Dynamics on the algebra of quantum observables.

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Classical mechanics

$f, g, \ldots : \mathbb{R}^{2n}_{p,x} \to \mathbb{R}$

classical observables

$\{f, g\} = \sum_j (f_{p_j} g_{x_j} - f_{x_j} g_{p_j})$

the Poisson bracket

$\dot f = \{h, f\}, \quad f|_{t=0} = f_0$

Hamiltonian equation

Quantum mechanics

$F, G, \ldots$ (self-adjoint) operators on $L_2(\mathbb{R}^n_x)$

quantum observables

$[F, G] = \frac{1}{-i\hbar}(F \circ G - G \circ F)$

commutator

$\dot F = [H, F], \quad F|_{t=0} = F_0$

Heisenberg equation

$$x_j \longmapsto \hat x_j = x_j \cdot (\,\cdot\,)$$

$$p_j \longmapsto \hat p_j = -i\hbar \frac{\partial}{\partial x_j}(\,\cdot\,)$$

$$\hat p_j \circ \hat x_j - \hat x_j \circ \hat p_j = -i\hbar$$

Associative algebras $\widetilde{QO}^{form}$, QO^{form}

Monomial: $z = z_k \circ \ldots \circ z_1$, $z_j \in \{\hat{x}_1, \ldots, \hat{x}_n, \hat{p}_1, \ldots, \hat{p}_n\}$

$\deg z = k$

Homogeneous form: $\tilde{F}_k = \sum_{\deg z = k} \tilde{f}_z z$, $\tilde{f}_z \in \mathbb{C}$

$\deg \tilde{F}_k = k$, $\tilde{F}_k \in \mathbb{F}_k$

$\widetilde{QO}^{form}$ is the space of formal series $\tilde{F} = \sum_{k=0}^{\infty} \tilde{F}_k$, $\tilde{F}_k \in \mathbb{F}_k$

associative algebra w.r.t. $\circ$

notation: $\Gamma_j = \hat{p}_j \circ \hat{x}_j - \hat{x}_j \circ \hat{p}_j$.

Consider the ideal $J \subset \widetilde{QO}^{form}$ generated by

$\hat{p}_j \circ \hat{p}_k - \hat{p}_k \circ \hat{p}_j$, $\hat{x}_j \circ \hat{x}_k - \hat{x}_k \circ \hat{x}_j$, $1 \leq j, k \leq n$

$\hat{p}_j \circ \hat{x}_k - \hat{x}_k \circ \hat{p}_j$, $k \neq j$

$\Gamma_j - \Gamma_k$, $1 \leq k, j \leq n$

$\Gamma_j \circ \hat{p}_j - \hat{p}_j \circ \Gamma_j$, $\Gamma_j \circ \hat{x}_j - \hat{x}_j \circ \Gamma_j$

Definition. $QO^{form} = \widetilde{QO}^{form} / J$

The natural projection $\pi : \widetilde{QO}^{form} \longrightarrow QO^{form}$
is a homomorphism of the associative algebras

$\Gamma := \pi(\Gamma_j)$ commutes with everything

Commutator on QO^{form}

For a monomial z we say that $\mathrm{type}(z) = (\alpha, \beta) \in \mathbb{Z}_+^{2n}$ if z contains $\hat{x}_j$ α_j times and $\hat{p}_j$ β_j times.

We define $\widetilde{aver}(z) = x^\alpha p^\beta$

Example: $z = \hat{p}_1^2 \circ \hat{x}_1 \circ \hat{p}_1 \circ \hat{p}_2 \Rightarrow \mathrm{type}(z) = (1, 0, 3, 1)$,

$$\widetilde{aver}(z) = x_1 p_1^3 p_2$$

Definition CO^{form} is the space of formal series in $x, p \in \mathbb{R}^n$ (classical formal observables).

$\widetilde{aver} : \widetilde{QO}^{form} \to CO^{form}$ is a homomorphism of the associative algebras

Prop 1 $\exists$ $aver : QO^{form} \to CO^{form}$ such that the diagram

$$\begin{tikzcd} \widetilde{QO}^{form} \arrow[r, "\pi"] \arrow[dr, "\widetilde{aver}"'] & QO^{form} \arrow[d, "aver"] \\ & CO^{form} \end{tikzcd}$$

is commutative

Prop 2 $F \in QO^{form}$, $aver\, F = 0 \Leftrightarrow \exists!\, F_0 \in QO^{form}$ $F = \Gamma \circ F_0$.

By Prop 2 the following map is well-defined:

$$\Omega : \mathrm{Ker}(aver) \to QO^{form}, \quad \Omega(F) = \frac{F}{\Gamma}.$$

$$F, G \in QO^{form} \Rightarrow [F, G] := \Omega(F \circ G - G \circ F).$$

Prop 3 $aver : QO^{form} \to CO^{form}$ is a homomorphism of the associative and Lie algebras.

Analyticity.

$$\widetilde{QO}^{form} \ni F = \sum_{(\alpha,\beta)\in \mathbb{Z}_+^{2n}} F^{\alpha,\beta}, \qquad F^{\alpha,\beta} = \sum_{type(z)=(\alpha,\beta)} f_z z$$

$F^{\alpha,\beta}$ is a homogeneous form of type (α,β)

$$\widetilde{aver}\, F = \sum_{\alpha,\beta} f_{\alpha,\beta}\, x^\alpha p^\beta, \qquad f_{\alpha,\beta} = \sum_{type(z)=(\alpha,\beta)} f_z,$$

$$Aver\, F = \sum_{\alpha,\beta} \varphi_{\alpha,\beta}\, x^\alpha p^\beta, \qquad \varphi_{\alpha,\beta} = \sum_{type(z)=(\alpha,\beta)} |f_z|.$$

Definition. $F \in \widetilde{QO}^{form}$ belongs to $\widetilde{QO}$ iff $Aver\, F$ is analytic at zero.

$F \in QO^{form}$ belongs to QO iff $\exists \tilde{F} \in \widetilde{QO}$ $\pi(\tilde{F}) = F$.

Implicit function theorem

Thm. Let $X_1(\hat{x},\hat{p}),\dots,X_n(\hat{x},\hat{p}),P_1(\hat{x},\hat{p}),\dots,P_n(\hat{x},\hat{p})\in QO$ be such that
$[X_j,X_k]=0=[P_j,P_k]$, $[P_j,X_k]=\delta_{jk}$ (canonical set of analytic quantum observables)
Then $\exists\, u_1,\dots,u_n, v_1,\dots,v_n\in QO$ such that
$\hat{x}_j=u_j(X,P)$, $\hat{p}_j=v_j(X,P)$.

Non-commutative Darboux theorem

Thm. Let $P_1,\dots,P_n\in QO$ be independent at zero and $[P_j,P_k]=0$.
Then $\exists X_1,\dots,X_n\in QO$ such that $X_1,\dots,X_n,P_1,\dots,P_n$ is a canonical set.

Globalization

1. To a domain $D\subset\mathbb{R}^{2n}$.
   Analytic continuation:
   $F\in QO(D)$ if for any $(x^\circ,p^\circ)\in D$
   F is a converging series in $\hat{x}-x^\circ$, $\hat{p}-p^\circ$
2. To a symplectic manifold M
   Non clear in general.
   $M=T^*N$ no problem

Liouville theorem.

$H \in QO(\mathcal{P})$ Suppose that $\exists\, F_1, \dots, F_n \in QO(\mathcal{P})$ such that

$$[H, F_j] = [F_j, F_k] = 0.$$

Then $f_j = \operatorname{aver} F_j$ are involutive first integrals of the classical Hamiltonian system with Hamiltonian $h = \operatorname{aver} H$

Let $D \subset \mathcal{P}$ be a domain on which classical angle-action variables $(\varphi, I) \in \mathbb{T}^n \times D_0$, $D_0 \subset \mathbb{R}^n$ are defined.

$(x, p) = \tau(\varphi, I)$ τ is a symplectic map $\Rightarrow$ we have the isomorphism of the algebras

$$\begin{array}{ccc} a : CO(D) & \longrightarrow & CO(\mathbb{T}^n \times D_0) \\ \Psi & & \Psi \\ f & \longmapsto & a(f) := f \circ \tau \end{array}$$

Thm Consider the algebra $QO(\mathbb{T}^n \times D_0)$ generated by $\hat{\varphi}, \hat{I}$. There exists an isomorphism $\mathcal{A} : QO(D) \longrightarrow QO(\mathbb{T}^n \times D_0)$ s. th.

$$\begin{tikzcd} QO(D) \arrow[r, "\mathcal{A}"] \arrow[d, "\operatorname{aver}"'] & QO(\mathbb{T}^n \times D_0) \arrow[d, "\operatorname{aver}"] \\ CO(D) \arrow[r, "a"] & CO(\mathbb{T}^n \times D_0) \end{tikzcd}$$

is commutative

and $\mathcal{A}(H) = \mathcal{H}(\hat{I})$, $\mathcal{A}(F_j) = \mathcal{F}_j(\hat{I})$.

Remark. In the new variables $\hat{\varphi}, \hat{I}$ the Heisenberg equation is easily solved:

$$\dot{G} = [\mathcal{H}, G], \quad G|_{t=0} = G_0(\hat{\varphi}, \hat{I}).$$

$$\Rightarrow \quad G = G_0(\hat{\varphi} + \omega(\hat{I})t, \hat{I}), \qquad \omega = \frac{\partial \mathcal{H}}{\partial \hat{I}}(\hat{I}).$$