THE FIELDS INSTITUTE
FOR RESEARCH IN MATHEMATICAL SCIENCES

ABSTRACTS 1.2

GREG W ANDERSON
University of Minnesota

An adelic version of Stirling's formula and a related conjecture

The notion of $F_q(T)$-soliton was introduced some years ago by the speaker and figured recently in the Anderson-Brownawell-Papanikolas analysis of algebraic relations among special Gamma-values over $F_q(T)$. It is safe to say that this notion appears to be distressingly ad hoc. Now let k be any global function field. Let Phi be a locally constant compactly supported integer-valued function on the adele ring of k. Consider the product of terms $x^Phi(x)$extended over all nonzero elements x of k. By the methods of Tate's thesis we work out completely the asymptotics of the valuations of such products at an arbitrary place of k, arriving at an adelic version of Stirling's formula which can also be seen as a sort of "log-twist" of the Poisson summation formula. The adelic Stirling formula then suggests a conjecture which conceptualizes and generalizes the notion of $F_q(T)$-soliton to the general base-field k.

GEORGE E. ANDREWS
The Pennsylvania State University

Engel Expansions and the Rogers-Ramanujan Identities

In his book, Irrationalzahlen, O. Perron presents several expansions of real numbers that are useful in irrationality questions. A. and J. Knopfmacher extended some of these expansions to Laurent series. It turns out that one these methods (the Engel expansion) leads to some surprising results concerning expansions of the Rogers-Ramanujan type. We shall briefly examine Engel's original idea and shall report on the most recent developments as envisioned by the Knopfmachers.

FRITS BEUKERS
University of Utrecht

Exploring E-functions

The subject of E-functions existed long before the advent of E-mail, E-commerce, E-government, etc. They were introduced around 1929 by C.L.Siegel who studied irrationality and transcendenceof values at algebraic points. Well-known examples are the exponential function and the Bessel functions. Dale Brownawell was very much involved in this subject in the 1980's. In this lecture we give an overview of developments in this area, discussing classical contributions of Shidlovski and Chudnovski, the differential galois theory approach, and also recent work by Yves Andre.

DAVID BOYD
University of British Columbia

The A-polynomials of Turk's Head Knots and other highly symmetric knots

The A-polynomial of a knot is a two variable polynomial $A(x, y)$ that is notoriously difficult to compute. We describe a method for computing this polynomial for families of highly symmetric knots including the Turk's Head knots and a family including one of the dodecahedral knots. The method, which was discovered experimentally, involves representing the knot complement as a cover of an orbifold obtained by Dehn surgery on a 2 component link and computing a three-variable relative of the A-polynomial, $G(x, y, w)$ for this link.

PETER BUNDSCHUH
University of Cologne

Algebraic independence over $\mathbb{Q}_p$

We report on joint work with Kumiko Nishioka. Let $f(z)$ be a power series $\sum_{n\geq 1} \zeta(n) z^{e(n)}$, where $(e(n))$ is a strictly increasing linear recurrence sequence of non-negative integers, and $(\zeta(n))$ a sequence of roots of unity in $\overline{\mathbb{Q}}_p$, the algebraic closure of $\mathbb{Q}_p$, satisfying an appropriate technical condition. Then we are mainly interested in characterizing the algebraic independence over $\mathbb{Q}_p$ of elements $f(\alpha_1), ..., f(\alpha_t)$ from $\mathbb{C}_p$ in terms of the distinct $\alpha_1, ..., \alpha_t \in \mathbb{Q}_p$ satisfying $0 < |\alpha_\tau|_p < 1$ for $\tau = 1, ..., t$. A striking application of our basic result says that, in the particular case $e(n) = n$, the set $\{f(\alpha) \mid \alpha \in \mathbb{Q}_p,\ 0 < |\alpha|_p < 1\}$ is algebraically independent over $\mathbb{Q}_p$ if $(\zeta(n))$ satisfies the technical condition. We shall end the talk by stating a conjecture concerning more general sequences $(e(n))$.

NORIKO HIRATA-KOHNO
Nihon University

Linear forms in p-adic elliptic logarithms and some related results

We present a refined explicit lower bound for linear forms in p-adic elliptic logarithms which is defined by using a reversed function of the Lutz-Weil elliptic p-adic function. We also discuss some preliminaries in case of Tate's elliptic p-adic function having a period, instead of the Lutz-Weil elliptic p-adic function having no period. Let K be an algebraic number field of finite degree D over the rational number field $\mathbb{Q}$. Consider $\mathcal{E}$ an elliptic curve defined over K, which is defined by the Weierstraß equation of the form: $y^2 = x^3 - ax - b \quad (a, b \in O_K)$ with $4a^3 \neq 27b^2$. Let p be a rational prime $\in \mathbb{Q}$. For a place v of K over p, denote K_v the completion of K by v. Put $\mathbb{C}_p$ the completion of the algebraic closure of K_v. We know that $\mathbb{C}_p$ is algebraically closed field of characteristic

0. We fix a valuation $|\cdot|_v$ on $\mathbb{C}_p$, normalized such that $|x|_v = p^{-ord_p(x)}$ for $x \in \mathbb{Q}$. Put $\lambda_p = \frac{1}{p-1}$ if $p \neq 2$, and $\lambda_2 = 3$. We set $\mathcal{C}_p := \{z \in \mathbb{C}_p \,:\, |z|_v < p^{-\lambda_p}\}$ and $\mathcal{C}_v := \mathcal{C}_p \cap K_v$. We recall the definition of the Lutz-Weil elliptic p-adic function. It is known that there exists an analytic function φ defined on $\mathcal{C}_v \to K_v$, satisfying $\varphi(0) = 0$, $\varphi'(0) = 1$ and the differential equation $(Y')^2 = 1 - aY^4 - bY^6$. We may also enlarge the domain of the definition of this function φ to $\mathcal{C}_p$. For the p-adic Lie-group $\mathcal{E}(\mathbb{C}_p)$ we have the exponential map $\mathcal{C}_p \to \mathcal{E}(\mathbb{C}_p)$ represented by

$$\exp_p(z) = \big(\varphi(z), \varphi'(z), \varphi^3(z)\big)$$

which is called the Lutz-Weil elliptic p-adic function. Thus the elliptic curve is written by $Y^2Z = X^3 - aXZ^2 - bZ^3$ for $(X, Y, Z) = (\varphi, \varphi', \varphi^3)$. The difference between this p-adic exponential map and the complex one is the fact that φ is locally analytic only on $\mathcal{C}_p$, not on $\mathbb{C}_p$. Indeed, φ is an odd and injective function such that $|\varphi(z)|_v = |z|_v, |\varphi'(z)|_v = 1$ for any $z \in \mathcal{C}_p$, then $\exp_p$ has no period. There are corresponding addition formula and derivation formula like the Weierstraß elliptic function $\wp$. For an algebraic number, write $h(\cdot)$ as the absolute logarithmic projective height. Put $h = \max_{1\le i\le k}\{h(1, a_i, b_i), 1\}$. For $1 \le i \le k$, let

$$0 \neq u_i \in \{u \in \mathcal{C}_v \,:\, \exp_p(u) \in \mathcal{E}_i(K)\}.$$

Define $U_i = \dfrac{p^{-\lambda_p}}{|u_i|_v} \quad (> 1)$ and V_i by

$$\log V_i \ge \max\{h(\exp_p(u_i)),\, \frac{1}{D}\} \quad (1 \le i \le k)$$

where we may suppose

$$U_1 = \max(U_i),\, V_1 = \max(V_i), \quad 1 \le i \le k.$$

Let $\beta_1, \cdots \beta_k \in K - \{0\}, \quad |\beta_i|_v \le 1\,(1 \le i \le k)$ and put

$$\log B \ge \max_{1\le i\le k}\{1, h(\beta_i)\}.$$

If $\beta_1 u_1 + \cdots + \beta_k u_k \neq 0$, *then there exists an effective constant* $C > 0$ *depending only on* k, p *such that*

$$\log|\beta_1 u_1 + \cdots \beta_k u_k|_v \ge$$
$$-C \cdot D^{2k+2}\,(\log B + h + \log\log V_1 + \log DU_1)$$
$$\times\,(\log\log V_1 + h + \log DU_1)^{k+1} \times \prod_{i=1}^{k}(h + \log V_i + \log U_i)\,.$$

(These log's mean the usual Archimedean logarithms.) We have an explicit form of $C > 0$.

WINNIE LI
Penn State University

Modular Forms for Noncongruence Subgroups

Coauthors: Winnie Li, Ling Long and Zifeng Yang Unlike modular forms for congruence subgroups. the arithmetic properties of modular forms for noncongruence subgroups are little understood. The study of modular forms for noncongruence subgroup was initiated by Atkin and Swinnerton-Dyer in 1971. From less than a handful of examples they computed, they came up with a conjecture on the congruence relations of the Fourier coefficients of such forms. No progress was made until mid 1980's, when Scholl associated motives to the space of cusp forms for a noncongruence subgroup of given weight. He also showed that the cusp forms collectively satisfy a certain kind of congruence relations, which are coarser than the ASD conjecture. Recently we obtain a certain noncongruence subgroup whose weight 3 cusp forms not only satisfy the ASD conjecture, but also have very interesting connections with modular forms for some congruence subgroup. Our example arises from the study of the modularity conjecture for a certain elliptic surface. This will be explained in detail in the talk.

DAVID MASSER
University of Basle, Switzerland

On functions and polynomials: an arithmetician's miscellany

We mention some results and open problems, both old and new, on topics of common interest to the speaker and the Jubilar, including algebraicity questions for elliptic functions, zero estimates, and abcology; the last also with reference to certain mixing problems in ergodic theory.

MATTHEW PAPANIKOLAS
Texas A&M University

Transcendence in positive characteristic

Coauthors: Greg Anderson,Dale Brownawell In this talk we will survey the contributions of Dale Brownawell to the area of function field transcendence. Transcendence problems over function fields in characteristic p enjoy a rich history going back to Carlitz and Wade in the 1930's and 1940's. In many respects conjectures about the transcendence and algebraic independence of periods and logarithms of Drinfeld motives have mirrored the conjectures for their counterparts in characteristic 0. However, with some notable exceptions, until recently theorems that could be proved in positive characteristic were essentially the ones that could also be proved over number fields (even if their proofs were quite different!).

In 2002, Greg Anderson, Dale Brownawell, and I proved that all algebraic relations among special Γ-values in positive characteristic can be derived from the functional equations of the Γ-function. We will outline the main theorems leading up to this result and discuss the fundamental components of its proof. We will also discuss how the foundations of this work have led to recent progress on other problems, including Shimura's conjecture over function fields and the algebraic independence of Carlitz logarithms.

PATRICE PHILIPPON
Institut de Mathmatiques de Jussieu

Approximation properties in the functional case

Algebraic independence theory over the field of complex numbers involves some uniform properties of approximation of complex numbers by algebraic ones. For example, the Nullstellensatz or criteria of Gelfond's type provide useful informations. We will discuss aspects of the functional analogue of these questions, replacing the field of rational numbers by a field of rational fractions in one variable over an algebraically closed field.

ALF VAN DER POORTEN
ceNTRe for Number Theory Research, Sydney

Elliptic sequences and continued fractions

In general, the coefficients to the Padé approximants to algebraic functions increase explosively in height with the degree of the approximating polynomials. This is a bad thing for the purposes of diophantine approximation, but it makes it all the more fun nonetheless to extract arithmetic information from the continued fraction expansion of a quadratic irrational function defined over, say, $\mathbb{Q}$.

For example, the sequence defined by $A_{h-2}A_{h+2} = A_{h-1}A_{h+1} + A_h^2$ and $A_0 = A_1 = A_2 = A_3 = 1$ arises from the curve $V^2 - V = U^3 + 3U^2 + 2U$ by reporting the denominators of the points $M + hS$, with $M = (-1, 1)$ and $S = (0, 0)$. The recursion $B_{h-3}B_{h+3} = B_{h-2}B_{h+2} + B_h^2$ and $B_0 = B_1 = B_2 = B_3 = B_4 = B_5 = 1$ arises from adding multiples of the divisor at infinity on the Jacobian of the curve $Y^2 = (X^3 - 4X + 1)^2 + 4(X - 2)$ of genus 2 to the divisor given by $[(f, 0), (g, 0)]$; it will please adherents to the cult of Fibonacci to learn that here f is the golden ratio (and g its conjugate).

STEPHEN H. SCHANUEL
S.U.N.Y.,Buffalo

Objective algebraic number theory

When we speak of square numbers, we remember that natural numbers are isomorphism classes of finite sets, and that the sum and product of natural numbers derive from the objective sum and product of the sets. Lawvere and the speaker have been investigating the surprising array of familiar categories in which the arithmetic of addition and multiplication of objects shares many properties with natural numbers. In particular, a simple property of the addition, called the extensive law, together with a weak finiteness condition, gives rise to a theory of spectra, generalizing Galois theory in that the category of interest is opposite to a finite category which plays the role taken by G-sets in Galois theory.

T.N. SHOREY
Tata Institute of Fundamental Research

Diophantine equations involving arithmetic progressions

A theorem of Erdos and Selfridge states that a product of two or more consecutive positive integers is never a power. We shall consider analogous questions for products of terms in arithmetic progression.

JOSEPH SILVERMAN
Brown University

Real and p-adic Properties of Elliptic Divisibility Sequences and Elliptic Division Polynomials

Coauthors: Nelson Stephens (University of London)

Elliptic divisibility sequences (EDS) are sequences of integers defined by the nonlinear recurrence satisfied by the division polynomials of an elliptic curve. I will discuss the behavior of EDS in complete fields, including the sign variation of EDS over $\mathbb{R}$ (joint work with N. Stephens) and convergence of EDS subsequences in the ring $\mathbb{Z}_p$ of p-adic integers.

ROBERT TIJDEMAN
Leiden University

Irrationality of infinite sums of rational numbers

Coauthors: J. Hancl

The lecture deals with a method which not only implies the irrationality of numbers such as $\sum_{N=1}^{\infty}[N^a]/N!$ $(a \geq 0)$, $\sum_{N=1}^{\infty}[N\log^b N]/N!$ $(b \in \mathbb{R})$ and $\sum_{N=1}^{\infty}[N^a \exp(\log^b N)]/N!$ $(a \geq 0, 0 < b < 1)$, but also that such numbers are linearly independent over the rationals.

ROB TUBBS
University of Colorado

An illustrated history of α^β

Some of the most interesting transcendence results of the second half of the twentieth century were elaborations and refinements of results from the 1930's concerning the arithmetic nature of α^β. In this lecture we will offer an appropriately selective history of the study of α^β and its analogues.

JEFFREY D. VAALER
University of Texas

Mahler Measure and the ABC Inequality

We will describe an upper bound for the Mahler measure of the Wronskian of a collection of N+1 linearly independent polynomials with complex coefficients. If the coefficients of the polynomials are algebraic numbers a similar inequality holds at non-archimedean completions. Together these lead to an inequality for the absolute Weil heights of the roots of the polynomials. This later inequality is analogous to Mason's ABC inequality for polynomials. We will also discuss some applications to Diophantine problems.

PAUL VOJTA
University of California, Berkeley

Arithmetic jet bundles

Jet differentials are differentials intended to capture information only available by taking higher derivatives of functions. When defined as iterated Kähler differentials, though, they are not suitable for working on varieties in positive characteristic or on schemes of mixed characteristic (such as arithmetic varieties). Instead, it is better to use Hasse-Schmidt divided differentials $d_n x$ (think: $(1/n!)d^n x$). These will be discussed briefly (for fuller details, see math.berkeley.edu/ vojta/jets.pdf). These differentials allow one to define jet spaces for arbitrary scheme morphisms $X \to Y$; such spaces are analogous to generalizations of the relative tangent bundle but they incorporate information on higher derivatives.

In his 1995 talk at Santa Cruz, J.-P. Demailly discussed compactified quotient jet spaces due originally to J. G. Semple and others. These correspond to certain closed subspaces of the iterated space of lines in the tangent bundle of a complex manifold: X, $\mathbb{P}(\Omega^1_{X/\mathbb{C}})$, $\mathbb{P}(\Omega^1_{\mathbb{P}(\Omega^1_{X/\mathbb{C}})/\mathbb{C}})$, etc.

I will discuss current work in progress on extending the Semple-Demailly theory to arbitrary morphisms of schemes, via Hasse-Schmidt differentials. I will also say a few words about possible diophantine applications.

KUNRUI YU
Hong Kong University of Science and Technology, Hong Kong

P-adic logarithmic forms

This is a continuation of my work presented on 2003 Tijdeman Conference. I shall report on p-adic estimates, where the assumption on the multiplicative independence among the algebraic numbers $\alpha_1, \ldots, \alpha_n$, the assumption that the α's are p-adic units and the assumption on the number field K, which contains the α's, are all removed. I shall also discuss the new features of these p-adic estimates.

YURI ZARHIN
Pennsylvania State University

Endomorphism algebras of superelliptic jacobians

We construct explicitly a plenty of superelliptic jacobians, whose endomorphism algebra is a (product of) cyclotomic field(s).