1. Ramsey theory and Banach spaces, Fields Institute, Fall 2002. Problem Set #2
September 26, 2002

Hand in the solutions to exercises #1(4), #2, #3, #4 (1), #5 and #6.
Due date is **October 7, 2002**.

Exercise 1. *Use your favorite instance of compactness, Ramsey's theorem and/or Erdös–Rado theorem to prove the following:*

(1) *For all triples of natural numbers n, d and k there is a natural number m such that for every $f\colon [m]^d \to k$ there is an f-homogeneous $S \subseteq m$ of size n.*
(2) *For all pairs of natural numbers d, k there is a natural number m such that for every function $f\colon [m]^d \to k$ there is an f-homogeneous $S \subseteq m$ such that $|S| \geq \min(S)$.*
(3) *A function $f\colon [m]^d \to m$ is* regressive *if $f(s) < \min s$ for all s. A set S is* min-homogeneous *for f if $f(s) = f(t)$ for all s and t in $[S]^d$ such that $\min(s) = \min(t)$. Prove that for all natural numbers n and d there is a natural number m such that for every regressive $f\colon [m]^d \to m$ there is a min-homogeneous $S \subseteq m$ of size n.*
(4) *Prove that for every natural number d there is a natural number m such that for every regressive $f\colon [m]^d \to m$ there is a min-homogeneous $S \subseteq m$ such that $|S| > \min(S)$.*

By T we denote the so-called *Tsirelson's space*, as defined in class. Note that this is the dual of the space originally constructed by B. Tsirelson. By $\|x\|_T$ we denote the T-norm of a vector x. By $\{e_i\}_{i=1}^{\infty}$ we denote the standard basis for $\mathbb{R}^{\mathbb{N}}$. So for a vector $x = \sum_{i=1}^{\infty} a_i e_i$ we can write $\|x\|_{\ell_1}$, $\|x\|_T$, to denote its ℓ_1-norm or its T-norm, respectively. (Note that some of the values may be infinite.)

Exercise 2. *Find a spreading model for T.*

Exercise 3. (1) *Prove*[1] *that for every $\varepsilon > 0$ there is a vector $x = \sum_{i=1}^{\infty} a_i e_i$ such that $\varepsilon\|x\|_{\ell_1} > \|x\|_T$.*
(2) *For each $\varepsilon > 0$ explicitly find a vector x such that $\varepsilon\|x\|_{\ell_1} > \|x\|_T$.*

Exercise 4. *Let X be a compact Hausdroff space. Consider X^X with the product topology and the composition operation.*
(1) *Show that X^X is a right-topological semigroup.*
(2) *Show that X^X is not a left-topological semigroup unless X is discrete.*
(3) *For what functions $g \in X^X$ is the operation $f \mapsto g \circ f$ continuous?*

Exercise 5. (1) *Describe the idempotents in X^X.*
(2) *Give some examples of left and right ideals of X^X.*
(3) *Find a minimal left ideal of X^X.*

Exercise 6. *Describe the ordering $<$ on the idempotents of X^X. (That is, give a description of when $f < g$ holds for two idempotents f and g of X^X.) What are the minimal idempotents of X^X?*

For $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ define $A - n = \{m - n : m \in A, m - n > 0\}$. For ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ define

$$\mathcal{U} + \mathcal{V} = \{A \subseteq \mathbb{N} : \{n \in \mathbb{N} : A - n \in \mathcal{U}\} \in \mathcal{V}\}.$$

Exercise 7. *Show that the map $f\colon \beta\mathbb{N} \to \beta\mathbb{N}$ defined by $f(\mathcal{U}) = \mathcal{U} + \mathcal{U}$ is not continuous.*

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[1] You may use the properties of T proved in the summer course.