Name: Emily Redelmeier
Student Number: 990740016
Course: APM446H
Assignment 1

38/40

(1) Let $\varphi_1, \varphi_2 \in C_c^\infty(\mathbb{R}^d)$, and let $c \in \mathbb{R}$. Then

$$\begin{aligned}\langle \rho(t), \varphi_1 + c\varphi_2 \rangle &= \int_\Omega (\varphi_1 + c\varphi_2)(\mathcal{X}(t,x))\,dx \\ &= \int_\Omega (\varphi_1(\mathcal{X}(t,x)) + c\varphi_2(\mathcal{X}(t,x)))\,dx \\ &= \int_\Omega \varphi_1(\mathcal{X}(t,x))\,dx + c\int_\Omega \varphi_2(\mathcal{X}(t,x))\,dx \\ &= \langle \rho(t), \varphi_1 \rangle + c\langle \rho(t), \varphi_2 \rangle\end{aligned}$$

So for all t, $\rho(t)$ is a distribution over $\mathbb{R}^d$, so $\rho \in \mathcal{D}'(\mathbb{R}^d)$.

you still have to show it is continuous

Furthermore,

$$\begin{aligned}|\langle \rho(t), \varphi \rangle| &= \left|\int_\Omega \varphi(\mathcal{X}(t,x))\,dx\right| \\ &\le |\Omega| \sup_{x\in\Omega} |\varphi(\mathcal{X}(t,x))| \\ &= |\Omega| \sup_{x\in\mathcal{X}[\Omega]} |\varphi(x)| \\ &\le |\Omega| \sup_{x\in\mathbb{R}^d} |\varphi(x)|\end{aligned}$$

this will do

5/5

(2) Since $\mathcal{X}$ is a diffeomorphism, its Jacobian always exists and is never singular. So $|D\mathcal{X}(t,x)|^{-1}$ is always defined. Furthermore, $\mathcal{X}^{-1}$ is defined on Ω'. Then for every t,

$$\begin{aligned}\langle \rho(t), \varphi \rangle &= \int_\Omega \varphi(\mathcal{X}(t,x))\,dx \\ &= \int_{\mathcal{X}[\Omega]} \varphi(y) \frac{1}{|D(\mathcal{X}(t, \mathcal{X}^{-1}(t,y)))|}\,dy \\ &= \int_{\Omega'} \frac{1}{|D(\mathcal{X}(t, \mathcal{X}^{-1}(t,y)))|} \varphi(y)\,dy\end{aligned}$$

$$\therefore \langle \rho, \varphi \rangle = \int_{\Omega'} \bar{\rho}(y)\varphi(y)\,dy$$

where $\bar{\rho}(y) = \frac{1}{|D(\mathcal{X}(t, \mathcal{X}^{-1}(t,y)))|}$.

~~3/3~~ 5/5

(3)

$$\begin{aligned}\langle \partial_t \rho, \varphi \rangle &= \int_\Omega \nabla\varphi(\mathcal{X}(t,x)) \cdot \partial_t \mathcal{X}(t,x)\,dx \\ &= \int_\Omega \frac{d}{dt} \varphi(\mathcal{X}(t,x))\,dx \\ &= \frac{d}{dt}\int_\Omega \varphi(\mathcal{X}(t,x))\,dx \\ &= \frac{d}{dt}\langle \rho, \varphi \rangle\end{aligned}$$

you would need to justify this.

3/3

(4) Let $\varphi_1, \varphi_2 \in C_c^\infty(\mathbb{R}^d)$, and let $c \in \mathbb{R}$. Then

$$\begin{aligned}\langle \partial_t \rho(t), \varphi_1 + c\varphi_2 \rangle &= \int_\Omega \nabla(\varphi_1 + c\varphi_2)(\mathcal{X}(t,x)) \cdot \partial_t \mathcal{X}(t,x)\,dx \\ &= \int_\Omega (\nabla\varphi_1(\mathcal{X}(t,x)) + c\nabla\varphi_2(\mathcal{X}(t,x))) \cdot \partial_t \mathcal{X}(t,x)\,dx\end{aligned}$$

$$= \int_\Omega \left(\nabla\varphi_1(X(t,x))\cdot\partial_t X(t,x) + c\nabla\varphi_2(X(t,x))\cdot\partial_t(t,x)\right)dx$$
$$= \int_\Omega \nabla\varphi_1(X(t,x))\cdot\partial_t X(t,x)\,dx + c\int_\Omega \nabla\varphi_2(X(t,x))\cdot\partial_t(t,x)\,dx$$
$$= \langle \partial_t\rho(t), \varphi_1\rangle + c\langle\partial_t\rho(t), \varphi_2\rangle$$

so $\partial_t\rho \in \mathcal{D}'(\mathbb{R}^d)$.

Furthermore, again you need to show it is continuous. and again this will suffice?

$$|\langle\partial_t\rho(t),\varphi\rangle| = \left|\int_\Omega \nabla\varphi(X(t,x))\cdot\partial_t X(t,x)\,dx\right|$$
$$\le |\Omega| \sup_{x\in\Omega} |\nabla\varphi(X(t,x))\cdot\partial_t X(t,x)|$$
$$\le |\Omega| \sup_{x\in\Omega} |\nabla\varphi(X(t,x))||\partial_t X(t,x)|$$
$$\le |\Omega| \sup_{y\in\mathbb{R}^d}|\nabla\varphi(y)| \sup_{x\in\Omega}|\partial_t(t,x)|$$

5/5

BTW why is this true?

5 a) If we have $t_1, t_2 \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^d$ such that $t_1 = t_2$ and $X(t_1, x_2) = X(t_2, x_2)$ then since X is a diffeomorphism, $t_1 = t_2$ and $x_1 = x_2$. Then $\partial_t X(t_1, x_1) = \partial_t X(t_2, x_2)$. So we can define a function $v(t): \Omega' \to \mathbb{R}^d$ such that $\partial_t X(t,x) = v(t, X(t,x))$.

1/2

what is it? $\partial_t X(t, X^{-1})$ $\quad \Omega \xrightarrow{X} \Omega'$

b) Unravelling the terms, the first term of the integrand is $\frac{\partial\varphi(t,x)}{\partial t}(t,x)\bar\rho(t,x)$. Then

$$\int_{[0,T]\times\mathbb{R}^d} \frac{\partial\varphi(t,x)}{\partial t}(t,x)\bar\rho(t,x)\,dx\,dt = \int_{\mathbb{R}^d}\int_0^T \frac{\partial\varphi(t,x)}{\partial t}(t, X(t,x))\,dt\,dx$$
$$= \int_{\mathbb{R}^d}\left(\varphi\Big|_{X(0,x)}^{X(T,x)}\right)dx$$

Since $\varphi \in C_c^\infty(]0,T[\times\mathbb{R}^d)$, $\varphi(0,x) = \varphi(T,x) = 0$ for any x. So this integrand is zero, and hence so is the integral.

~~true not for any x. only those outside supp φ~~

The integral of the second term is

$$\int_{[0,T]\times\mathbb{R}^d} \nabla_x\varphi(t,x)\cdot v(t,x)\bar\rho(t,x)\,dx\,dt = \int_0^T\int_{\mathbb{R}^d} \nabla_x\varphi(t, X(t,x))\cdot v(t, X(t,x))\,dx\,dt$$
$$= \int_0^T\int_{\mathbb{R}^d}\nabla_x\varphi(t, X(t,x))\cdot\partial_t X(t,x)\,dx\,dt$$
$$= \int_0^T \langle\partial_t\rho(t),\varphi\rangle\,dt$$
$$= \int_0^T \frac{d}{dt}\langle\rho,\varphi\rangle\,dt \quad \text{nice.}$$
$$= \langle\rho,\varphi\rangle\Big|_0^T$$
$$= \langle\rho,\varphi(T)\rangle - \langle\rho,\varphi(0)\rangle$$
$$= \langle\rho,0\rangle - \langle\rho,0\rangle$$
$$= 0$$

5/5

you can also write ~~integral~~ integrand as $\partial_t\varphi(t,X)$

So the integral of the second term of the integrand is also zero. So the entire integral is zero.

7) Let $\phi_1, \phi_2 \in (C_c^\infty(\mathbb{R}^d))^d$ and let $c \in \mathbb{R}$. Then

$$\langle(\phi_1 + c\phi_2,)(t)\rangle = \int_\Omega (\phi_1 + c\phi_2)(X(t,x))\cdot\partial_t X(t,x)\,dx$$
$$= \int_\Omega (\phi_1(X(t,x)) + c\phi_2(X(t,x)))\cdot\partial_t X(t,x)\,dx$$

$$= \int_\Omega (\phi_1(\mathcal{X}(t,x)) \cdot \partial_t \mathcal{X}(t,x) + c\,\phi_2(\mathcal{X}(t,x)) \cdot \partial_t \mathcal{X}(t,x))\,dx$$
$$= \int_\Omega \phi_1(\mathcal{X}(t,x)) \cdot \partial_t \mathcal{X}(t,x)\,dx + c\int_\Omega \phi_2(\mathcal{X}(t,x)) \cdot \partial_t \mathcal{X}(t,x)\,dx$$

Furthermore, **Again continuity.**

$$|\langle \phi, j\rangle| = \left|\int_\Omega \phi(\mathcal{X}(t,x)) \cdot \partial_t \mathcal{X}(t,x)\,dx\right|$$
$$\leq \int_\Omega |\phi(\mathcal{X}(t,x))||\partial_t \mathcal{X}(t,x)|\,dx$$
$$\leq \sup_{x\in\Omega} |\partial_t \mathcal{X}(t,x)| \left|\int_\Omega |\phi|(\mathcal{X}(t,x))\,dx\right|$$
$$= \sup_{x\in\Omega} |\partial_t \mathcal{X}(t,x)| |\langle \rho(t), |\phi|\rangle|$$

✓ **5/5**

⑧ As shown in 5b, $\int_{[0,T]} \langle \partial_t \varphi(t), \rho(t)\rangle dt = 0$.
Then,
$$\langle \nabla\varphi(t), j(t)\rangle = \int_\Omega \nabla\varphi(\mathcal{X}(t,x)) \cdot \partial_t \mathcal{X}(t,x)\,dx$$
$$= \langle \partial_t \rho, \varphi\rangle$$
$$= \frac{d}{dt}\langle \rho, \varphi\rangle$$

so
$$\int_0^T \langle \nabla\varphi(t), j(t)\rangle dt = \int_0^T \frac{d}{dt}\langle \rho,\varphi\rangle dt$$
$$= \langle \rho, \varphi\rangle \Big|_0^T$$
$$= \langle \rho, \varphi(T)\rangle - \langle \rho, \varphi(0)\rangle$$
$$= 0 - 0$$
$$= 0$$

✓ **5/5**

⑨ Since $f \in C'(\mathbb{R})$, f' is continuous, and is hence bounded on $[0, C']$. So for any $x \in [0, C']$,
$$|f(x) - f(0)| \leq \sup_{y\in[0,C']} |f'(y)|x$$
$$\therefore |f(x)| \leq \sup_{y\in[0,C']} |f'(y)|x$$ **Excellent.**
$$\therefore |f(\rho(t,x))| \leq \sup_{y\in[0,C']} |f'(y)|\rho(t,x)$$

Since $|f(\rho(t,x))|$ is bounded above by a constant multiple of $\rho(t,x)$, and any constant multiple of $\rho(t,x)$ must have a finite integral (since $\rho(t,x)$ does), $f(\rho(t,x))$ must have a finite integral.

$$\frac{d}{dt}\int f(\rho(t,x))\,dx = \int \frac{d}{dt} f(\rho(t,x))\,dx$$
$$= \int f'(\rho(t,x))\partial_t \rho(t,x)\,dx$$
$$= -\int f'(\rho(t,x))(\nabla\cdot(\rho v))\,dx$$
$$= -\int f'(\rho(t,x)) \sum_{i=1}^d \frac{\partial(\rho v_i)}{\partial x_i}\,dx$$
$$= -\int f'(\rho(t,x))\left(\sum_{i=1}^d \frac{\partial \rho}{\partial x_i} v_i + \sum_{i=1}^d \rho \frac{\partial v_i}{\partial x_i}\right)dx$$
$$= -\int f'(\rho(t,x))(\nabla_x \rho \cdot v + \rho\, \nabla_x \cdot v)\,dx$$

$$= -\int f'(\rho(t,x))\, \nabla_x \rho(t,x) \cdot v(t,x)\, dx$$

$$= -\int \nabla_x F(\rho(t,x)) \cdot v(t,x)\, dx$$

integrate by parts

$$= \int f(\rho)\, \nabla \cdot v$$

But $\nabla \cdot v = 0$.

9/10