RENORMALIZATION GROUP APPROACH IN SPECTRAL ANALYSIS AND PROBLEM OF RADIATION

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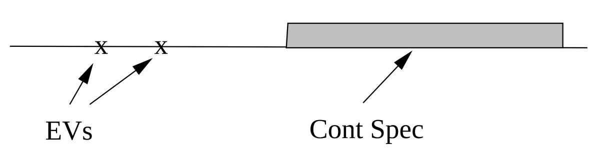
The text below contains the slides of the talk I have given at CRM on November 10, 2000. A related talk I gave at the Fields Institute on November 4, 2000. I completed sentences indicated on the slides, added a few explanations of the notation and concepts presented which I gave orally during the talks and inserted brief literature comments and a list of references. Apart from this I changed nothing. As a result the paper retains the informal style of the talk.

My gratitude goes to my collaborators and friends Volker Bach and Juerg Fröhlich, the joint work with whom is at the heart of this talk, to Volodya Buslaev, Stephen Gustafson, Peter Hislop, Walter Hunziker, Marco Merkli, Yuri Ovchinnikov, and Avy Soffer, joint work with whom was touched upon here or influenced my understanding of the questions presented.

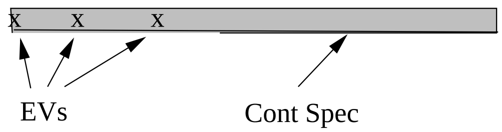
SPECTRAL ANALYSIS

I want to address the problem of perturbation of spectra of operators. For example, consider the problem of perturbation of a *single eigenvalue*. There are two possible cases:

1 Isolated eigenvalues



2 Embedded eigenvalues



In physical applications the second situation is generic, while the first one arises as a crude idealization when one considers a small part of a system in question.

Let us consider several examples of the second case.

HOPF BIFURCATION FROM SOLITONS

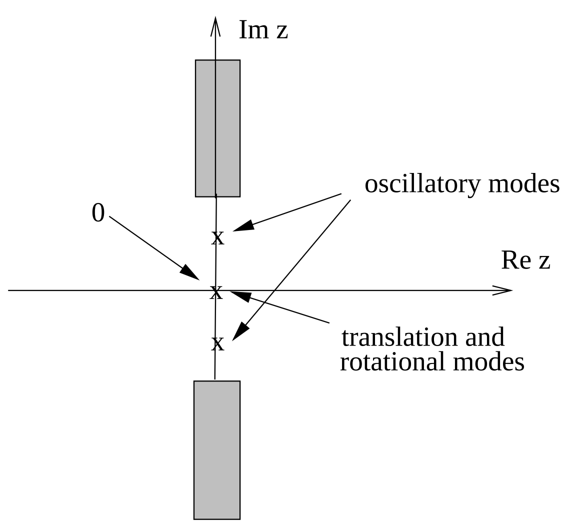
Consider the nonlinear Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + g\big(|\psi|^2\big)\psi,$$

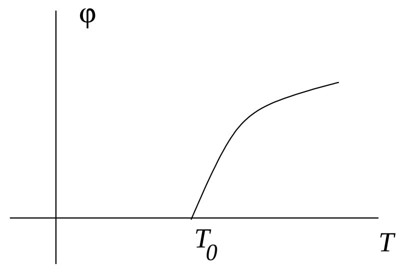
where ψ: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$. This equation has soliton solutions

$$\psi_{\rm sol}(x,t) = e^{i\Phi(x,t)}\ f(x - vt)$$

where $\Phi(x,t)$ is some real phase depending on the velocity v. The spectrum of fluctuations around $\psi_{\rm sol}$, i.e. of the linearization, $L_{\psi_{\rm sol}}$, of the r.h.s. around $\psi_{\rm sol}$, is



Do oscilatory modes lead to the bifurcation of time-periodic solutions?



Following the Hopf bifurcation analysis we have to consider the Floquet operator

$$-T^{-1}\frac{\partial}{\partial t} + L_{\psi_{\mathrm{sol}}} \quad \text{on} \quad L^2(\mathbb{R}^n \times S^1) \ ,$$

where S^1 is the unit circle and T is an unknown period of the bifurcating periodic solution we are looking for. The spectrum of this operator is

$$\mathrm{spec}(L_{\psi_{\mathrm{sol}}}) + iT^{-1}\mathbb{Z}, \tag{$*$}$$

where $\mathrm{spec}(L_{\psi_{\mathrm{sol}}})$ is shown on the figure preceding the one above. Spectrum $(*)$ consists of a continuum filling in the entire imaginary axis and translation/rotation and oscillatory eigenvalues repeated periodically and embedded into this continuum.

Thus the answer to the question of what kind of solution bifurcates from oscillatory modes depends on an understanding of what happens to embedded oscillatory modes under a nonlinear perturbation.

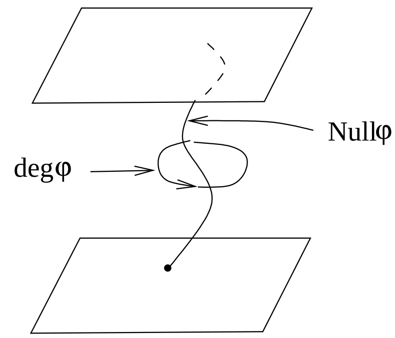
VORTEX SPECTRUM

Consider the Ginzburg-Landau equation

$$\Delta\varphi + (1 - |\varphi|^2)\varphi = 0$$

$$\varphi : \mathbb{R}^3 \to \mathbb{C}$$

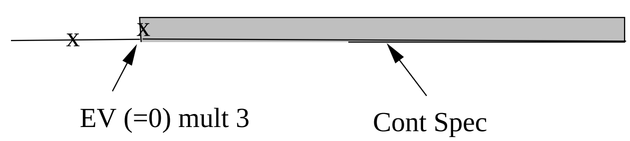
with the boundary condition that $|\varphi| \to 1$ as $|x^{\perp}| \to \infty$, where $x^{\perp} = (x_1, x_2)$ for $x = (x_1, x_2, x_3)$. Solutions of this equation can be specified by smooth curves of zeros of φ and a topological degree of φ with respect to these curves.



This equation has special–equivariant–solutions called *vortices*

$$\varphi_n = f_n(r)e^{in\theta} \ .$$

where (r, θ) are cylindrical coordinates. The spectrum of the linearized equation (i.e. of vortex fluctuations) is



(The negative eigenvalues are present for $|n| > 1$ and absent for $|n| = 1$.)

A detailed analysis of perturbation of the zero embedded eigenvalue is a key to understanding the dynamics of many (interacting) vortices.

QUANTUM SPECTRUM OF GEODESICS

Consider a space of curves given by their parameterizations, φ. Let $V(\varphi)$ be an energy of a curve φ. A quantization of $V(\varphi)$ yields the Schrödinger operator

$$-\Delta_\varphi + V(\varphi) \quad \text{on} \quad L^2(S', d\mu_C) \ , \tag{$*$}$$

where $d\mu_C$ is a Gaussian measure on the Schwartz space $S' = S(\mathbb{R}^n)$ and the meaning of the "Laplacian", Δ_φ, acting on functionals of the field $\varphi \in S'(\mathbb{R}^n)$ will be alluded at later.

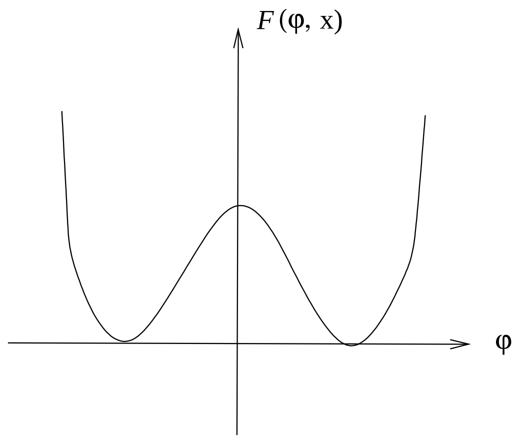
Now, let φ_{CP} be a critical point of $V(\varphi)$. The question we want to ask is: What are the *quantum corrections to the energy of φ_{CP}?*

Answering this question involves understanding the low energy spectrum of $(*)$ near the classical energy $V(\varphi_{CP})$ which in turn leads to a perturbation of embedded eigenvalues and the nearby spectrum.

In a special situation φ_{CP} could be a geodesic or, more generally, a minimal submanifold.

An important example of the situation above is that of quantum vortices. In this case φ: $\mathbb{R}^3 \to \mathbb{C}$ and $V(\varphi)$ is of the form

$$V(\varphi) = \int \frac{1}{2}|\nabla\varphi|^2 + F(\varphi, x) \tag{$**$}$$



double-well potential

The (line) vortices arise as critical points of $V(\varphi), \varphi : \mathbb{R}^3 \to \mathbb{C}$, satisfying certain topological conditions (see above). The latter conditions imply that the null sets of these critical points are curves which are geodesics in a certain Riemannian metric (see a figure above).

One can think of the dynamics of vortices as motion of their centers – Null φ – with relatively rigid vortex rigging around them.

Another interesting case is that of functional $(**)$ with x-independent $F \geq 0$ and for $\varphi : [0, 1] \to \mathbb{R}^m$. In this case, critical points of $V(\varphi)$ are (modulo parametrization) geodescics in the Riemannian metric $ds^2 = F(y)\, dy^2$ (Jacobi metric). The latter fact is related to Maupertuis principle in Classical Mechanics.

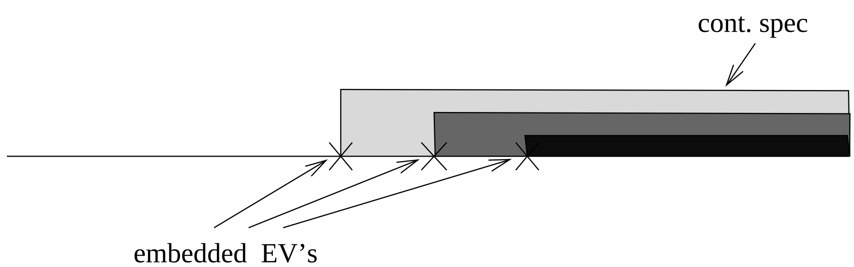
PROBLEM OF RADIATION

I want to present an example of a common physical situation when a small system (with finite number of degrees of freedom) is coupled to a large system (of infinite number of degrees of freedom) – the problem of radiation. This problem is reduced to finding the low energy spectrum of the quantum *Hamiltonian for the system of matter and radiation*

$$H(e) = \sum \frac{1}{2m_j} p_{j,eA}^2 + V(x) + H_{\mathrm{rad}}$$

on $\mathcal{H}_{\mathrm{matter}} \otimes \mathcal{H}_{\mathrm{rad}}$ (Schrödinger equation coupled to quantized Maxwell equations).

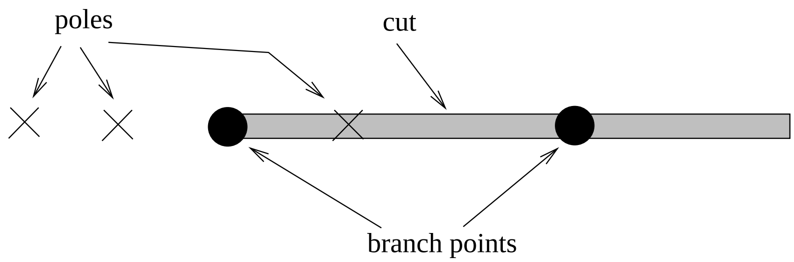
SPEC $H(0)$



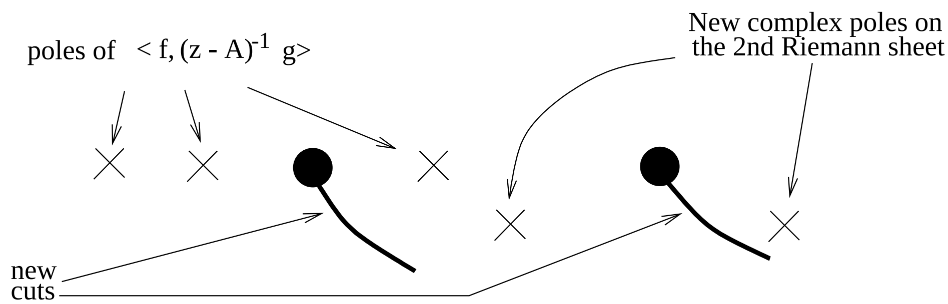
The spectrum of the unperturbed (=uncoupled) Hamiltonian $H(0)$ contains eigenvalues sitting on the top of the thresholds of continuous spectrum. They correspond to bound states of an atom in a vacuum. Are these bound states stable or unstable (when $e \neq 0$)?

REFINEMENT OF NOTION OF SPECTRUM

Standard notions of spectral analysis are insufficient for treating perturbation of embedded eigenvalues. We extend the notion of spectrum as follows. Consider a self-adjoint operator H on a Hilbert space $\mathcal{H}$. Then *point and continuous spectra are poles and cuts of* $\langle f, (z-H)^{-1} g\rangle \; \forall f, g \in \mathcal{H}$



Consider the *Riemann surface* of $\langle f, (z-H)^{-1}g\rangle$ for f and g in some dense set $\mathcal{D} \subset \mathcal{H}$. In other words we want to continue this analytic function from, say, $\mathbb{C}^+$ across the cut (continuous spectrum of H) into the second Riemann sheet:



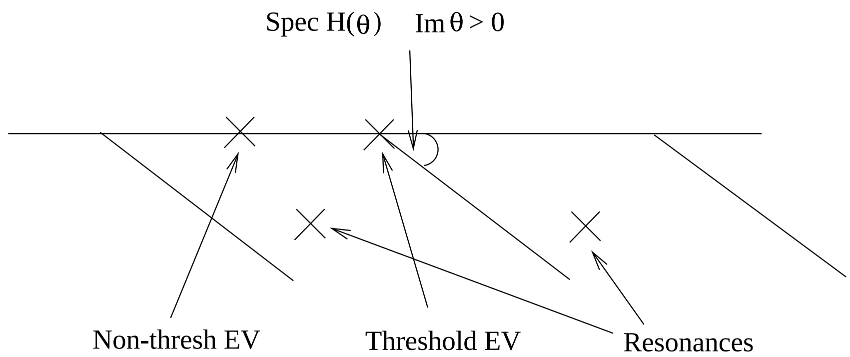
We see that non-threshold eigenvalues of H become isolated poles of this analytic continuation while new complex poles, not seen before, are revealed. Clearly, real poles coming from embedded eigenvalues and complex poles must be treated on the same footing.

DEFORMATION OF SPECTRA

Now I outline a constructive tool used in the study of the Riemann surface for a given operator H–the spectral deformation method. It goes as follows. Consider the orbit

$$H \to H(\theta) = U(\theta)HU(\theta)^{-1}$$

of H under a one-parameter group, $U(\theta)$, of unitary operators, s.t. $H(\theta)$ has an *analytic continuation in θ into a neighbourhood of $\theta = 0$.* The spectrum of such a continuation looks typically as on the figure below.

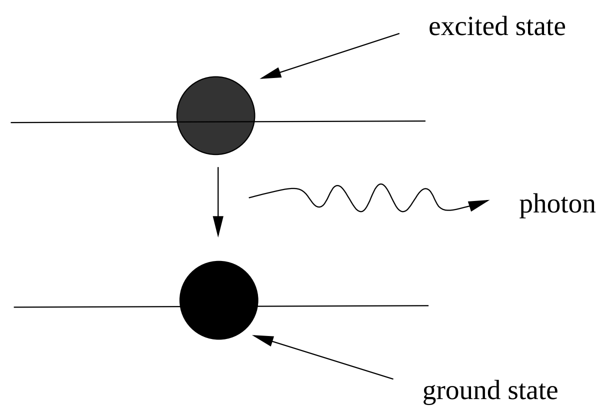


The resolvent $\big(H(\theta) - z\big)^{-1}$ provides the desired information about the Riemannian surface of the operator H. In particular, the real eigenvalues of $H(\theta)$ coincide with the eigenvalues of H, i.e. with the real poles mentioned above, while the complex eigenvalues of $H(\theta)$ are related to the complex poles on the second Riemann sheet. These complex eigenvalues are called the *resonances* of H.

Thus the problem of understanding the behaviour of embedded eigenvalues and the continuous spectrum of H under a perturbation is reduced to the problem of understanding the complex spectrum of the operator $H(\theta)$ for complex θ's.

MATHEMATICAL PROBLEM OF RADIATION

The goal here is to construct a *mathematical theory of emission and absorption of electro-magnetic radiation by systems of non-relativistic matter s.a. atoms and molecules:*



Mathematically, this translates into the problem of understanding the bound state–resonance structure of the quantum Hamiltonian of a system of quantum matter coupled to quantum radiation.

QUANTIZED MAXWELL EQUATIONS

I review quickly a mathematical framework of quantum theory of radiation. First, I describe the quantized Maxwell equations and then their coupling to quantum matter.

The quantized Maxwell equations can be presented as the Schrödinger equation $\frac{\partial \phi}{\partial t} = H_{\rm rad}\phi$ with the quantum Hamiltonian operator

$$H_{\rm rad} := \frac{1}{2}\int : E^{\rm op}(x)^2 + \big({\rm curl} A^{\rm op}(x)\big)^2 : d^3x$$

acting on $\mathcal{H}_{\rm rad} = L^2(S', d\mu_C)$. Here S' is the Schwartz space of the transverse vector fields, $A(x)$, ${\rm div} A(x) = 0$, on $\mathbb{R}^3$, $d\mu_C$ is the Gaussian measure on S' with the mean 0 and covariance $C = (-\Delta)^{-\frac{1}{2}}$, $A^{\rm op}(x)$ and $E^{\rm op}(x)$ are the quantum operators of the vector potential and electric field in the Couloub gauge,

$$\begin{aligned} A^{\rm op}(x) &= \text{operator of multiplication by } A(x) \\ E^{\rm op}(x) &= -i\frac{\delta}{\delta\, A(x)} + iC^{-\frac{1}{2}}A(x)\ , \end{aligned}$$

and the double colons signify the Wick, or normal, ordering, i.e. some sort of deformation of the quantization procedure.

Now we explain briefly an origin of this Hamiltonian.

The Maxwell equations in a vacuum is an infinitely dimensional Hamiltonian system with the *Hamiltonian functional*

$$H(A, E) = \frac{1}{2}\int E^2 + ({\rm curl}\, A)^2$$

defined on the phase-space $H_1^{\rm transv} \times L_2^{\rm transv}$ equipped with a standard symplectic structure. Here $A(x)$ is the vector potential in the Coulomb gauge and $E(x)$ is the electric field (the field conjugate to $A(x)$), which are transverse vector fields on $\mathbb{R}^3$ (i.e. ${\rm div}\, A(x) = 0$ and ${\rm div}\, E(x) = 0$), and $H_1^{\rm transv}$ and $L_2^{\rm transv}$ are the Sobolev space of order 1 and L_2-space of transverse vector fields.

A naive quantization of this dynamical system patterned on the quantization of the Newton equations goes as follows.

Concept	**CFT**	**QFT**
Phase/state space	$H_1^{\text{transv}} \otimes L_2^{\text{transv}}$	"$L^2(H_1^{\text{transv}}, DA)$"
Symplectic structure	Poisson brackets	commutators
Canonic. variables (w.r. to the symplectic structure)	$A(x)$	$A^{\text{op}}(x) =$ operator of multiplication by $A(x)$
	$E(x)$	$E^{\text{op}}(x) = -i\frac{\delta}{\delta A(x)}$
Observables	Real functionals $f(A, E)$ on $H_1^{\text{transv}} \otimes L_2^{\text{transv}}$	Self-adjoint operators $A = f(A^{\text{op}}, E^{\text{op}})$ on "$L^2(H_1^{\text{transv}}, DA)$"
Dynamics	Ham.fnctnl $H(A, E)$	Ham.opr $H(A, E)$

The third column does not make sense mathematically, but its natural modification leads to the formulation presented at the beginning of this section.

MATTER

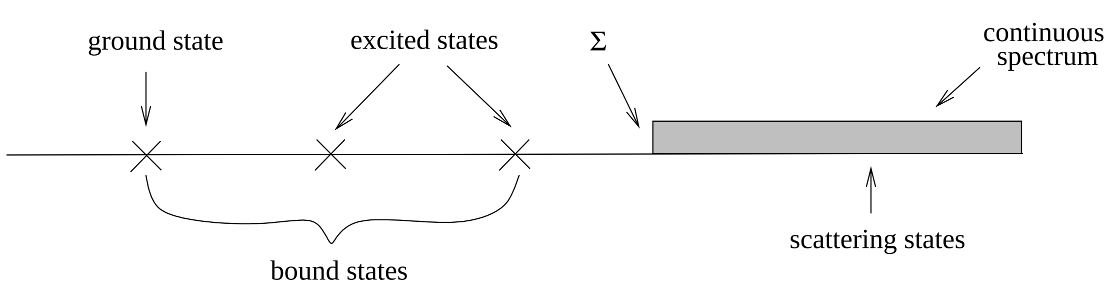
Quantum non-relativistic matter is described by a Schrödinger operator of the form (in units: $\hbar = 1$, $c = 1$, and $m_e = 1$)

$$H_{\text{matter}} = \sum_1^N \frac{1}{2m_j} p_j^2 + V(x)$$

on $\mathcal{H}_{\text{matter}}$(e.g. $L^2(\mathbb{R}^{3N})$). Here $p_j = -i\nabla_{x_j}$, $m_j > 0 \ \forall j$ and $x = (x_1, \ldots, x_N)$.

SPECTRUM OF H_{matter}

Typically, the spectrum of H_{matter} is of the form (Hunziker-van Winter-Zhislin Theorem)



MATTER + RADIATION

To introduce the coupling between matter and radiation, we think about the vector potential $A(x)$ as a *quantum connection* on $\mathbb{R}^3$ and pass to the covariant derivatives

$$p \rightarrow p_A = p - eA(x) \ .$$

This leads to the *Hamiltonian for the system of matter and radiation*

$$H(e) = \sum \frac{1}{2m_j} p_{j,A}^2 + V(x) + H_{\text{rad}}$$

on $\mathcal{H}_{\text{matter}} \otimes \mathcal{H}_{\text{rad}}$. This operator is not well defined. To remedy this we introduce

Ultraviolet cut-off: $\quad A(x) \rightarrow \chi * A(x), \ \int |\chi|^2 d^3k < \infty$

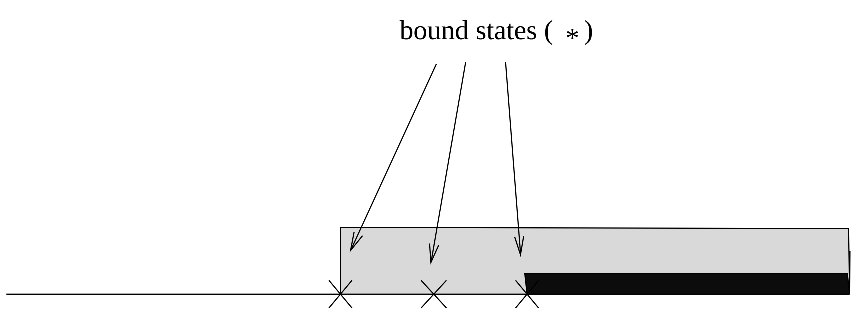
in the interaction terms $p_{j,A}^2$. The resulting operator (which we still denote by the same symbol $H(e)$) is *self-adjoint* and is bounded from below.

MATHEMATICAL PROBLEM

Problem of Radiation: Fate of the bounded states of matter

$$\Big(\underbrace{\varphi_j}_{\substack{\text{bound state of matter} \\ \text{with energy } E_j}} \otimes \underbrace{\Omega}_{\substack{\text{vacuum of} \\ \text{quantized EM field}}} \Big) e^{-iE_j t} \ , \ \forall j \tag{$*$}$$

as charges are “turned on”.



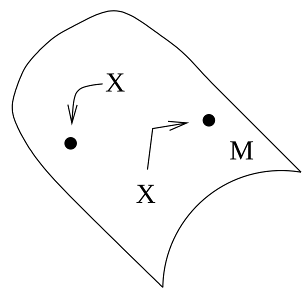
Spec $H(0)$

E.g. one would like to show that an atom in an excited state in a vacuum is unstable, that it emits a photon and descends into the stable ground state.

RENORMALIZATION GROUP APPROACH

Assume we want to study a part of the spectrum of the operator $H(e)$ near E_0. We proceed as follows:

- Pass to a metric space M of operators
- Construct a flow Φ_τ on M s.t.
  - Φ_τ eliminates "inessential" degrees of freedom
  - Φ_τ is isospectral in $|z - E_0| \leq e^{-\tau}$
- Find fixed points of Φ_τ and their stability.



Observe now that

- Isospectrality of Φ_τ allows us to transfer the spectral information we have about fixed points to the initial operator $H(e)$;
- Classify possible behaviour of physical systems in question according to the fixed points to which they are attracted.

Since ϕ_τ eliminates "inessential" degrees of freedom (a kind of partial dissipation) we expect that the Hamiltonians $H_\tau := \phi_\tau(H)$ at levels τ simplify as $\tau \to \infty$ and in particular that the fixed points are especially simple.

DECIMATION MAP

In order to define the RG-flow we first define a map, called the decimation map, which *eliminates inessential degrees of freedom.* First, we observe that the map

$$H \to P_\tau H P_\tau \ ,$$

where P_τ is the spectral projector associated with the inequality $|H(0) - E_0| \leq e^{-\tau}$, eliminates the part of H acting on $(\mathrm{Ran} P_\tau)^\perp$ but it *distorts Spec* H. We modify this map in order to restore the spectral fidelity in a small neighbourhood of $z = 0$:

$$D_\tau : H \to P_\tau(H - HR^\perp H)P_\tau \ ,$$

where $R^\perp = P_\tau^\perp (P_\tau^\perp H P_\tau^\perp)^{-1} P_\tau^\perp$ and $P_\tau^\perp = \mathbf{1} - P_\tau$. This new map is called the *decimation map.*

Trade-off: D_τ is isospectral at $z = 0$, but is *non-linear*.

RESCALING

Next, we introduce the rescaling map S_τ acting on operators by a unitary conjugation, $S_\tau : H \to U_\tau H U_\tau^{-1}$, which rescales the photon momenta as $k \to e^{-\tau} k$. As a result we have

$$S_\tau : |H(0) - E_0| \le e^{-\tau} \to |H(0) - E_0| \le 1 \ .$$

RG - FLOW

Now we are ready to define the RG-flow:

$$\Phi_\tau = E_\tau \circ S_\tau \circ D_\tau \ ,$$

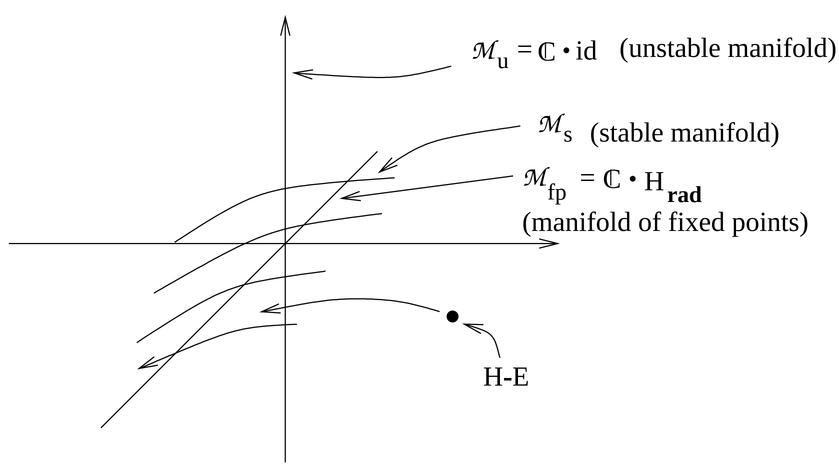
where $E_\tau(A) = e^\tau A$, a normalization map.

Observe that Φ_τ has the following properties

- Φ_τ is a semi-flow
- Φ_τ projects out $|H(0) - E_0| \ge e^{-\tau}$ and magnifies the result
- Φ_τ is isospectral in $\{z \in C||z| \le e^{-\tau}\}$ modulo the factor e^τ.

RG FLOW DIAGRAM

The figure below shows fixed point, stable, and unstable manifolds of Φ_τ. The point here is that the fixed point manifold is very simple: $\mathcal{M}_{fp} = \mathbb{C} \cdot H_{\rm rad}$, while the stable manifold, $\mathcal{M}_s$, has the codimension 2. Hence given an operator H there is a complex number $E = E(H)$, s.t. $H - E \in M_s$.



Now proceed as follows. Apply the RG-flow to $H - E$. Then $\Phi_\tau(H - E)$ converges to the fixed point manifold so that for τ sufficiently large

$$\Phi_\tau(H - E) \approx w H_{\rm rad} \text{ for some } w \in \mathbb{C}$$

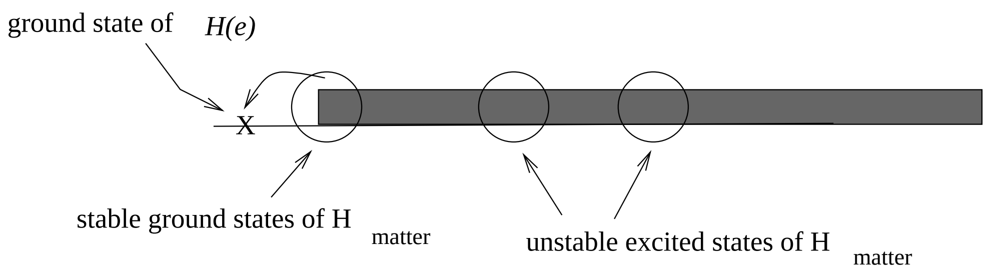
and as a result H is isospectral to $wH_{\mathrm{rad}} - E$ in $\{|z - E| \leq e^{-\tau}\}$. This way we transfer the spectral information about H_{rad}, which is available to us, to the operator H.

MATHEMATICAL RESULTS (Bach-Fröhlich-IMS)

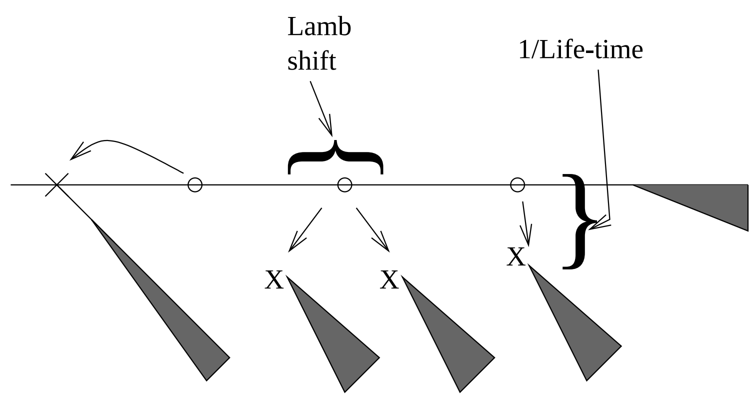
Assume that $|e|$ is sufficiently small and, in the third statement below, that the particle potential, $V(x)$, is confining, i.e. $V(x) \to \infty$ as $|x| \to \infty$. Then we have

I. **Binding.** $H(e)$ has a ground state ψ. $\|e^{\alpha|x|}\psi\| < \infty$ $\alpha > 0$.

II. **Instability.** $H(e)$ has no EVs near the excited EVs of H_{matter}.



III. **Resonances.** The excited states of H_{matter} bifurcate into resonances of $H(e)$.



Projection of Riemann surface of $H(e)$ onto $\mathbb{C}$

(EV stands for "eigenvalue".)

REMARKS

The result on the vortex spectrum mentioned is equivalent to the property of (linearized) stability/instability of vortices (see [Gus, GS, LL, M, OS1]). Recent results on dynamics of vortices are reviewed in [OS2].

The spectrum of critical points of the potential in Quantum Mechanics was found [Sim, BCD, Sj].

One can also try to understand spectra of critical points of the action functional

$$S(\phi) = \int \left(\int \frac{1}{2} |\dot{\phi}|^2 \, dx - V(\phi) \right) dt$$

which are periodic in time.

The method of spectral deformation and the theory of resonances based on it were proposed by Aguilar, Balslev, Combes and Simon and extended in works of Balslev, Hunziker, Jensen, Sigal, Simon and others. See [HisSig, HunSig] for recent reviews. A different approach was proposed by Helffer and Sjöstrand (see [HeSj, HeM]).

For a text on rigorous quantum field theory see [GJ] and for a physical discussion of the problem of radiation, [C-TD-RG].

A recent review of the quantum theory of many particle systems can be found in [HunSig].

The renormalization group approach and the results on the radiation problem presented here were obtained in [BFS1-3], where the reader can find many references to the earlier or simultaneous work (see work [HuSp] for a review of this and related work). Some improvements of these results are given in [BFS4]. The result on the ground state was improved in [AH1, AH2, G, GLL, H, Hirosh, Sp]. Further important progress involving scattering theory was made in [DG, FGS].

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