preliminary numerical comparisons between the static case (no rebalancing of the debt portfolio) with the dynamic case (continuous and costless rebalanc-

β_s	$\rho = \frac{1}{2}$	$\rho = 1$	$\rho = 2$	$\rho = 4$	$\rho = 8$
static case	1.231	0.6187	0.3100	0.1551	0.07759
constant λ	0.2320	0.1160	0.05800	0.0290	0.0145
constant $f_s(t)$	-0.4478	-0.2239	-0.1120	-0.05598	-0.02799
increasing $f_s(t)$	1.7119	0.8602	0.4298	0.2150	0.1075
decreasing $f_s(t)$	-2.615	-1.308	-0.6536	-0.3668	-0.1635

Table 8: Optimal initial fractions of floating rate debt β_s calculated from the result in proposition 6 and compared with the results of the previous static model in table (5) for the base case parameters.

ing) indicate, perhaps surprisingly, low increase in 'welfare' in dynamic situation compared to static situation. At least this is the case for high levels of relative risk aversion.

In the dynamic case the optimal initial fractions of floating rate debt are partly determined by the initial forward interest rates, which do not influence the corresponding optimal fraction in the static case. Therefore, we do not learn anything about the optimal floating rate debt fraction in the dynamic case from the static case. Even if we are willing to assume that the market price of interest rate risk is constant the initial optimal fractions of floating rate debt are different in the static and dynamic cases.

The research in this article can be extended in a number of ways. First, realism can be improved by introducing a multi-factor interest rate model. Also, transaction costs can be introduced, in the spirit of Davis and Norman (1990), Korn (1998), Øksendal and Sulem (2000), and Zakamouline (2002) in order to make the set-up closer to real world situations. Finally, this set-up may be used to study the effect of a stochastic collateral ($\bar{W}$).

A Appendix

In this appendix a number of detailed calculations is collected.

From equation (2) direct calculations give

$$\int_t^T \lambda_s ds = \frac{q}{v}\left[(m - r_t^x)(T-t) + \frac{1}{q}(r_t - f_t(T))\right] - \frac{q}{v}\left[\frac{v^2}{4q^3}(2q(T-t) - 1 + e^{-2q(T-t)})\right] \quad (32)$$

and

$$\int_t^T \lambda_s e^{qs} ds = \frac{q}{v}e^{qT}\left[\frac{m}{q}(1 - e^{-q(T-t)}) - \frac{1}{q}(f_t(T) - r_t e^{-q(T-t)})\right] + \frac{q}{v}e^{qT}\left[\frac{v^2}{2q^3}(2e^{-q(T-t)} - e^{-2q(T-t)} - 1)\right]. \quad (33)$$

From the equations (2) and (4) by using (32) and (33) it follows that

$$\mathrm{Cov}\left(\int_s^T r_t dt, \int_s^T \lambda_s(t) dB_t | \mathcal{F}_s\right) = \\ (r_s - m)\frac{1-e^{-q(T-s)}}{q} + (m - r_s^x)(T-s) - \frac{1}{2}\sigma_{s,T}^2. \quad (34)$$

It is now straight forward to calculate

$$\begin{aligned} V_{s,T}^2 &= \mathrm{Var}\left(\int_s^T r_t dt + \int_s^T \lambda_s(t) dB_t | \mathcal{F}_s\right) \\ &= \sigma_{s,T}^2 + \int_s^T \lambda_s(t)^2 dt + 2\mathrm{Cov}\left(\int_s^T r_t dt, \int_s^T \lambda_s(t) dB_t | \mathcal{F}_s\right) \\ &= \int_s^T \lambda_s(t)^2 dt + 2\left[(r_s - m)\frac{1-e^{-q(T-s)}}{q} + (m - r_s^x)(T-s)\right]. \end{aligned} \quad (35)$$

The partial derivative of $V_{s,T}^2$ is

$$\frac{\partial V_{s,T}^2}{\partial s} = -[\lambda_s(t)^2 + 2[(r_s - m)e^{-q(T-s)} + m - r_s^x]]. \quad (36)$$

In the case where $\bar{\lambda}$ is constant we obtain from equation (4)

$$\mathrm{Cov}\left(\int_s^T r_t dt, \int_s^T \bar{\lambda} dB_t | \mathcal{F}_s\right) = \mu_{s,T} - \frac{1}{2}\sigma_{s,T}^2 - r_s^x(T-s). \quad (37)$$

and

$$V_{s,T}^2 = \mathrm{Var}\left(\int_s^T r_t dt + \int_s^T \bar{\lambda} dB_t | \mathcal{F}_s\right) = \bar{\lambda}^2(T-s) + 2(\mu_{s,T} - r_s^x(T-s)). \quad (38)$$

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