Partition Theory and Banach Spaces

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Notation

Given a set X, we denote by $[X]^2$ the set of all unordered pairs in X:

$$[X]^2 = \{\{a,b\} : a \neq b \in X\}$$

Similarly we will define $[X]^d$ for all $d \in \mathbb{N} \cup \{\omega\}$. We will identify $[\mathbb{R}]^2$ with the set

$$\{\langle a,b\rangle \in \mathbb{R}^2 : a < b\}.$$

In this case, given some $\{a,b\} \in [\mathbb{R}]^2$, we will write $\{a,b\}_<$ to denote that $a < b$. Similar notations and conventions will be used for $[\mathbb{R}]^d$ for all $d \in \mathbb{N}$. Given some $a = \{a_0, a_1, \ldots, a_{d-1}\}_< \in [\mathbb{R}]^d$, and some $X \subset d$ we will write

$$a \restriction_X = \{a_i : i \in X\}.$$

Let $A \in [\mathbb{N}]^\omega$. Given $m \in \mathbb{N}$, we denote $A/m = A \setminus (m+1) = A \setminus \{0,1,\ldots,m\}$. Given some finite $S \subset \mathbb{N}$, we denote $A/S = A/(\max S)$. If $m > \max S$ we write $S^\frown m = S \cup \{m\}$.

Ramsey's Theorem

Theorem (Ramsey's Theorem). *If $k, d \in \mathbb{N}$ and $f : [\mathbb{N}]^d \to k$, then there is an $A \in [\mathbb{N}]^\omega$ such that $f \restriction_{[A]^d}$ is constant. We call the set A **f-homogeneous**, or, if $f\left[[A]^d\right] = \{i\}$, it will be called **$i$-homogeneous**.*

In symbols, this is expressed $\omega \to (\omega)^d_k$.

Proof 1. This proof will be for the specific case where $d = k = 2$, though it can easily be modified for any $k \in \mathbb{N}$. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. For each $m \in \mathbb{N}$ and $i \in \{0,1\}$ set

$$A_{m,i} = \{n \neq m : f(\{m,n\}) = i\}.$$

If neither $A_{m,0}$ nor $A_{m,1}$ is in $\mathcal{U}$, then $\{m\} = (\mathbb{N} \setminus A_{m,0}) \cap (\mathbb{N} \setminus A_{m,1}) \in \mathcal{U}$, contradicting the fact that $\mathcal{U}$ is free. Thus for each m there is an $i_m \in \{0,1\}$ such that $A_m = A_{m,i_m} \in \mathcal{U}$. There is then some $i \in \{0,1\}$ such that $A = \{m \in \mathbb{N} : i_m = i\} \in \mathcal{U}$.

Choose some $n_1 \in A$. Suppose that for some $k \in \mathbb{N}$ that $n_1, \ldots, n_k \in A$ have been chosen so that $n_l \in A_{n_j}$ for all $1 \le j < l \le k$. Choose $n_{k+1} \in \left(A \cap \bigcap_{j=1}^k A_{n_j}\right) / n_k$. Having done this for all $j \in \mathbb{N}$, set $B = \{n_j : j \in \mathbb{N}\}$.

Given $j < l \in \mathbb{N}$ we have that $i_{n_j} = i$, and $n_l \in A_{n_j}$ so $f(\{n_j, n_l\}) = i$, and B is i-homogeneous. □

Definition. *A set $\mathcal{F} \subset [\mathbb{N}]^\omega$ is called **dense** if $\mathcal{F}$ is closed under countably infinite subsets, and for each $A \in [\mathbb{N}]^\omega$ there is some $B \in [A]^\omega \cap \mathcal{F}$.*

Theorem (Diagonalization Lemma). *1. If $\mathcal{F}_m$ is dense for all $m \in \mathbb{N}$, then there is an $A \in [\mathbb{N}]^\omega$ such that $A/m \in \mathcal{F}_m$ for all $m \in A$.*

2. If $\mathcal{F}_S$ is dense for all finite $S \subset \mathbb{N}$, then there is an $A \in [\mathbb{N}]^\omega$ such that $A/S \in \mathcal{F}_S$ for all finite $S \subset \mathbb{N}$ with $\max S \in A$.

Proof. To prove (1), we will recursively find $B_1 \supset B_2 \supset \dots$ and $n_1 < n_2 < \dots$ as follows. Set $n_1 = 1$, and pick $B_1 \in \mathcal{F}_1$. Suppose that $B_1 \supset \dots \supset B_m$ and $n_1 < \dots < n_m$ have been chosen for some $m \in \mathbb{N}$ such that for $1 \le i < m$ we have $n_{i+1} \in B_i$, and $B_i \in \mathcal{F}_i$ for $1 \le i \le m$. Let $n_{m+1} = \min(B_m/n_m)$, and choose

$$B_{m+1} \in [B_m]^\omega \cap \bigcap_{i=1}^{n_{m+1}} \mathcal{F}_i.$$

which can be done, as any finite intersection of dense sets can be shown to be dense.

Now set $A = \{n_i : i \in \mathbb{N}\}$. For $i \in \mathbb{N}$ we have

$$A/n_i = \{n_j : j > i\} \subset B_i \in \mathcal{F}_{n_i}.$$

We will now demonstrate that (1) implies (2). Given a collection $\{\mathcal{F}_S : S \in [\mathbb{N}]^{<\omega}\}$ of dense sets, for each $m \in \mathbb{N}$ set

$$\mathcal{F}'_m = \bigcap_{S \subset m+1} \mathcal{F}_S.$$

Then $\{\mathcal{F}'_m : m \in \mathbb{N}\}$ is a collection of dense sets, so by (1) there is an $A \in [\mathbb{N}]^\omega$ such that $A/m \in \mathcal{F}'_m$ for all $m \in A$. Given some finite $S \subset \mathbb{N}$ with $m = \max S \in A$, we have

$$A/S = A/m \in \mathcal{F}'_m \subset \mathcal{F}_S.$$

□

Proof 2 of Ramsey's Theorem. This proof is by induction on d, and requires the following fact: suppose that $f : [\mathbb{N}]^d \to k$ has a homogeneous set, then the family of f-homogeneous sets in dense.

The case $d = 1$ is the Pigeonhole Principle. Assume that the case $d = n$ has been proved, and fix some $f : [\mathbb{N}]^{n+1} \to k$. For each $m \in \mathbb{N}$ define $g_m : [\mathbb{N}/m]^n \to k$ by

$$g_m(S) = f(\{m\} \cup S).$$

By the induction hypothesis, $\mathcal{F}_{g_m} = \{A \in [\mathbb{N}/m]^n : A \text{ is } g_m\text{-homogeneous}\}$ is dense. By the Diagonalization Lemma, let $A \in [\mathbb{N}]^\omega$ be such that $A/m \in \mathcal{F}_{g_m}$ for all $m \in A$.

For each $m \in A$ let $i_m < k$ be such that (A/m) is i_m-homogeneous. Let $i < k$ be such that $B = \{m \in A : i_m = i\}$ is infinite. Given $S \in [B]^{n+1}$, let $m = \min S$, and let $S' = S \setminus \{m\} = S/m$. Then $S' \in [B/m]^n \subset [A/m]^n$ and so $f(S) = g_m(S') = i$. Thus B is i-homogeneous. □

Proposition. *If $\langle x_n \rangle_{n \in \mathbb{N}}$ is a sequence in $[0,1]$, then there is a convergent subsequence.*

Proof 1. Define $f : [\mathbb{N}]^2 \to 2$ by

$$f(\{m,n\}_<) = \begin{cases} 0, & \text{if } x_m < x_n \\ 1, & \text{if } x_m \ge x_n. \end{cases}$$

Then there is an f-homogeneous $A \in [\mathbb{N}]^\omega$, and $\langle x_n \rangle_{n \in A}$ is monotonic. The proposition follows from the fact that every bounded monotonic sequence converges. □

Proof 2. This proof will apply equally well to any compact metric space X.

Define $f : [\mathbb{N}]^3 \to 2$ by

$$f\left(\{k, m, n\}_<\right) = \begin{cases} 1, & \text{if } d(x_m, x_n) < \frac{1}{k} \\ 0, & \text{if } d(x_m, x_n) \geq \frac{1}{k}. \end{cases}$$

Then there is a f-homogeneous set $A \in [\mathbb{N}]^\omega$. If A is 1-homogeneous, then $\langle x_n \rangle_{n \in A}$ is a Cauchy sequence, and therefore convergent.

If A is 0-homogeneous, let $k = \min A$. Then for all $\{m, n\} \in [A/k]^2$ we have $d(x_m, x_n) \geq \frac{1}{k}$. From this it follows that X cannot be covered by finitely many balls of radius $\frac{1}{2k}$, contradicting the compactness of X. □

Canonical Partitions

We will now discuss what we can say about functions $f : [\mathbb{N}]^d \to \mathbb{N}$, or more generally $f : [\mathbb{N}]^d \to Y$.

Definition (Erdös-Radó). *A partition $f : [\mathbb{N}]^d \to Y$ is **canonical** if there is an $X \subset d$ such that for all $S = \{s_0, \ldots, s_{d-1}\}_<, T = \{t_0, \ldots, t_{d-1}\}_< \in [\mathbb{N}]^d$ we have*

$$f(S) = f(T) \Leftrightarrow \{s_i : i \in X\} = \{t_i : i \in X\}.$$

Example. *Let $f : [\mathbb{N}]^d \to Y$ be a canonical partition with witness $X \subset d$.*

If $d = 1$, then either $X = \emptyset$ or $\{0\}$. If $X = \emptyset$, then it must be that f is a constant function. If $X = \{0\}$, then f is injective.

If $d = 2$, then $X = \emptyset, \{0\}, \{1\}$, or $\{0, 1\}$. If $X = \emptyset$, then f is constant. If $X = \{0, 1\}$, then f is injective. If $X = \{0\}$, then f depends only on $\min S$ for all $S \in [\mathbb{N}]^d$, and similarly if $X = \{1\}$ then f depends only on $\max S$.

Theorem (Erdös-Radó). *If $f : [\mathbb{N}]^d \to Y$, then there is an $A \in [\mathbb{N}]^\omega$ such that $f \restriction_{[A]^d}$ is canonical.*

Proof. The proof is by induction on d.

If $d = 1$, then either f takes on infinitely many values, or it takes a single value infinitely often. In the former case, we may choose A to contain a single element from the preimage of each element in the range of f. In the latter case we choose A to be the preimage of that element.

Assume that the case $d = n$ has been proved, and fix $f : [\mathbb{N}]^{d+1} \to Y$. For each $m \in \mathbb{N}$, let $\mathcal{F}_m$ be the set of all infinite $B \subset \mathbb{N}$ such that for all (not necessarily distinct) $s, t \in [m]^d$, either

(a) $f(s^\frown i) = f(t^\frown j)$ for all $i, f \in B/m$, or

(b) $f(s^\frown i) = f(t^\frown j) \Leftrightarrow i = j$ for all $i, j \in B/m$, or

(c) if $m = \max t$, and $j \mapsto f(t^\frown j)$ is not constant on B/m, then $f(s^\frown i) \neq f(t^\frown j)$ for all $i \in B/s$ and all $j \in B/t$.

Claim. *$\mathcal{F}_m$ is dense.*

Proof of Claim. To do this, we will show that for any particular $s, t \in [m]^d$ the set $\mathcal{F}_m^{s,t}$ defined similarly to the above is dense, and thus, as $\mathcal{F}_m = \bigcap_{s,t \in [m]^d} \mathcal{F}_m^{s,t}$ we will have our result.

It can easily be shown that $\mathcal{F}_m^{s,t}$ is closed under countably infinite subsets by the fact that the conditions (a), (b), (c) are all individually closed under countably infinite subsets.

Define $g : [\mathbb{N}/m]^2 \to 2$ by

$$g(\{i,j\}_<) = \begin{cases} 1, & \text{if } f(s^\frown i) = f(t^\frown j) \\ 0, & \text{otherwise.} \end{cases}$$

Then there is a g-homogeneous set $B \in [\mathbb{N}]^\omega$.

If B is 1-homogeneous, then it can be shown that $A = B/\min B$ satisfies (a).

If B is 0-homogeneous, we consider the function $i \mapsto f(t^\frown i)$ defined on B. If this takes on some value $y \in Y$ infinitely often on B, then by setting A to be the inverse image of y under the maping on B, we have that A will trivially satisfy (c).

If the mapping takes on infinitely many values on B, then we may let B' be a choice set for the inverse image of the range of the mapping. Consider finally $C = \{i \in B' : f(s^\frown i) = f(t^\frown i)\}$. If C is infinite, then C will satisfy (b). If C is finite, then we may construct an infinite $A \subset B' \setminus C$ which satisfies (c).

It follows that for all g-homogeneous $B \in [\mathbb{N}]^\omega$ there is an $A \in [B]^\omega$ in $\mathcal{F}_m^{s,t}$, and thus each $\mathcal{F}_m^{s,t}$ is dense, and then so, too, is

$$\mathcal{F}_m = \bigcap_{s,t \in [m]^d} \mathcal{F}_m^{s,t}.$$

◇

By the Diagonalization Lemma, take $B \in [\mathbb{N}]^\omega$ such that $B/m \in \bigcap_{i \leq m} \mathcal{F}_i$ for all $m \in B$. Define $g : [B]^d \to Y \cup \{Y\}$ by

$$g(s) = \begin{cases} a \in Y, & \text{if } f(s^\frown i) = a \text{ for all } i \in B/s \\ Y, & \text{if } f(s^\frown i) \neq f(s^\frown j) \text{ for all } i \neq j \in B/s. \end{cases}$$

(By the fact that $B/\max s \in \mathcal{F}_{\max s} \subset \mathcal{F}_{\max s}^{s,s}$ it can be shown that g is well-defined.) By the induction hypothesis, let $C \in [B]^\omega$ be such that g is canonical on C with witness $X \subset d$. An application of Ramsey's Theorem allows us to assume without loss of generality that either $g(s) \neq Y$ for all $s \in [C]^d$, or that $g(s) = Y$ for all $s \in [C]^d$.

Case 1: If $g(s) \neq Y$ for all $s \in [C]^d$, then let $u = s^\frown i, v = t^\frown j \in [C]^{d+1}$. We have

$$f(u) = f(s^\frown i) = g(s)$$
$$f(v) = f(t^\frown j) = g(s)$$

and these are equal iff $u \restriction_X = s \restriction_X = t \restriction_X = v \restriction_X$ iff $f(u) = f(v)$, and so f is canonical on C.

Case 2: If $g(s) = Y$ for all $s \in [C]^d$, we make the following claim:

Claim. *Given $u, v \in [C]^{d+1}$, if $\max u \neq \max v$, then $f(u) \neq f(v)$.*

Proof. Letting $u = s^\frown i$ and $v = t^\frown j$, we may assume that $\max t \geq \max s$. If $f(s^\frown i) = f(t^\frown j)$, then condition (c) fails for $m = \max t$.

Since $i \mapsto f(t^\frown i)$ is not constant, as $t \in [C]^d$, we arrive at a contradiction. ◇

For $s \in [C]^d$, let

$$\mathcal{E}_s = \{t \in [C]^d : \forall i \in C/(s \cup t), f(s^\frown i) = f(t^\frown i)\}.$$

For $s \neq t \in [C]^d$, we either have $\mathcal{E}_s \cap \mathcal{E}_t = \emptyset$, or $\mathcal{E}_s = \mathcal{E}_t$. Define $g : [C]^d \to \{\mathcal{E}_s : s \in [C]^d\}$ by $g(s) = \mathcal{E}_s$. Find $D \in [C]^\omega$ such that g is canonical on D. Then there is some $\overline{X} \subset d$ such that for all $s, t \in [D]^d$,

$$\mathcal{E}_s = \mathcal{E}_t \Leftrightarrow s\restriction_{\overline{X}} = t\restriction_{\overline{X}}$$

Let $X = \overline{X} \cup \{d\}$. Let $u = s^\frown i, v = t^\frown j \in [D]^{d+1}$.

If $f(u) = f(v)$ for $u, v \in [D]^{d+1}$, then $\max u = \max v$, by then above claim, and $\mathcal{E}_s = \mathcal{E}_t$. We have $\mathcal{E}_s = \mathcal{E}_t$ iff $s\restriction_{\overline{X}} = t\restriction_{\overline{X}}$, and so in all we have $u\restriction_X = v\restriction_X$.

If $u\restriction_X = v\restriction_X$, then in particular $s\restriction_{\overline{X}} = t\restriction_{\overline{X}}$, and so $\mathcal{E}_s = \mathcal{E}_t$. As $i = \max u = \max v = j$, we have $f(u) = f(s^\frown i) = f(t^\frown j) = f(v)$.

Thus f is canonical on D. $\square$