Test for entanglement: realignment criterion, entanglement witness and positive maps

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CQIQC, Toronto, July 2004

We develop a very simple method to assess the inseparability of bipartite quantum systems, which is based on a realigned matrix constructed from the density matrix. It shows dramatic ability to identify many of the bounded entangled states discussed in the literature. Based on this criterion and the Peres-Horodecki criterion [i.e., PPT (positive partial transposition) criterion], we develop further a more powerful method to identify entangled states for any bipartite system through a universal construction of the witness operator. The method also gives a new family of positive but non-completely positive maps of arbitrary high dimensions, which provide a much better test than the witness operators themselves.

In cooperation with

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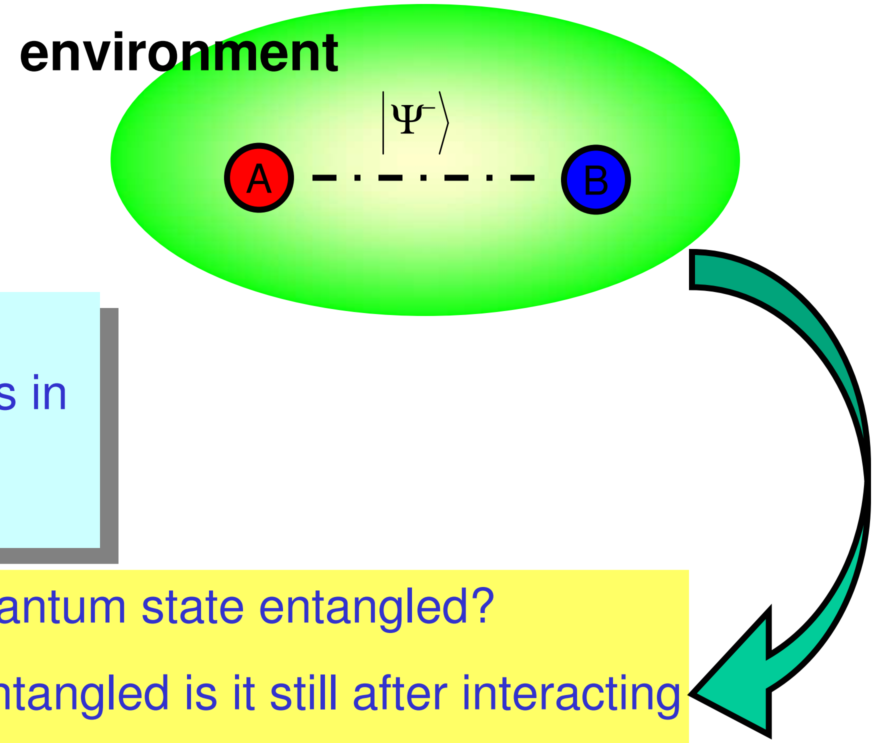
References: quant-ph/0205017,0208058, 0306041, 0312185

Decoherence



The separability problem:
one of the basic and emergent problems in present and future quantum information processing

Is a quantum state entangled?

How entangled is it still after interacting with a noisy environment?

A separable state is a quantum state which can be prepared in a ***local*** or ***classical*** way (LOCC): (local operations and classical communications)

$$\rho_{AB\cdots Z} = \sum_i p_i \rho_i^A \otimes \rho_i^B \otimes \cdots \otimes \rho_i^Z$$

Otherwise, it is entangled

(Werner 89)

A strong separability criterion for mixed state

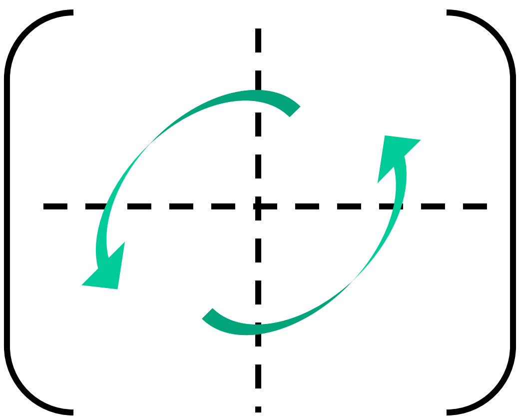
Positive partial transpositions(PPT)

(Peres 96)

$$\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i \geq 0 \;\Rightarrow\; \rho^{T_A} = \sum_i p_i (\rho_A^i)^T \otimes \rho_B^i \geq 0$$

An example of 2x2 state□

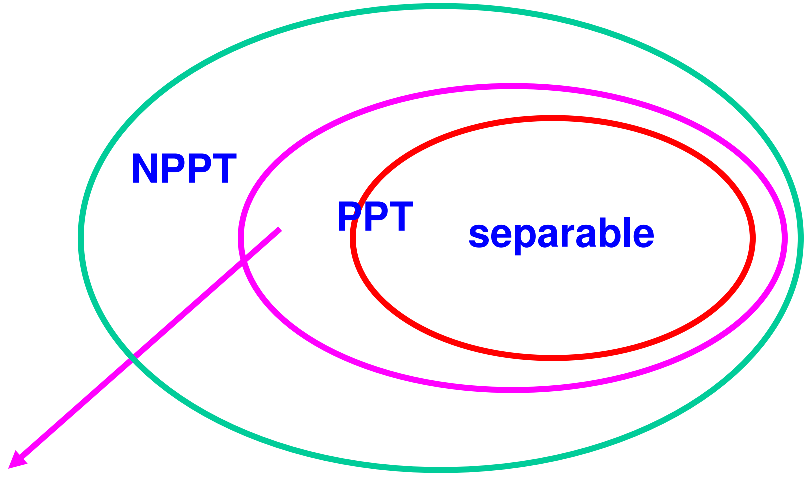
$$\rho = \left(\begin{array}{cc|cc} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \hline \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{array}\right)$$



$$\rho^{T_A} = \left(\begin{array}{cc|cc} \rho_{11} & \rho_{12} & \rho_{31} & \rho_{32} \\ \rho_{21} & \rho_{22} & \rho_{41} & \rho_{42} \\ \hline \rho_{13} & \rho_{14} & \rho_{33} & \rho_{34} \\ \rho_{23} & \rho_{24} & \rho_{43} & \rho_{44} \end{array}\right)$$

Present status for the separability problem

Generic state



-Low rank

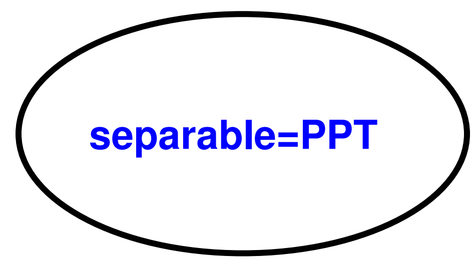
-Operational necessary or sufficient conditions (Lewenstein, Cirac, Horodecki, Albeverio, Fei et al,2000, 2001)

Bounded entangled states (BES) which can not be distilled to be EPR pair: un-distillable

The main progress

- Bell inequalities (Bell, 1964)
- Entanglement of formation for two qubits (Wootters,1998)
- The reduction criterion (Horodecki, Cerf et al 1999)
- Low rank cases (Lewenstein,Cirac,Horodecki, Albeverio, Fei et al 2000, 2001)
- The necessary and sufficient criterion(Y.D. Zhang and C.Z. Li 2000,2001)
- The majorization criterion (Nielsen and Kempe,2001)
- Entanglement witnesses (Horodecki, Terhal, Lewenstein et al,1996,2000)
- PPT extension (Doherty et al,2002)

2x2 and 2x3



(Horodecki and Peres 96)

Most of them are weaker than PPT and are unable to distinguish BES!
Some of them are operationally complicated.

A matrix realignment method for recognizing entanglement

define realignment operation:

If Z is an mxm block matrix with block size nxn,

$$Z = \begin{pmatrix} Z_{11} & \cdots & Z_{1m} \\ \vdots & \ddots & \vdots \\ Z_{m1} & \cdots & Z_{mm} \end{pmatrix} \longrightarrow R(Z) = \tilde{Z} = \begin{pmatrix} vec(Z_{11})^T \\ \vdots \\ vec(Z_{m1})^T \\ \vdots \\ vec(Z_{1m})^T \\ \vdots \\ vec(Z_{mm})^T \end{pmatrix}$$

$$A = [a_{ij}] \longrightarrow vec(A) = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \\ \vdots \\ a_{1m} \\ \vdots \\ a_{mm} \end{pmatrix}$$

A 2x2 example:

$$\rho = \left(\begin{array}{cc|cc} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \hline \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{array}\right) \longrightarrow R(\rho) = \tilde{\rho} = \left(\begin{array}{cccc} \rho_{11} & \rho_{21} & \rho_{12} & \rho_{22} \\ \hline \rho_{31} & \rho_{31} & \rho_{32} & \rho_{42} \\ \hline \rho_{13} & \rho_{23} & \rho_{14} & \rho_{24} \\ \hline \rho_{33} & \rho_{43} & \rho_{34} & \rho_{44} \end{array}\right)$$

The realignment criterion

For any bipartite separable state, we have

$$\|\tilde{\rho}\| \leq 1$$

necessary criterion for separability

Here $\|\tilde{\rho}\|$ is the sum of all the singular values of $\tilde{\rho}$, or sum of the square roots of eigenvalue for $\tilde{\rho}\tilde{\rho}^{\dagger}$

Recognizing entangled states

$$\|\tilde{\rho}\| > 1 \longrightarrow \rho$$ **is entangled**

sufficient criterion for entanglement

This criterion is strong enough to distinguish many of BES in the literature!

Universal construction of the witness operator

1. Universal construction of the witness operator from the realignment criterion

$$W = Id - (\mathcal{R}^{-1}(U^* V^T))^T$$

where U,V are unitary matrices that yield the singular value decomposition (SVD) of $\mathcal{R}(\rho)$ i.e., $\mathcal{R}(\rho) = U \Sigma V^\dagger$

2. Universal construction of the witness operator from the *PPT* criterion

$$W = Id - (VU^\dagger)^{T_A}$$

where U,V are unitary matrices that yield the singular value decomposition (SVD) of ρ^{T_A} i.e., $\rho^{T_A} = U \Sigma V^\dagger$

Positive maps connected to entanglement witnesses

Jamiołkowski isomorphism

$$W = (Id_A \otimes \Lambda)P_+^m$$

where $P_+^m = |\Phi\rangle\langle\Phi|$ and $|\Phi\rangle = \frac{1}{\sqrt{m}}\sum_{i=1}^m |ii\rangle$

Thus

$$\Lambda(|i\rangle\langle j|) = \langle i|W|j\rangle$$

If

$(Id_A \otimes \Lambda)\rho \ngeq 0$ → ρ **is entangled**

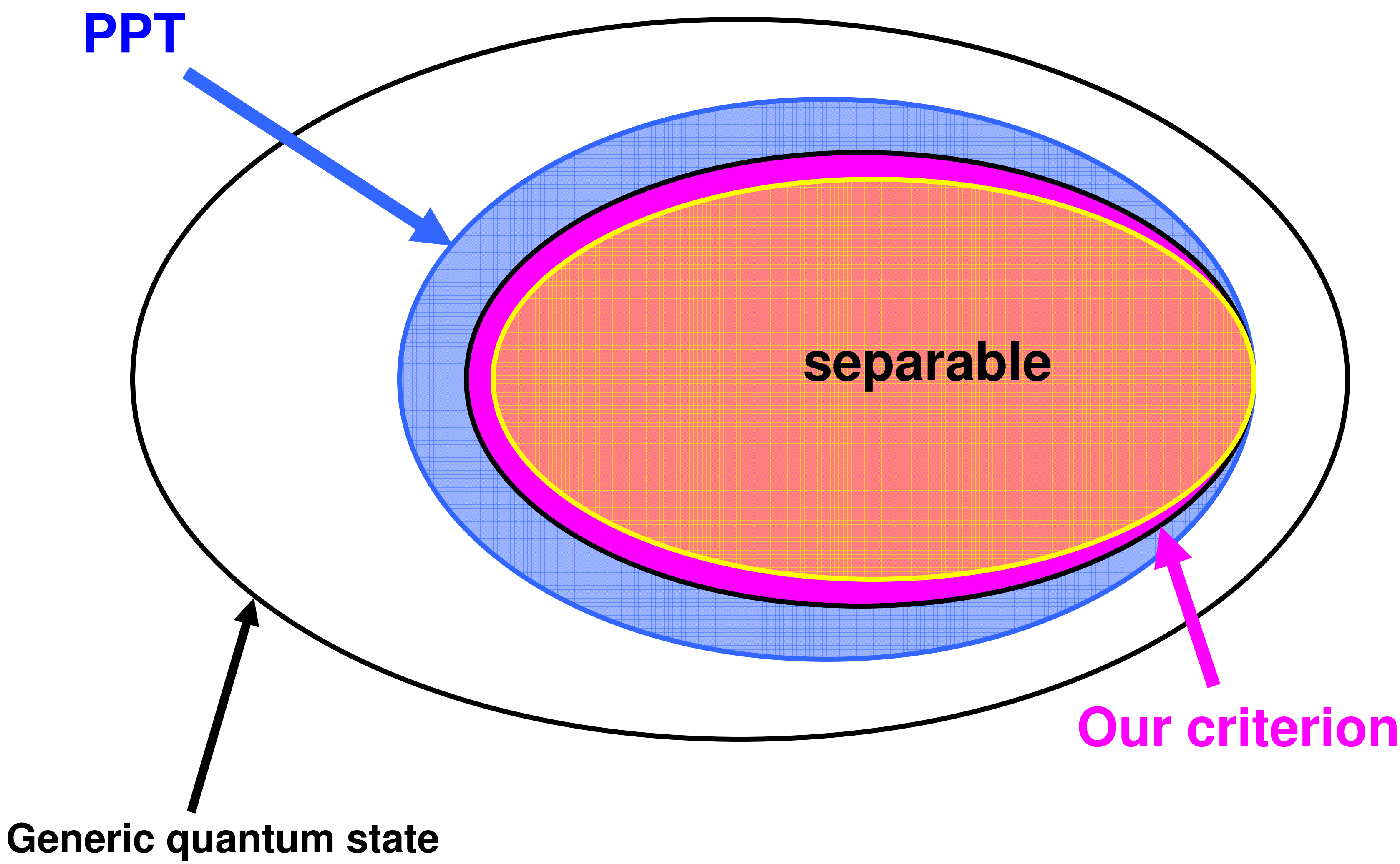
Results

1. Entanglement witness operators generated from the realignment criterion and PPT criterion are more powerful than the two criteria to identify entanglement
2. Positive map (not completely positive) constructed from these entanglement witnesses (EW) are further powerful than the EWs

Significance

1. Offer a more power operational method to recognize entanglement, in particular, the bounded entanglement
2. Provide a powerful new method to detect entanglement, since the entanglement witnesses are physical observables and may be measured locally
3. Gives a new systematic way to obtain positive but non-CP maps

Comparison of separability criteria



Conclusions and discussions

- The separability of a quantum state and quantitative character for entanglement become two of the most basic problems in quantum Information theory
- Multipartite systems and higher dimensions make a rich structure but with more complexity
- The realignment criterion and the corresponding witness operators and positive maps significantly expand our ability to recognize directly the entanglement
- The final solution needs better ideas and is still full of challenge

Kai Chen, Ling-An Wu, Quantum Information and Computation, Vol.3, No.3 (2003) 193-202, quant-ph/0205017; Physics Letters A 306 (2002) 14-20, quant-ph/0208058; Phys. Rev. A 69, 022312 (2004), quant-ph/0306041
S. Albeverio, K. Chen, S.M. Fei, Phys. Rev. A 68 (2003) 062313, quant-ph/0312185.

Note: **It should be remarked that Rudolph has independently done work similar to quant-ph/0205017 in quant-ph/0202121 [Oliver Rudolph, "Further results on the cross norm criterion for separability"] where he called it the computable cross norm criterion.**