IS QUANTUM ERROR CORRECTION FEASIBLE?

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Quantum computers could overpower classical ones only if feasible schemes of error reduction and correction exist!

See discussion of "chemical computer" which executes factoring algorithm R. A., quant-ph/0306103

The theory of fault-tolerant quantum computation - threshold results

E. Knill *et.al*, Introduction to Quantum Error Correction, quant-ph/0207170, 30 Jul 2002

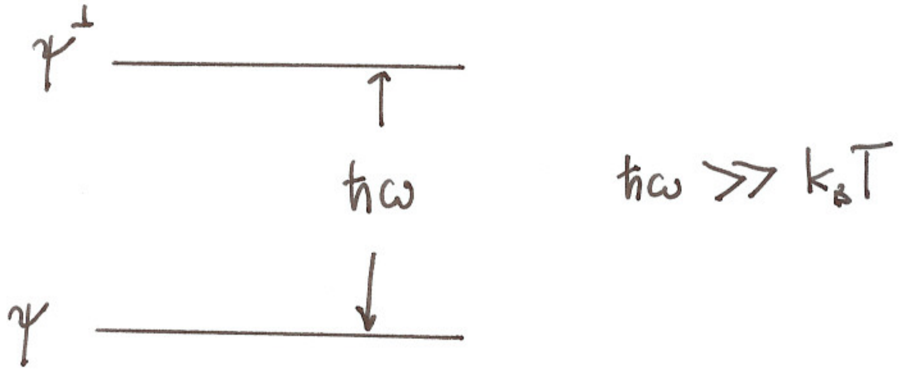
E. Knill, R. Laflamme and W. Żurek, Resilient quantum computation, *Science*, 279, 342 (1998)

D. Aharonov and M. Ben-Or, Fault-tolerant quantum computation with constant error, quant-ph/9906129, (1999)

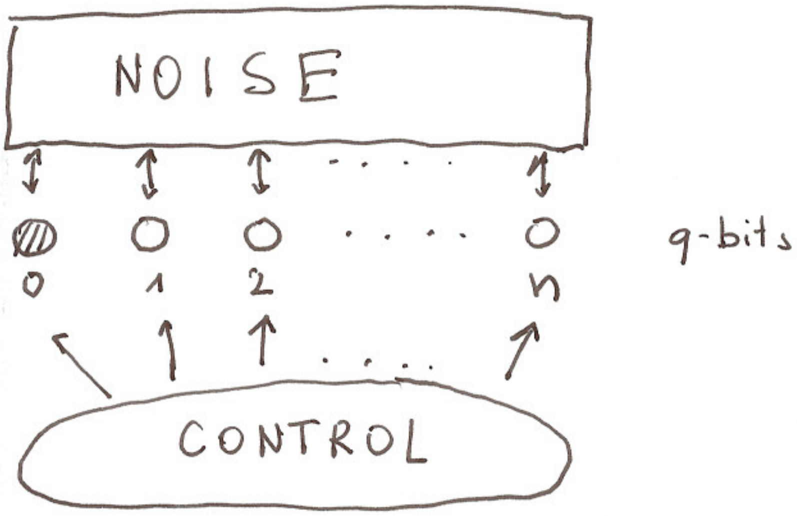
We consider a simpler problem: maitaining a single uknown qubit state for an arbitrarily long period of time

Protecting unknown qubit state
in the environment at the temperature T

Remark: Known qubit state ψ can be protected with exponentially small error



A physical model



Theoretical description

A) Phenomenological

B) Hamiltonian

Phenomenological error model and error corrections

Initial state

$$\rho_{in} = |\psi><\psi| \otimes \rho_A \quad , \quad \psi - \text{unknown qubit state}$$

Discrete time evolution

$$\Gamma = \Lambda_k \mathcal{U}_k \Lambda_{k-1} \mathcal{U}_{k-1} \cdots \Lambda_1 \mathcal{U}_1$$

where $\mathcal{U}_m \rho = U_m \rho U^\dagger$ -unitary gates, Λ_m - error CP maps.

Final state

$$\rho_{out} = \Gamma \rho_{in}$$

Error

$$\epsilon = 1 - <\psi, (\mathrm{Tr}_A \rho_{out}) \psi>$$

Threshold results

Any quantum state can be efficiently maintained for an arbitrarily long period of time at arbitrarily small error ϵ provided the decoherehce rate due to the interaction with an environment is lower that a certain threshold value.

or in a weaker form

Any quantum state can be efficiently maintained for an arbitrarily long period of time at the error ϵ arbitrarily close to the initial error ϵ_0 provided the decoherehce rate due to the interaction with an environment is lower that a certain threshold value.

"Efficiently" - using polynomial in the number of time steps resources i.e. "ancillas" and gates

Drawbacks of phenomenological models

1) Discrete time model $\neq$ continuous time model

Example: Pure dephasing

$$P_j = |j><j| \quad , \quad |j> -\text{basis in Hilbert space}$$

Discrete time

$$\Lambda\rho = (1-p)\rho + p\sum_j P_j \rho P_j \quad , \quad \sum_j P_j = I$$

If $[U_m, P_j] = 0$ and $[\rho_{in}, P_j] = 0$ then

$$\rho_{out} = \mathcal{U}_k \mathcal{U}_{k-1} \cdots \mathcal{U}_1 \rho_{in}$$

noise disapears!

Continuous time ($\hbar \equiv 1$)

$$\frac{d}{dt}\rho_t = -i[H(t), \rho_t] - \gamma \sum_j [P_j, [P_j, \rho_t]]$$

Noise does not disapear and strongly depends on $H(t)$!

2) Quantum noise is non-Markovian

Qubit-bath interaction

$$H_{int} = \lambda \sigma^k \otimes R$$

Spectral density

$$\mathrm{Tr}\big(\rho_B\, R\, R(t)\big) = \int_{-\infty}^{\infty} \hat{R}(\omega) e^{-i\omega t} d\omega.$$

Strictly Markovian noise

$$\mathrm{Tr}\big(\rho_B\, R\, R(t)\big) \sim \delta(t)$$

or

$$\hat{R}(\omega) = \text{constant}$$

produces (bistochasic) semigroup satisfying

$$\frac{d}{dt}\rho_t = -i[H(t), \rho_t] - \gamma[\sigma^k, [\sigma^k, \rho_t]]$$

KMS- condition

$$\hat{R}(-\omega) = e^{-\omega/k_B T}\hat{R}(\omega)$$

contradicts strict Markov property ("quantum memory" $\tau_Q = \hbar/k_B T$)

MME in the weak coupling limit (for constant H)

$$\frac{d}{dt}\rho_t = \frac{-i}{2}\omega[\sigma^3, \rho_t] +$$

$$\frac{1}{2}\lambda^2\Big\{R(\omega)\big([\sigma^-, \rho_t\sigma^+] + [\sigma^-\rho_t, \sigma^+]\big) + R(-\omega)\big([\sigma^+, \rho_t\sigma^-] + [\sigma^+\rho_t, \sigma^-]\big)\Big\}$$

Dissipative part depends on the Hamiltonian!

Hamiltonian model

Single qubit -0, the error correcting n-qubit system A and the bath B.

Interaction Hamiltonian

$$H_{int} = \lambda \sum_{\alpha=0}^{n} \sum_{k} \sigma_{\alpha}^{k} \otimes R_{k}^{\alpha}$$

Spectral density $\hat{R}_{kl}^{\alpha\beta}(\omega)$

$$\mathrm{Tr}\big(\rho_B\, R_k^{\alpha}\, R_l^{\beta}(t)\big) = \int_{-\infty}^{\infty} \hat{R}_{kl}^{\alpha\beta}(\omega) e^{-i\omega t} d\omega.$$

KMS-condition

$$\hat{R}_{kl}^{\alpha\beta}(-\omega) = e^{-\omega/k_B T}\, \hat{R}_{lk}^{\beta\alpha}(\omega)$$

Non-decoupling condition for all (relevant) $\omega \geq 0$

$$\big[\hat{R}_{kl}^{\alpha\beta}(\omega)\big] \geq \gamma[\delta_{\alpha\beta}\delta_{kl}] > 0\ .$$

$$\tau_D = \frac{1}{\lambda^2 \gamma} \quad - \text{ decoherence time}$$

Total Hamiltonian

$$H(t) = H_{0A}(t) + H_R + H_{int}$$

initial state

$$\rho(-\tau/2) = |\psi><\psi| \otimes \rho_A \otimes \rho_B\ .$$

Partial results

1) $T = \infty$ **and Markovian model,** i.e. $\hat{R}^{kl}_{\alpha\beta}(\omega) \sim \delta_{\alpha\beta}\delta_{kl}$

$$\frac{d}{dt}\rho_t = -i[H(t), \rho_t] - \gamma \sum_{k=1}^{3} \sum_{\alpha=0}^{n} [\sigma^k_\alpha, [\sigma^k_\alpha, \rho_t]]$$

Define $I(\rho) = \log d - S(\rho)$ (d- dim of the Hilbert space)

Lemma

$$I(\rho_t) \leq e^{-4\gamma t} I(\rho_0)$$

The entropy of the qubit-0 satisfies

$$S(\rho_t^{(0)}) \geq \log 2 \big(1 - e^{-4\gamma t}(n+1)\big)$$

To keep $S(\rho_t^{(0)}) \leq \epsilon$ we need at least

$$n(t) \geq \big(1 - \frac{\epsilon}{\log 2}\big) e^{4\gamma t}$$

exponentially large number of *ancillas.*

Compare with Aharonov et.al. quant-ph/9611028 - constant input of "fresh qubits" necessary

2. Error formula in Born approximation

R.A. and M. Horodecki, ,P. Horodecki and R. Horodecki, Phys.Rev. **A 65**, *062101 (2002)*
R.A., Controlled Quantum Open Systems, in Irreversible Quantum Dynamics LNP 622, Springer, Berlin (2003)

Reduced time evolution of ρ_{0A}

$$\Gamma^*(\rho_{0A}) = \hat{U}_{0A}\Big(\rho_{0A} + \lambda^2\Phi^*(\rho_{0A}) - \frac{\lambda^2}{2}\{K, \rho_{0A}\}\Big)$$

where

$$\hat{U}_{0A} = \mathbf{T}\exp\Big(-i\int_{-\tau/2}^{\tau/2} \hat{H}_{0A}(t)\, dt\Big)$$

with $\hat{H} = [H, \cdot]$ and $K = \Phi(\mathbf{1})$

$$\hat{U}_{0A} = \mathbf{1} \otimes \hat{U}_A\ .$$

Φ^* - *error map* is completely positive

$$\frac{d}{dt}\sigma_\alpha^k(t) = -i[H_{0A}(t), \sigma_\alpha^k(t)]$$

$$\Phi^*(\rho_{0A}) = \sum_{\alpha,\beta}\int_{-\tau/2}^{\tau/2} ds \int_{-\tau/2}^{\tau/2} du\, R_{kl}^{\alpha\beta}(s-u)\,\sigma_\beta^l(s)\,\rho_{0A}\,\sigma_\alpha^k(u)$$

The error is given by

$$\epsilon = 1 - <\psi|\mathrm{Tr}_A\big(\Gamma^*(\rho_{0A})\big)|\psi>$$

Simplified non-ergodic Markovian model

We assume $T = \infty$ and keep only the terms with σ^1_α. This makes states commuting with σ^1_α invariant and allows "fresh qubits".

Then introducing

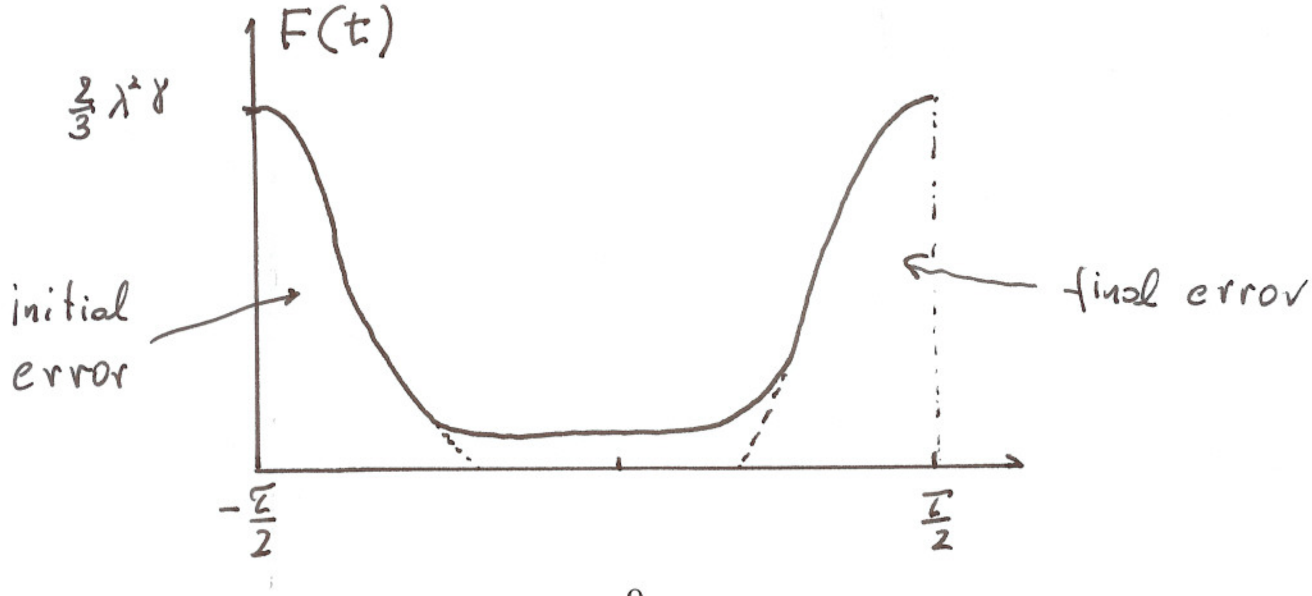
$$A_0^\alpha(t) = \frac{\mathbf{1}}{2} \otimes \mathrm{Tr}_0(\sigma^1_\alpha(t)) \quad , \quad \alpha = 0, 1, 2, ..., n$$

and averaging over an initial qubit-0 state one obtains

$$\bar{\epsilon} \geq \frac{2}{3}\lambda^2\gamma \sum_{\alpha=0}^{n} \mathrm{Tr}\Big(\int_{-\tau/2}^{\tau/2} dt[\mathbf{1} - A_0^\alpha(t)^2]\rho_A\Big) = \int_{-\tau/2}^{\tau/2} F(t)dt$$

$$F(t) \geq 0 \quad , \quad F(-\tau/2) = F(\tau/2) = \frac{2}{3}\lambda^2\gamma$$

There exist unitary maps (encodings) $U(t)$ for which $F(t) = 0$. But initial and final errors cannot be avoided. Moreover, $F(t) = 0$ for perfect tuning of all control parameters what is also not possible. As $F(t) \geq 0$ **errors cannot be corrected but only prevented**. Non-negative error production- a new face of the second law ?



Thermodynamics of open systems

0-th Law: Return to equlibrium

$$\lim_{t\to\infty} \rho(t) = \rho_\beta = Z^{-1} e^{-\beta H}$$

I-st Law: Energy conservation

$$dE = dQ - dW \ , \ \frac{dQ}{dt} = \mathrm{Tr}\big(\frac{d\rho_t}{dt} H(t)\big) \ , \ \frac{dW}{dt} = -\mathrm{Tr}\big(\rho_t \frac{dH(t)}{dt}\big)$$

II-nd Law: Non-negative entropy production

$$\frac{dS}{dt} = \kappa(t) + \beta\frac{dQ}{dt} \ , \ \kappa(t) \geq 0$$

???- Law: Information about uknown state cannot be efficiently protected

Any (efficient) action on a single qubit which can be described in Hamiltonian terms cannot reduce the error below the value ϵ_0 *depending on the physical implementation of the qubit and its environment.*

Essentially proven by the example of above.

Any (efficient) action on a single qubit which can be described in Hamiltonian terms cannot reduce the error below the value $\epsilon_0 + c\tau$ *(for* $\epsilon_0 + c\tau << 1$*) where* τ *is the period of time and c is a strictly positive constant depending on the physical implementation of the qubit and its environment.*

To be proven rigorously.