A Completion of the Bohr and Rosenfeld's Seminal Article on the Problem of Measuring the Free Quantum Electromagnetic Field

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It has been decided by general consensus that in 1933, Bohr and Rosenfeld showed that the order of magnitude of precision in the measurement of the free quantum electromagnetic components imposed by their uncertainty relations could be estimated with thought experiments. The ingenious article by Bohr and Rosenfeld describes the complicated measuring devices that would be needed, but does not include a graphic representation of these. Neither does it show the measurement of a magnetic component in a given space-time region, nor the measurement of two parallel magnetic components, nor the measurement of two non-parallel magnetic components, nor the parallel measurement of one magnetic and one electric component, nor the non-parallel measurement of one electric and one magnetic component, nor the non-parallel measurement of one magnetic and one electric component in two different space-time regions. This article covers these pending matters and the graphics of cases studied by them.

Eighteen years after the publication of the fundamental work of Bohr and Rosenfeld, Corinaldesi, a doctoral student of Rosenfeld's, found an error in the estimation of the order of magnitude of a critical field in a situation considered by Bohr and Rosenfeld as the significant one. They used their estimate to conclude that it was not necessary to take into account the vacuum fluctuations in the situation considered.

In the present paper, it will be shown through a comparative analysis between the results of Bohr and Rosenfeld and Corinaldesi, that in the cases studied and not studied by Bohr and Rosenfeld, the error discovered by the latter does not affect the conclusions of the former.

1. INTRODUCTION

It has been decided by general consensus that in 1933, Bohr and Rosenfeld (1) demonstrated by means of thought experiments that the physical possibilities of measurement were in accordance with what had been predicted by the formalism of free quantum electrodynamics. That is to say, the order of magnitude of precision in the measurement of the free quantum electromagnetic components imposed by their uncertainty relations could be estimated by ideal experiments, distinguishing clearly between the effects of the structure of the test bodies and the quantum-mechanical effects which arise from their manipulation.

In the commutation relations (24) of the free quantum electromagnetic field components in two defined space-time points, only the constants $\hbar$ and c and space-time derivatives of the Dirac δ-function are required. Also, with $\hbar$ and c there is no fundamental space-time scale association and the Dirac δ-function is only well-defined when integrate over so, Bohr and Rosenfeld concluded that only the space-time averages of the field components make sense, and that their measurement should be carried out with charge and/or monopole distributions, and the possibility of constructing them was studied in detail by Bohr and Rosenfeld, who opted for the measurement of an electric component in a given space-time region. They also studied the measurement of two parallel electric components, the measurement of two non-parallel electric components and the parallel measurement of one electric and one magnetic component in different space-time regions.

Bohr and Rosenfeld's ingenious article describes in words the complicated measuring devices that would be needed, but does not include a graphic representation of these, nor does it show the measurement of a magnetic component in a given space-time region, nor the measurement of two parallel magnetic components, nor the measurement of two non-parallel magnetic components, nor the parallel measurement of one magnetic and one electric component, nor the non-parallel measurement of one electric and one magnetic component, nor the non-parallel measurement of one magnetic and one electric component in two different space-time regions.

This article covers the pending cases and the illustrations of the cases already studied. Recently Compagno and Persico [(2), (3), (25), (27)] have cast doubt on the validity of the calculation that Bohr and Rosenfeld carried out with respect to the self-field of the test body in the measuring of an electric component within a space-time region.

Nonetheless, Compagno and Persico's re-analysis of Bohr and Rosenfeld's procedure has been criticized by Hnizdo [(4), (5), (6), (26), (28)], who declares that Compagno and Persico's results are incorrect. Bearing in mind this debate, and remembering [(7)] that some eighteen years after the publication of Bohr and Rosenfeld's original work, Corinaldesi [(16), (17)], who was one of Rosenfeld's own doctoral students, discovered a significant error in a critical field used by them to show that it was not necessary to take into account vacuum fluctuations in the measurement problem under discussion.

This makes it important to check the conclusions made by Bohr and Rosenfeld in the light of Corinaldesi's finding, so that subsequent developments can be fully justified. This paper shows that from a conceptual point of view, Bohr and Rosenfeld have no need to respond to Corinaldesi. Furthermore, using the calculations of order of magnitude which arise from the comparison of the Bohr and Rosenfeld and Corinaldesi-type of cases, we can see that the correction of the latter does not have decisive consequences on the analysis made by the former, or on the cases of measurement which were not analyzed by Bohr and Rosenfeld in their article.

2. UNCERTAINTY RELATIONS

Some of the uncertainty relations predicted by the formalism for the average components of the free quantum electromagnetic field are [(1), (24)]

$$\Delta\overline{E}_x^{(I)}\Delta\overline{E}_x^{(II)} = \Delta\overline{H}_x^{(I)}\Delta\overline{H}_x^{(II)} \approx \hbar\left|\overline{A}_{xx}^{(I,II)} - \overline{A}_{xx}^{(II,I)}\right|, \tag{2.1}$$

$$\Delta\overline{E}_x^{(I)}\Delta\overline{E}_y^{(II)} = \Delta\overline{H}_x^{(I)}\Delta\overline{H}_y^{(II)} \approx \hbar\left|\overline{A}_{xy}^{(I,II)} - \overline{A}_{xy}^{(II,I)}\right|, \tag{2.2}$$

$$\Delta\overline{E}_x^{(I)}\Delta\overline{E}_z^{(II)} = \Delta\overline{H}_x^{(I)}\Delta\overline{H}_z^{(II)} \approx \hbar\left|\overline{A}_{xz}^{(I,II)} - \overline{A}_{xz}^{(II,I)}\right|, \tag{2.3}$$

$$\Delta\overline{E}_x^{(I)}\Delta\overline{H}_x^{(II)} = \Delta\overline{H}_x^{(I)}\Delta\overline{E}_x^{(II)} = 0\,, \tag{2.4}$$

$$\Delta\overline{E}_x^{(I)}\Delta\overline{H}_y^{(II)} = \Delta\overline{H}_x^{(I)}\Delta\overline{E}_y^{(II)} \approx \hbar\left|\overline{B}_{xy}^{(I,II)} - \overline{B}_{xy}^{(II,I)}\right|, \tag{2.5}$$

$$\Delta\overline{E}_x^{(I)}\Delta\overline{H}_z^{(II)} = \Delta\overline{H}_x^{(I)}\Delta\overline{E}_z^{(II)} \approx \hbar\left|\overline{B}_{xz}^{(I,II)} - \overline{B}_{xz}^{(II,I)}\right|, \tag{2.6}$$

where, for example

$$\overline{E}_x^{(I)} = \frac{1}{V_I T_I}\int_{T_I} dt_1 \int_{V_I} E_x \; dv_1\,, \tag{2.7}$$

is the average of E_x in the space-time region I, with volume V_I and duration T_I.

Additionally, writing $\vec{r}_1 = (x_1, y_1, z_1)$, $\vec{r}_2 = (x_2, y_2, z_2)$ and $r = |\vec{r}_2 - \vec{r}_1|$ we have

$$\overline{A}_{xx}^{(I,II)} = -\frac{1}{V_I V_{II} T_I T_{II}}\int_{T_I} dt_1 \int_{T_{II}} dt_2 \int_{V_I} dv_1 \int_{V_{II}} dv_2 \left(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{c^2 \partial t_1 \partial t_2}\right)\left\{\frac{1}{r}\delta\left(t_2 - t_1 - \frac{r}{c}\right)\right\}, \tag{2.8}$$

$$\overline{A}_{xy}^{(I,II)} = -\frac{1}{V_I V_{II} T_I T_{II}}\int_{T_I} dt_1 \int_{T_{II}} dt_2 \int_{V_I} dv_1 \int_{V_{II}} dv_2 \frac{\partial^2}{\partial x_1 \partial y_2}\left\{\frac{1}{r}\delta\left(t_2 - t_1 - \frac{r}{c}\right)\right\}, \tag{2.9}$$

$$\overline{A}_{xz}^{(I,II)} = -\frac{1}{V_I V_{II} T_I T_{II}}\int_{T_I} dt_1 \int_{T_{II}} dt_2 \int_{V_I} dv_1 \int_{V_{II}} dv_2 \frac{\partial^2}{\partial x_1 \partial z_2}\left\{\frac{1}{r}\delta\left(t_2 - t_1 - \frac{r}{c}\right)\right\}, \tag{2.10}$$

$$\overline{B}_{xy}^{(I,II)} = -\frac{1}{V_I V_{II} T_I T_{II}}\int_{T_I} dt_1 \int_{T_{II}} dt_2 \int_{V_I} dv_1 \int_{V_{II}} dv_2 \frac{\partial^2}{c\partial t_1 \partial z_2}\left\{\frac{1}{r}\delta\left(t_2 - t_1 - \frac{r}{c}\right)\right\}, \tag{2.11}$$

$$\overline{B}_{xz}^{(I,II)} = -\frac{1}{V_I V_{II} T_I T_{II}}\int_{T_I} dt_1 \int_{T_{II}} dt_2 \int_{V_I} dv_1 \int_{V_{II}} dv_2 \frac{\partial^2}{c\partial t_1 \partial y_2}\left\{\frac{1}{r}\delta\left(t_2 - t_1 - \frac{r}{c}\right)\right\}. \tag{2.12}$$

Therefore, it must be possible to take the measurement of only one field component with arbitrary precision, the averages of all the component pairs over the same space-time region commute and must be exactly and independently measurable. Also, the averages of the two components of the same type over two space-time regions with equal times commute, and similarly the averages of the components of different types commute in two space-time regions when the spatial regions coincide.

From the above uncertainty relations, there are at least ten representative and different cases of measurement, from which the measurement cases considered by Bohr and Rosenfeld were

(1) $\overline{E}_x^{(I)}$,

(2) $\overline{E}_x^{(I)}$ and $\overline{E}_x^{(II)}$,

(3) $\overline{E}_x^{(I)}$ and $\overline{E}_y^{(II)}$ and

(4) $\overline{E}_x^{(I)}$ and $\overline{H}_x^{(II)}$;

and the measurement cases not considered by Bohr and Rosenfeld which will be treated here are

(5) $\overline{H}_x^{(I)}$,

(6) $\overline{H}_x^{(I)}$ and $\overline{H}_x^{(II)}$,

(7) $\overline{H}_x^{(I)}$ and $\overline{H}_y^{(II)}$,

(8) $\overline{H}_x^{(I)}$ and $\overline{E}_x^{(II)}$,

(9) $\overline{E}_x^{(I)}$ and $\overline{H}_y^{(II)}$ and

(10) $\overline{H}_x^{(I)}$ and $\overline{E}_y^{(II)}$.

3. PRESUPPOSITIONS FOR ELECTROMAGNETIC FIELD MEASUREMENTS

3.1 ELECTRIC FIELD MEASUREMENTS

Taking into account that quotations from different authors have been marked with quotations marks where they begin and end, it is reasonable to accept [8], along with Bohr, "that to have scientific experiences it is necessary to develop a language with no ambiguity. He maintained that there is only one language - human language - and the only one we will ever have. In its everyday use this language is full of ambiguities but nevertheless it can be refined, and rid of all ambiguities. For Bohr, classical physics is only a refinement of this everyday language".

Within Bohr's approach [31] "a crucial epistemological role is imparted to the instruments of measurement. In Bohr's words they must be described in purely classical terms. Concerning them it is essential to note that at the place where Bohr made a conceptual analysis of some thought experiments involving movable diaphragms, what he suggested in this connection is that instruments and their parts are to be considered either as classical or as quantum systems not according to their physical properties", for example, be macroscopic or be microscopic, "but just according to our point of view. They are quantum systems if we carry on observations on them by means of other instruments. They are classical systems if we use them as instruments of observation" so, taking into account that the measurement of electromagnetic field quantities rests by definition on the transfer of linear momentum to suitable electric or magnetic test bodies situated in the field, the interactions between the field which is measured and the test body which defines the measurement must be described in classical terms. Equally, the test body as a field source during its linear momentum measurements must also be described in these terms in spite of the Heisenberg's uncertainty principle restricts the above linear momentum measurements.

"This, of course, raises some delicate problems, concerning both the notation of object and that of instrument. What part of a phenomenon can properly be called the (quantum) object?

This question has been made the theme of some debate. Here let it simply is noted how Roldán [(8)] contributed to somewhat clarify this point: he noted first that, of course, within any well-specified phenomenon we feel it necessary to distinguish between the instrument of observation and all the rest. But his idea is that we must be careful not to rashly identify this rest with the object, as we might well be tempted to do. Following him, let us just call it the inside of the phenomenon. His point is that, as we just noted, the cut between the instrument and the inside depends on what phenomenon is considered: in a two-slit experiment with a movable diaphragm (and of the type Bohr considered), it leaves the diaphragm on the side of the inside; on the contrary, in the case in which the same diaphragm is fixed, it leaves it on the instrument side. Hence it is only in the latter case that the inside coincides with the diffracted particle and can therefore properly be called the object. To get at, within Bohr's views, a suitable general definition of the object we must therefore make one more move. Roldán suggests defining the object as the feature of the inside that remains the same when two complementary phenomena are considered. In the example under study the phenomena with fixed and movable diaphragms are, in a way, complementary in Bohr's sense and their insides do have a common part, namely the particle. We see therefore, what was not obvious at the start that even within Bohr's approach the notion of a quantum object can consistently be kept".

In order to have a definite case in mind we consider the measurement of the average of the electric field component in the *x*-direction.

Choosing the region of field measurement with volume $V = L^3$ and time T, and observing that that measurement remains by definition in the linear momentum transfer to the test body situated in the field during instants Δt, which occur in each individual linear momentum measurement at the beginning t^i and at the end t^f of interval T; it is concluded that Δt must take itself as small as possible, especially if the field measured varies quickly with time. Therefore we have that Δt << T. By also adding a large mass to the test body, its acceleration due to the action of the field can obviously be made very small.

Supposing that the measurements of the linear momentum of the test body are made in the *x*-direction, and that its uncertainty is Δp_x, which is accompanied by a loss Δx << L in the knowledge of the position of the body in question; in spite of the fact that a sufficiently massive test body can be chosen to reduce the acceleration acquired within the action of the field measured, the body can nevertheless acquire a velocity v_x during Δt, which can be greater still if Δp_x and Δt are small. Indeed, according to relations $\Delta H = v_x \Delta p_x$ and $\Delta H \Delta t \approx \hbar$, where ΔH is the energy uncertainty of the test body during the time of measurement Δt, we obtain $v_x \approx \dfrac{\hbar}{\Delta p_x \Delta t}$. This velocity, as we shall see later can be compensated by means of a mechanical device as a lever moved by a spring, which, after each linear momentum measurement, gives a counter-impulse to the test body so that the velocity is reduced to zero. We can therefore consider that, with the exception of time lapse Δt which each measurement takes, the test body is always at rest, and subsequently its position practically never changes.

It is helpful to say that what Bohr and Rosenfeld had in mind here is what is now called a quantum non-demolition measurement [30].

The description of the rigid test body is necessary to ensure that the only effect of the field measured on the test body is the global displacement of the body. Nevertheless, as the body has a finite extension, the effects of retardation due to the finite propagation of the interactions must be taken into account. In particular, it is required that cT < L, because otherwise, the field measured will have time to spread from one extreme to the other of the test body during time T of exposure to the field. This propagation will have the tendency of masking the temporal dependence which is also to be measured. The opposite case cT ≥ L, still in the classical domain, is of little interest, because all the peculiarities of the undulatory fields within volume $V = L^3$ are at the end reduced by the averaging process in the propagation during time interval T.

Now, given that the speed of propagation of the interactions is c, the time required to give to an extensive test body of longitude L, considered as a whole, a well-defined linear momentum, is in the order of magnitude $\delta t \approx \frac{L}{c} >$ T. Therefore the measurement time Δt will not meet the condition $\Delta t << T$ because this would imply that $\Delta t << \delta t$ and the body, as a whole, will not be able during the time of measurement to acquire a well defined linear momentum by the action of the field. We must therefore think about a test body which consists of a system of τ individual components, each with dimension L_τ and sufficiently small so that all retardation effects will not be felt, that is $L_\tau << c\Delta t$, and such that the distance between them does not change during the measurement process.

The test bodies can be considered as uniformly charged rigid bodies, with density ρ, uniformly extended over all the volume $V = L^3$ under investigation.

Following strictly the Bohr and Rosenfeld's [1] words, "we must follow the behavior of the test body as accurately as possible during the whole measuring process. It turns out that for this purpose it is necessary first of all to know accurately the position of each test body at all times before and after its use in the measurement. This is achieved most expediently by having the test body firmly attached to a rigid frame serving as a spatial reference system, except in the time interval during which the momentum transfer to the test body from the field is to be determined. At the beginning of this interval the attachment must be disconnected and the momentum component of the test body in the direction of the field component that is to be determined must be measured. We always assume that by an immediately following counter-impulse, as discussed above, the body is brought back to rest with accuracy inversely proportional to its mass, at a position which is not accurately predictable. At the end of the time interval and after renewed measurement of the momentum component in question, the firm attachment is re-established; here it turns out to be not unessential that the test body be brought back into exactly the same position as it had originally.

Still more far-reaching demands on idealization with respect to the construction and handling of the test body system are obviously needed to measure field averages over two partially overlapping space-time regions. For in this case we must have test bodies at our disposal which can be displaced inside each other without mutual mechanical influence. In order that the electromagnetic field to be measured is disturbed as little as possible by the presence of the test body system, we shall imagine, moreover, that every electric or magnetic component body is placed adjacent to a neutralization body with exactly the opposite charge".

So, at any attempt to construct idealized diagrams of the arrangements of the sort that will be offered here, the positive and negative charge distributions could be completely overlap and/or be simply adjacent to one another according as what part of the arrangements we are considering, those in the two space-time regions of measurement as a whole, or in each one of regions of measurement. The present paper's pseudo-realistic diagrams do not take into account the fact that the test bodies must be made of a large number of independent small bodies and the implied complications in the momentum measurement and counter-impulse processes. Presumably, this is precisely the reason why Bohr and Rosenfeld did not use such diagrams.

Remember that according to classical electrodynamics the interaction between the test body and the field measured $\overline{E}_x$ can be written as

$$p_x^f - p_x^i = \rho\, \overline{E}_x VT\,, \tag{3.1}$$

where p_x^i and p_x^f are the linear momenta of the charged body at the beginning and at the end of interval T, respectively.

In 1951 Corinaldesi [16], [17] discovered that Bohr and Rosenfeld [1] had made a mistake in the calculation of a critical field related to vacuum fluctuations. It has been argued [18], [7] that this was surprising in their famous article they used themselves [19] in the measurement of the charge and current quantities, and by Peres and Rosen [20], DeWitt [21], von Borzeszkowski and Treder [22] and Bergmann and Smith [23] among others [29], in the measurement of the free quantum gravitational field.

We will now show that Corinaldesi's correction does not change the conclusions made by Bohr and Rosenfeld in their article, neither in the cases studied by them, nor in the cases not considered by them, and at the same time we will justify the subsequent studies based on Bohr and Rosenfeld's pioneering article from a very different perspective to that used by Hnizdo [4], [5], [6], [26], [28] in the defense of Bohr and Rosenfeld's work, under attack from Compagno and Persico [2], [3], [25], [27].

Bohr and Rosenfeld designated the square root of any of the expressions (2.1), (2.2), (2.3), (2.5) or (2.6) as a critical field $\Delta_{\text{Bohr-Rosenfeld}} = \Delta_{\text{BR}}$, for every electromagnetic field that may be considered, if the latter, being much greater than Δ_{BR}, was close to the classical description mode.

Similarly, Bohr and Rosenfeld defined another critical quantity $\Delta o_{\text{Bohr-Rosenfeld}} = \Delta o_{\text{BR}}$, as the square root of the mean quadratic fluctuation of each quantity of electromagnetic field, when the number of photons is defined and is equal to zero. This situation corresponds to the vacuum, in which the expected value of all the field averages is certainly null, but not their mean quadratic fluctuations. The quantity Δo_{BR} is critical in the sense that when field averages are considered which are much greater than it, the vacuum fluctuations can be neglected.

Taking into account that the electrical component of the reaction field given by Bohr and Rosenfeld (1) according to the expression (4.11) of Section 4.1, has an approximate value

$$E_x^r \approx \rho \Delta x \, , \qquad (3.2)$$

which, replacing it in (3.1), allows the calculation of the approximate linear momentum transferred by radiation from the test body

$$\delta_r p_x \approx \rho^2 V \Delta x \Delta t \, . \qquad (3.3)$$

From the relation $\Delta p_x \Delta x \approx \hbar$ we obtain, based on the equation (3.1) that

$$\Delta \overline{E}_x \approx \frac{\hbar}{\rho \Delta x V T} \, . \qquad (3.4)$$

In the interesting physical case L > cT Bohr and Rosenfeld found (1) that

$$\Delta_{BR} \approx \sqrt{\frac{\hbar}{L^3 T}}, \; \Delta o_{BR} \approx \frac{\sqrt{\hbar c}}{L^2}, \qquad (3.5)$$

that is to say that $\left(\frac{\Delta_{BR}}{\Delta o_{BR}}\right)^2 \approx \frac{L}{cT} > 1$, and in the limiting case L >> cT, the critical quantity Δ_{BR} is much greater than Δo_{BR} and, therefore, after showing the consequences of the formalism, the vacuum fluctuations can be neglected.

Darrigol (18) has obtained the following simple values for Corinaldesi's (16), (17) critical fields, if L > cT

$$\Delta_{Corinaldesi} = \Delta_C \approx c\sqrt{\frac{\hbar T}{L^5}}, \; \Delta o_{Corinaldesi} = \Delta o_C \approx \frac{\sqrt{\hbar c}}{L^2}, \qquad (3.6)$$

from which $\left(\frac{\Delta_C}{\Delta o_C}\right)^2 \approx \frac{cT}{L} < 1$, and in the limiting case L >> cT, Δ_C is much less than Δo_C, and the vacuum fluctuations cannot be neglected.

In the case of cT ≥ L, Bohr and Rosenfeld and Corinaldesi obtained similar values, that is

$$\Delta_{BR} \approx \Delta o_{BR} \approx \Delta_C \approx \Delta o_C \approx \frac{\sqrt{\hbar c}}{LcT}, \qquad (3.7)$$

obtaining that $\left(\frac{\Delta_{BR}}{\Delta o_{BR}}\right) \approx \left(\frac{\Delta_C}{\Delta o_C}\right) \approx 1$, without being able to neglect the vacuum fluctuations either.

Note that in the case L > cT the critical fields Δ_{BR} and Δ_C do not coincide. This was the big flaw in the calculations of Bohr and Rosenfeld which led them to conclude that the vacuum fluctuations Δo_{BR} in comparison with Δ_{BR} could be neglected.

In reality, as Corinaldesi [16], [17], demonstrated, the vacuum fluctuations $\Delta_{OC} \approx \Delta o_{BR}$ could not be neglected and are greater than the true critical field Δ_C.

Nor in the case cT ≥ L can the vacuum fluctuations be neglected, resulting in an impediment in the measurement which is inherent in the formalism.

The vacuum fluctuations of the field are equal or larger than the non-commutation effects that we would look for to verify the predictions of the free quantum electrodynamics formalism. Therefore, it seems that Bohr and Rosenfeld may have been wrong to claim that their ideal measurements could reveal the quantum nature of the field.

It could be thought that no measurement is therefore possible, and besides, the results obtained by Bohr and Rosenfeld and those obtained by other authors based on these, have no justification, or should be changed, taking into account the true value Δ_C and not Δ_{BR}.

Although it is possible that the exact value Δ_{Exact} falls typically in the middle between the Corinaldesi and Bohr and Rosenfeld estimates ($\Delta_C < \Delta_{Exact} < \Delta_{BR}$), for purposes of comparison, they would respectively play the role of minimum limit and maximum limit and so, the Corinaldesi and Bohr and Rosenfeld estimates seem very useful when one is speaking about the typical extreme magnitude orders.

In the following paragraphs aspects corresponding to the possible modification of Bohr and Rosenfeld's calculations will be analyzed, followed later by the conceptual analysis.

Beginning now with a process of comparison [10] and returning to the interesting physical case L > cT, we arrive at $\left(\frac{\Delta_C}{\Delta_{BR}}\right)^2 \approx \left(\frac{cT}{L}\right)^2 < 1$, obtaining $\Delta_C \approx (cT/L)\,\Delta_{BR}$ and therefore

$$\Delta_C < \Delta_{BR}. \qquad (3.8)$$

Following the own definitions of Δ_{BR} and Δ_C, equation (3.8) suggests that it is easier to arrive at the classical description mode in Corinaldesi's case than in Bohr and Rosenfeld's case.

We have seen that $\Delta\overline{E}_{x_{BR}} \approx \frac{\hbar}{\rho V T \Delta x}$, but as $\Delta_{BR} \approx \sqrt{\frac{\hbar}{L^3 T}}$ therefore $\Delta\overline{E}_{x_{BR}} \approx \frac{\Delta_{BR}^2}{\rho\Delta x} = \left(\frac{\Delta_{BR}}{\rho\Delta x}\right)\Delta_{BR}$. It is also required by the own definition of Δ_{BR} that $\Delta\overline{E}_{x_{BR}} << \Delta_{BR}$ so that the quantum domain is tackled in the $\Delta\overline{E}_{x_{BR}}$ measurement. The latter is the same as saying that the expression defined by $\frac{\Delta_{BR}}{\rho\Delta x} = \lambda_{BR}$ must satisfies $\lambda_{BR} << 1$, or that $\Delta_{BR} \approx \frac{\Delta\overline{E}_{x_{BR}}}{\lambda_{BR}}$ is a value of certainty in the measurement. We finally conclude that $\Delta\overline{E}_{x_{BR}} \approx \lambda_{BR}\Delta_{BR}$.

From the demands with which $\lambda_{BR} << 1$, L > cT, $\Delta x < L$ and $\alpha = \sqrt{\frac{e^2}{\hbar c}} < 1$, being α the fine-structure constant, Bohr and Rosenfeld [1] showed that the number of elemental charges N_{BR} of which the total charge of the test body is composed must be very big. That is to say $N_{BR} = (\rho V/e) = \frac{\Delta_{BR} V}{e\lambda_{BR}\Delta x} = \frac{L}{\lambda_{BR}\Delta x}\sqrt{\frac{L}{cT}}\sqrt{\frac{\hbar c}{e^2}} >> 1$. Note that the expression for N_{BR} does not directly depend on the value that Δ_{BR} could have, and therefore its order of magnitude will not change when the quantity Δ_C is replaced for comparison.

The intention is to clarify with calculations the extent to which some of the expressions used by Bohr and Rosenfeld will be affected, in particular those which contain in one way or another quantity Δ_{BR} and not Δ_C.

Corinaldesi's correction implies similar results to those obtained above, that is:

$$\Delta\overline{E}_{x_{BR}} \approx \frac{\Delta^2_{BR}}{\rho\Delta x} \approx \left(\frac{L}{cT}\right)^2 \frac{\Delta^2_C}{\rho\Delta x} = \Delta\overline{E}_{x_C}$$, from which

$$\Delta\overline{E}_{x_C} \approx \Delta\overline{E}_{x_{BR}}. \tag{3.9}$$

That is to say, there is equal uncertainty in the measurement in Corinaldesi's case compared with Bohr and Rosenfeld's case.

In analogy with the above argument, we can have, for the quantity defined by $\lambda_C = \frac{\Delta_C}{\rho\Delta x}\left(\frac{L}{cT}\right)^2 \approx \frac{\Delta\overline{E}_{x_C}}{\Delta_C}$, that $\Delta\overline{E}_{x_C} \approx \lambda_C\Delta_C$. Another immediate result is that $\frac{\lambda_C}{\lambda_{BR}} = \frac{\Delta_C}{\Delta_{BR}}\left(\frac{L}{cT}\right)^2 \approx \left(\frac{cT}{L}\right)\left(\frac{L}{cT}\right)^2 = \frac{L}{cT} > 1$, that is to say

$$\lambda_C > \lambda_{BR}. \tag{3.10}$$

We conclude from (3.10) that it is easier to tackle the quantum domain in the Bohr and Rosenfeld's measurement case than in Corinaldesi's.

The value of the number of elemental charges that the total charge of the test body should have in Corinaldesi's case will be $N_C = (\rho\ V/e) = \frac{\Delta_C V}{e\lambda_C\Delta x}\left(\frac{L}{cT}\right)^2$. Comparing expressions N_{BR} and N_C we obtain $\frac{N_C}{N_{BR}} = \frac{\lambda_{BR}}{\lambda_C}\left(\frac{L}{cT}\right)^2\frac{\Delta_C}{\Delta_{BR}} \approx \left(\frac{L}{cT}\right)^2\left(\frac{cT}{L}\right)^2 \approx 1$, that is to say

$$N_C \approx N_{BR}. \tag{3.11}$$

Equation (3.11) indicates that the test body, electrically speaking remains the same or that approximately the number of elemental charges of the test body is almost the same in Corinaldesi's and Bohr and Rosenfeld's cases.

The fact that ρ must make itself very large in order to reduce quantity (3.4), and likewise for the field of the test body ($E^r_x \approx \rho\ \Delta x$) to be very big, does not imply that E^r_x is classical. What is required is an estimation of the frequency involved in its magnitude. The region to be averaged has linear dimension of order L and temporal extension of order T < (L /c). That is, (1 / T) > (c / L) and the minimum photon frequency will be $\nu_{min} \approx (c / L)$, with minimum photon energy $\hbar\nu_{min}$, from which the maximum number of photons according to Bohr and Rosenfeld [(1)] is $n_{BR} = \frac{(E^r_x)^2 V}{\hbar \upsilon_{min}} \approx \rho^2 \Delta x^2 V \frac{L}{\hbar c} = \lambda_{BR}^{-2}\left(\frac{L}{cT}\right)$. This indicates that the more precise the measurement is ($\lambda_{BR} << 1$), the more classical the test body field becomes. In Corinaldesi's case $n_C \approx \rho^2 \Delta x^2 V \frac{L}{\hbar c} = \lambda_C^{-2}\left(\frac{cT}{L}\right)^3$ and, on comparing, $(n_C / n_{BR}) \approx \left(\frac{\lambda_C^{-2}}{\lambda_{BR}^{-2}}\right)\left(\frac{L}{cT}\right)^2 \approx 1$, from which

$$n_C \approx n_{BR} . \quad (3.12)$$

Therefore the number of photons in the test body field, according to (3.12) would not vary when making the comparison between Corinaldesi's and Bohr and Rosenfeld's cases.

For equation (3.3) Bohr and Rosenfeld [(1)] also obtained the expression $\delta_r p_{x_{BR}} \approx \rho^2 \Delta x V \Delta t \approx \Delta p_x \lambda_{BR}^{-2} \frac{\Delta t}{T}$, which implies that for any desired accuracy of the field measurement ($\lambda_{BR} << 1$), the influence of the electromagnetic reaction on the momentum measurement of the test body can be neglected if only Δt is chosen sufficiently small in comparison with T ($\Delta t << T$).

In Corinaldesi's case it is concluded that $\delta_r p_{xC} \approx \left(\frac{L}{cT}\right)^2 \Delta p_x \lambda_C^{-2} \frac{\Delta t}{T}$ and comparing once again we conclude that $(\delta_r p_{x_C} / \delta_r p_{x_{BR}}) \approx 1$, from which

$$\delta_r p_{x_C} \approx \delta_r p_{x_{BR}} , \quad (3.13)$$

provided the linear momentum transferred by radiation from the test body is the same in the two cases.

Remember that these results are obvious, based on expressions that depend on neither Δ_{BR}, Δ_C, λ_{BR} nor λ_C, but the fact that Bohr and Rosenfeld themselves transformed the above results according to the previous quantities, could suggest that these results could be amended by Corinaldesi's correction.

We can therefore conclude that all the preceding results which depend on one or another way of Δ_{BR}, Δ_C, λ_{BR} or λ_C, remain with the same status attributed by Bohr and

Rosenfeld, after taking into account Corinaldesi's correction in both cases L > cT and $L \leq cT$.

"Certainly [18] Bohr and Rosenfeld fail to provide the agreement between the possibilities of field definition and the possibilities of field transformation which is the essence of complementary mode of description, because the methods of field preparation of which Bohr and Rosenfeld are aware, as we will see later, exclude quantum states that fluctuate less than the vacuum state. This lack of agreement occurs even without the Corinaldesi's correction. However, that agreement can be reestablished by allowing more general field preparations which are now possible in the laboratory, the called squeezed light [32] ".

3.2 MAGNETIC FIELD MEASUREMENTS

In the measurement of $\overline{H}_x$ not studied [9] by Bohr and Rosenfeld we would have the following linear momentum balance if we use a current density j_z instead of a monopole distribution

$$p_y^f - p_y^i = j_z VT\overline{H}_x , \tag{3.14}$$

in complete analogy with (3.1).

Perhaps Bohr and Rosenfeld did not consider it necessary to give an explicit treatment of the measurements of magnetic field components for the simple reason that, using the traditional concept of magnetic poles (or monopoles), magnetic field measurements are entirely analogous to the electric field measurements. It is true that, until now, isolated magnetic poles do not exists in nature, and for this reason the formalism of monopoles is not given anymore in modern text on electricity and magnetism.

An alternative, more physical, treatment of the magnetic field case would be the use of a test body with a given density of magnetic dipoles, but then the field must be measured in terms of the torque it exerts on the test body, and the nice parallel with the electric case would not be possible. Our treatment in terms of the force on a current density component j_z seems flawed because the currents cannot run in just one direction, they must form loops, and therefore the force of the magnetic field on the test body cannot be due to a current running only in one direction.

Nevertheless, it can be assumed without loss of generality that inside the measurement region is a distribution of current sufficiently large for being straight there, and although closed outside the region too.

Due to expression (4.33) of Section 4.2, the order of magnitude of the magnetic component in the *x*-direction of the reaction field of the test body is

$$H^r_{j_z} \approx j_z \Delta y , \tag{3.15}$$

which, replacing it in (3.14) allows us to find the order of magnitude of the linear momentum transferred by radiation from the test body

$$\delta_{j_z} p_y \approx j_z^2 V \Delta t \Delta y . \tag{3.16}$$

From the equation $\Delta p_y \Delta y \approx \hbar$ and using equation (3.14) it is possible to obtain

$$\Delta \overline{H}_x \approx \frac{\hbar}{j_z \Delta y V T} . \tag{3.17}$$

From equation (3.17) and in analogy with Bohr and Rosenfeld's treatment, it can be concluded that $\Delta \overline{H}_{x\,BR} \approx \frac{\hbar}{j_z \Delta y V T}$, but since $\Delta_{\mathrm{BR}} \approx \sqrt{\frac{\hbar}{\mathrm{L^3 T}}}$ therefore $\Delta \overline{H}_{xBR} \approx \frac{\Delta^2_{BR}}{j_z \Delta y} = \left(\frac{\Delta_{BR}}{j_z \Delta y}\right) \Delta_{BR}$. It is also required by the own definition of Δ_{BR} that $\Delta \overline{H}_{x\,BR} << \Delta_{BR}$ so that the quantum domain is tackled in the $\overline{H}_{x\,BR}$ measurement.

The latter is the same as saying that the inequality $\frac{\Delta_{BR}}{j_z \Delta y} = \vartheta_{BR} << 1$ must be satisfied, or that $\vartheta_{BR} \approx \frac{\Delta \overline{H}_{xBR}}{\Delta_{BR}}$ is a value of certainty in the measurement. We finally conclude that $\Delta \overline{H}_{x\,BR} \approx \vartheta_{BR} \Delta_{BR}$.

A relation for obtaining the number of elemental currents that a magnetic test body must have cannot be found because there are no experimental tests of magnetic monopoles. What is noted is that, as has already been proved, N_{BR} and N_C being the numbers of elemental charges of which the total charges of the very large electric test bodies are composed, the current densities j_{zBR} and j_{zC} must also be very large and of the same order of magnitude because of equation (3.11).

Now, $\Delta\overline{H}_{xBR} \approx \frac{\Delta^2{}_{BR}}{j_z \Delta y} \approx \left(\frac{L}{cT}\right)^2 \frac{\Delta^2{}_C}{j_z \Delta y} = \Delta\overline{H}_{xC}$, from which

$$\Delta\overline{H}_{xC} \approx \Delta\overline{H}_{xBR}\,. \qquad (3.18)$$

That is to say, there is equal uncertainty in the measurement in the untreated Corinaldesi's case compared with the untreated Bohr and Rosenfeld's case.

In analogy with the above argument, we can have, for the quantity defined by $\vartheta_C = \frac{\Delta_C}{j_z \Delta y}\left(\frac{L}{cT}\right)^2 \approx \frac{\Delta\overline{H}_{xC}}{\Delta_C}$, that $\Delta\overline{H}_{xC} \approx \vartheta_C \Delta_C$.

Another immediate result is that $\frac{\vartheta_C}{\vartheta_{BR}} = \frac{\Delta_C}{\Delta_{BR}}\left(\frac{L}{cT}\right)^2 \approx \left(\frac{cT}{L}\right)\left(\frac{L}{cT}\right)^2 = \frac{L}{cT} > 1$, that is

$$\vartheta_C > \vartheta_{BR}\,. \qquad (3.19)$$

We conclude from (3.19) that it is easier to tackle the quantum domain in the untreated Bohr and Rosenfeld's measurement case than in Corinaldesi's.

The fact that j_z must make itself very large in order to reduce quantity (3.17), and likewise for the field of the test body ($H^r_{j_z} \approx j_z \Delta y$) to be very big, does not imply that $H^r_{j_z}$ is classical. What is required is an estimation of the frequency involved in its magnitude. The region to be averaged is of linear dimension of order L and temporal extension of order T< $\frac{L}{c}$. That is, $\frac{1}{T} > \frac{c}{L}$ and the minimum photon frequency will be $\nu_{\min} \approx \frac{c}{L}$, with minimum photon energy $\hbar\nu_{\min}$, from which the maximum number of photons according to the untreated Bohr and Rosenfeld's case is $n_{BR} = \frac{\left(H^r{}_{j_z}\right)^2 V}{\hbar\nu_{\min}} \approx$

$j_z^2 \Delta y^2 V \dfrac{L}{\hbar c} = \vartheta_{BR}^{-2}\left(\dfrac{L}{cT}\right)$. This indicates that the more precise the measurement is ($\vartheta_{BR} << 1$), the more classical the test body field becomes.

In the untreated Corinaldesi's case $n_C = \dfrac{\left(H^r{}_{j_z}\right)^2 V}{\hbar \nu_{min}} \approx j_z^2 \Delta y^2 V \dfrac{L}{\hbar c} = \vartheta_C^{-2}\left(\dfrac{L}{cT}\right)^3$ and, on comparing, $\dfrac{n_C}{n_{BR}} \approx 1$, from which

$$n_C \approx n_{BR}. \tag{3.20}$$

Therefore the number of photons in the test body field, according to (3.20) would not vary when making the comparison between Corinaldesi's and Bohr and Rosenfeld's untreated cases.

Using equation (3.16) we obtain $\delta_{j_z} p_{yBR} \approx j_z^2 V \Delta t \Delta y \approx \Delta p_y \vartheta_{BR}^{-2} \dfrac{\Delta t}{T}$, which implies that for any desired accuracy of the field measurement ($\vartheta_{BR} << 1$), the influence of the electromagnetic reaction on the momentum measurement of the test body can be neglected if only Δt is chosen sufficiently small in comparison with T ($\Delta t << T$).

In the untreated Corinaldesi's case it is concluded that $\delta_{j_z} p_{yC} \approx \left(\dfrac{L}{cT}\right)^2 \Delta p_y \vartheta_C^{-2} \dfrac{\Delta t}{T}$ and comparing once again we conclude that ($\delta_{j_z} p_{yC} \ / \ \delta_{j_z} p_{yBR}$) ≈ 1, from which

$$\delta_{j_z} p_{yC} \approx \delta_{j_z} p_{yBR}, \tag{3.21}$$

provided the linear momentum transferred by radiation from the test body is the same in the two untreated cases.

We can therefore conclude that all the preceding results which depend on one or another way of Δ_{BR}, Δ_C, ϑ_{BR} or ϑ_C, remain with the same status that could had been attributed by Bohr and Rosenfeld, after taking into account the untreated Corinaldesi's correction in both cases L > cT and $L \le cT$.

3.3 INTERPRETATION OF VACUUM FLUCTUATIONS

The problem of vacuum fluctuations can be divided in four topics which are:

(1) treatment of Corinaldesi's correction on Bohr and Rosenfeld's treated cases,

(2) treatment of Corinaldesi's untreated correction on Bohr and Rosenfeld's untreated cases,

(3) lack of Bohr and Rosenfeld's answer to Corinaldesi's finding and

(4) preparation of equal, larger or lesser fluctuating field states than vacuum and last Bohr and Rosenfeld's stand.

In this paper the topic (1) has been studied in the Section 3.1. Similarly the topic (2) has been considered in the Section 3.2. Now [(7)], as for the topic (3) and as was said before, Bohr and Rosenfeld made a calculation mistake in the case L > cT. That error led them to the erroneous conclusion that in the situation considered, the vacuum fluctuations could be neglected. Given that in the case $L \leq cT$ the vacuum fluctuations could also not be neglected, it would seem that the effect predicted by the formalism would always remain hidden. However, notwithstanding that it was not judged as physically interesting, Bohr and Rosenfeld made a profound analysis of the interpretation of the vacuum fluctuations for the case $L \leq cT$. That analysis was also necessary for the case of double measurements of field components in space-time regions that almost overlap, where, as can be seen from equations like (2.1), (2.2), (2.3), (2.5) or (2.6), the vacuum fluctuations would hide the measurement independently of the relation between L and cT. It is possible to consider as a deficiency of the Bohr and Rosenfeld work the fact that they never responded to the mistake found by Corinaldesi but in fact, as has been said above, Bohr and Rosenfeld studied a measurement situations, the problem of which was equal to the one that would be originated if the correction of Corinaldesi were taken into account.

Finally with respect to the topic (4), remember that the equation $n_C \approx n_{BR} = \frac{(\vec{E}_x)^2 V}{\hbar \upsilon_{min}} \approx \rho^2 \Delta x^2 V \frac{L}{\hbar c}$ is the maximum number of photons in the reaction field produced by the test body, and from the above it can be seen that $\frac{\rho^2 \Delta x^2}{n_C} \approx \frac{\rho^2 \Delta x^2}{n_{BR}} \approx \frac{\hbar c}{L^4}$ is an estimate of the square of the fluctuations of its own classical electromagnetic field, which only depends on the linear dimensions of the measurement domain and always remains finite without varying with ρ. Looking at equations (3.5), (3.6) and (3.7) it can be seen that for L > cT or cT ≥ L, the vacuum fluctuations of the free quantum electromagnetic field which is to be measured, be it in the case of Bohr and Rosenfeld or Corinaldesi, coincide therefore with the fluctuations of the classical electromagnetic field of the test body; that is to say, the fluctuations in the classical field produced by the test body are the same as the vacuum fluctuations.

"According to Bohr and Rosenfeld, it is possible to have three different types of field to measure [(18)]:

(a) classical description of field sources,

(b) quantum or photon composition of the field to be measured and

(c) measurement of the field with Bohr and Rosenfeld's devices".

After analyzing every preparation situation, they arrived at the conclusion that it is not possible to separate the measured field's fluctuations from those of disturbing fields, that is, those of the classical measuring device, which includes the test body.

"The three kinds of field preparation that Bohr and Rosenfeld could imagine all lead to quantum states that fluctuate at least as much as the vacuum state. If the Hilbert space of field states is restricted to such states, the formalism predicts, for the product of the variances in a double field measurement, a value always higher or equal to the square of vacuum fluctuations. Then the Bohr and Rosenfeld disregard of vacuum fluctuations of the compensating fields in the measurement process leads to a smaller value of the same quantity. And there is no agreement between formalism and possibilities of measurement.

What Bohr and Rosenfeld overlooked [(18)] is that by non-classical means field states can be prepared that fluctuate less than vacuum [(32)] (squeezed light of modern quantum optics). This reestablishes the agreement between formalism and measuring possibilities, as long as it is true that the vacuum fluctuations of compensating fields should be disregarded".

According to Bohr and Rosenfeld vacuum fluctuations are therefore an integral part of the measured field, and the results obtained by means of the Bohr and Rosenfeld experimental devices are the desired field averages. They reinforced this point of view arguing that, by definition, the measurement of all physical quantities must be based on the application of the classical concepts. As a consequence, any consideration of a limitation on the strict applicability of classical electrodynamics in the measurement of quantum fields would contradict the very concept of measurement.

To make observations, one has to interact with the system to be observed, which is then perturbed. Classical physics allows one to suppose that, in principle, the interaction between the measurement instrument and the observed system can be so well known that it is possible to consider the observed and the observer as separated. This aspect is what one can call the divisibility of the classical phenomena.

"Quantum mechanics is based on the quantization of action, h being the minimal action. Every quantum phenomenon is for Bohr [(8)] an indivisible whole, which includes the instrument of observation. The existence of h does not make it possible to define with precision, in a quantum phenomenon, the interaction between the system and the instrument of observation".

In the situations that have been considered in this paper, it would not be possible then to distinguish operationally between the fluctuations of the measured field and those belonging to the field produced by the test body that is a part of the measurement instrument. The field produced by the test body fluctuates in the same manner as the measured field.

The interpretation given by Bohr and Rosenfeld to the vacuum fluctuations allows them to conclude than in no case will those fluctuations hide the effect predicted by the formalism.

"The Bohr and Rosenfeld handling [18] of vacuum fluctuations requires further justification. Bohr himself had doubt on this handling, even after the Bohr and Rosenfeld's paper was published".

4. REACTION FIELD OF THE TEST BODY

4.1 CASE STUDIED BY BOHR AND ROSENFELD

We will follow strictly the Bohr and Rosenfeld's [1] treatment so, let us consider two space-time regions with volumes $V_I = L_I^3$ and $V_{II} = L_{II}^3$ and durations T_I and T_{II}, and let us ask for the electromagnetic field which is produced at a point $P_2 = (x_2, y_2, z_2, t_2)$ of region II by a measurement of the E_x's average over the region I. Thus, we assume that in volume V_I there are originally two electric charge distributions with the constant densities $+\rho_I$ and $-\rho_I$, that of the test body, and that of its neutralizing body. In the interval from t_I' to $t_I' + \Delta t_I'$ the first charge distribution experiences a simple non-uniform translation in the *x*-direction through a distance $D_x^{(I)}$ with uncertainty $\Delta x_I'$; in the interval from $t_I' + \Delta t_I'$ to t_I'' it remains at rest at the displaced position; finally, in the interval from t_I'' to $t_I'' + \Delta t_I''$ ($\Delta t_I' \approx \Delta t_I''$) it moves non-uniformly parallel to the *x*-axis back to its original position, which coincides with that of the neutralization distribution. We assume according to Section 3.1 that $\Delta t_I' << T_I = t_I'' - t_I'$ and $\Delta x_I' << D_x^{(I)} < L_I$.

The neutralizing body is fixed to the reference system by means of screws, and remains there during the complete measurement process.

The test body itself, before the first measurement of the linear momentum is taken, is fixed to the reference system by a mechanical arm set in such a way that it fits between a bolt secured with screws to the reference system, and another bolt fixed to the test body. All of these devices are neutral of charge and current, including the device which measures the linear momentum of the test body, and make no contribution to any electromagnetic field which could interfere with the field to be measured. The mechanical arm is controlled by a clock whose design allows it to secure or release both bolts, when required (see Figures 1 and 2).

In the interval between $t_I^{'}$ to $t_I^{'} + \Delta t_I^{'}$ the test body is detached from the reference frame, experiencing a simple non-uniform displacement caused by the first linear momentum measurement in the *x*-direction. The graduated ruler, fixed by screws to the reference system could serve respectively to compare the lengths of the experimental set-up and the spatial region of measurement. In the time from $t_I^{'} + \Delta t_I^{'}$ to $t_I^{''}$, the test body remains at rest in the displaced position.

The test body is chosen as a heavy test body which has to satisfy the uncertainty relation [24] $D_x^{(I)}\ \Delta p_x^{(I)}{}' \approx \hbar$, where $\Delta p_x^{(I)}{}'$ is the uncertainty in the first linear momentum measurement $p_x^{(I)}{}'$.

Furthermore, if we measure the momentum at the time $t_I^{'}$ within a short interval $\Delta t_I^{'}$ ($\Delta t_I^{'} << T_I$) a certain velocity $v_x^{(I)}{}'$ will be transferred to the test body according to the well-known relation (H′=energy) $\Delta H' \Delta t_I^{'} = v_x^{(I)}{}' \Delta p_x^{(I)}{}' \Delta t_I^{'} \approx \hbar$. Just then $v_x^{(I)}{}'$ is the velocity which is necessary to displace the test body by $D_x^{(I)}$ within the time $\Delta t_I^{'}$, that is, $v_x^{(I)}{}' \approx \dfrac{\hbar}{\Delta p_x^{(I)}{}' \Delta t_I^{'}} \approx \dfrac{D_x^{(I)}}{\Delta t_I^{'}}$. In contrast to the uncertainties $D_x^{(I)}$ and $\Delta p_x^{(I)}{}'$ the velocity $v_x^{(I)}{}'$ is a known quantity since the intervals $D_x^{(I)}$ and $\Delta t_I^{'}$ can be chosen (disregarding an uncertainty $\Delta\ v_x^{(I)}{}'$ which is small if the mass of the test body is large). In a similar way we measure the momentum at $t_I^{''}$ within a short interval $\Delta t_I^{''}$ and then bring the test body back to its original position so that it again covers the region V_I exactly.

For the body to stop a counterstroke is given during time $\Delta t_I^{'}$ by means of a lever triggered by a spring (whose elastic constant it is not necessary to specify here); it is also connected to the clock, which records measurement times. The stroke is effected on the test body on a rigid bar embedded in it, that is, on the inverted U–shaped protrusion on top of the test body.

The figures in this article were designed bordering on the grotesque and in a similar way to those constructed by Bohr [11] during his famous debate with Einstein, such that their role is merely to be a pseudo-realistic illustration of the measurement situations that these represent.

Certainly Bohr and Rosenfeld's argument require that $D_x^{(I)}$ be much smaller than L_I, which is not the case in the diagrams, but it is necessary to exaggerate the dimensions involved to make the diagrams clear.

Also, the placement of the V_I, for example, in the diagrams makes it seem that the entire volume of the laboratory is the volume under consideration, but this volume should be filled by the test body and its neutralizing body.

It might also be a good idea to include a depiction of the momentum-measuring device in the diagrams, but they do not have enough space for including it.

In the time interval from $t_I^{''}$ to $t_I^{''} + \Delta t_I^{''}$ the second linear momentum measurement $p_x^{(I)''}$ is carried out with uncertainty $\Delta p_x^{(I)''}$. The test body experiences another simple non-uniform displacement $D_x^{(I)}$ (unknown but constant) with uncertainty $\Delta x_I^{''} \approx \Delta x_I^{'}$ so, a certain velocity $v_x^{(I)''} \approx -\dfrac{D_x^{(I)}}{\Delta t_I^{''}}$ will be transferred to it. A second counterstroke is given during time $\Delta t_I^{''}$ to the test body for returning it to its original position.

In Figures 1 and 2 a hypothetical division has been made, and not a gap, between regions I and II with some pointed tiles situated down the apparatus, making a floor to lean it on. Regions I and II show a possible experimental set-up's configuration of the whole measurement process.

One further issue lies at a deeper foundational level. The question of precisely what is represented by $D_x^{(I)}$ is conceptually problematic and should at least be handled with some caution. Presumably this cannot represent an actual displacement of the test body, for this would imply that the body has a precise (though unknown) actual position as well a precise (known) momentum. This, of course, would commit us to a hidden variables interpretation of quantum mechanics and the perhaps unpalatable non-locality that such interpretation is committed to. Thus $D_x^{(I)}$ presumably is a measure of the potential displacement of the test body or the degree to which our classical concept of position fails to apply to the test body.

At any attempt to construct idealized diagrams of the arrangements of the sort that will be here, the positive and negative charge distributions could be completely overlap and/or be simply adjacent to one another according as what part of the arrangements we are considering, those in the two space-time regions V_I and V_{II} as a whole or in each one of regions (see Figures 4 and 5).

Finally, the sources of the reaction field are the following: (a) as we have seen above, during the whole time T_I the test body is displaced by an unknown quantity $D_x^{(I)}$. This gives rise to the field of an electric dipole with its moment $\rho_I\, V_I\, D_x^{(I)}$ in the *x*-direction. This dipole moment is distributed uniformly over V_I with a density $P_x^{(I)} = \rho_I\, D_x^{(I)}$. Using Dirac δ-function properties that is

$$P_x^{(I)} = \rho_I D_x^{(I)} \int_{t_I'}^{t_I''} \delta\, (t - t_1) dt_1 \,. \qquad (4.1)$$

The scalar potential at a space-time point P_2 is

$$\phi_x^{(I)}(P_2) = \rho_I D_x^{(I)} \int_{V_I} dv_1 \int_{T_I} dt_1 \frac{\partial}{\partial x_1}\left\{\frac{1}{r}\delta\, (t_2 - t_1 - \frac{r}{c})\right\}, \qquad (4.2)$$

with $r = |\vec{r}_2 - \vec{r}_1|$, $\vec{r}_2 = (x_2, y_2, z_2)$ and $\vec{r}_1 = (x_1, y_1, z_1)$.

(b) At the time t_I' of interval T_I the test body has a velocity $v_x^{(I)}{}'$ for a short time $\Delta t_I'$ ($v_x^{(I)}{}'$ is afterwards compensated). At the time t_I'' (the end of the interval T_I) the test body again has a velocity $v_x^{(I)}{}''$ for a short time $\Delta t_I''$ ($v_x^{(I)}{}''$ is afterwards compensated).

Assuming that $\Delta t_I'$ and $\Delta t_I''$ are infinitely small we can transform the current density $J_x^{(I)} = \rho_I D_x^{(I)} \left[\frac{1}{\Delta t_I'} - \frac{1}{\Delta t_I''}\right]$ into $J_x^{(I)} = \rho_I D_x^{(I)} \left[\delta(t - t_I') - \delta(t - t_I'')\right]$, that is

$$J_x^{(I)} = -\rho_I D_x^{(I)} \int_{t_I'}^{t_I''} \frac{\partial}{\partial t_1} \delta\, (t - t_1) dt_1 \,. \qquad (4.3)$$

This current density gives rise to a vector potential $\psi_x^{(I)}$ at a space-time point P_2

$$\psi_x^{(I)}(P_2) = -\rho_I D_x^{(I)} \int_{V_I} dv_1 \int_{T_I} dt_1 \frac{\partial}{c\partial t_1}\left\{\frac{1}{r}\delta\ (t_2 - t_1 - \frac{r}{c})\right\}. \qquad (4.4)$$

Based on the relations $\vec{E}^{(I)} = -\vec{\nabla}_2 \phi^{(I)} - \frac{\partial \vec{\psi}^{(I)}}{c\partial t_2}$ and $\vec{H}^{(I)} = \vec{\nabla}_2 x \vec{\psi}^{(I)}$ the following components are obtained

$$E_x^{(I)}(P_2) = \rho_I D_x^{(I)} \int_{T_I} dt_1 \int_{V_I} dv_1 A_{xx}^{(1,2)}, \qquad (4.5)$$

$$E_y^{(I)}(P_2) = \rho_I D_x^{(I)} \int_{T_I} dt_1 \int_{V_I} dv_1 A_{xy}^{(1,2)}, \qquad (4.6)$$

$$E_z^{(I)}(P_2) = \rho_I D_x^{(I)} \int_{T_I} dt_1 \int_{V_I} dv_1 A_{xz}^{(1,2)}, \qquad (4.7)$$

$$H_x^{(I)}(P_2) = 0, \qquad (4.8)$$

$$H_y^{(I)}(P_2) = \rho_I D_x^{(I)} \int_{T_I} dt_1 \int_{V_I} dv_1 B_{xy}^{(1,2)}, \qquad (4.9)$$

$$H_z^{(I)}(P_2) = \rho_I D_x^{(I)} \int_{T_I} dt_1 \int_{V_I} dv_1 B_{xz}^{(1,2)}. \qquad (4.10)$$

The averages of these field components over the region II are obtained by simple space-time integration

$$\overline{E}_x^{(I,II)} = D_x^{(I)} \rho_I V_I T_I \overline{A}_{xx}^{(I,II)}, \qquad (4.11)$$

$$\overline{E}_y^{(I,II)} = D_x^{(I)} \rho_I V_I T_I \overline{A}_{xy}^{(I,II)}, \qquad (4.12)$$

$$\overline{E}_z^{(I,II)} = D_x^{(I)} \rho_I V_I T_I \overline{A}_{xz}^{(I,II)}, \qquad (4.13)$$

$$\overline{H}_x^{(I,II)} = 0, \qquad (4.14)$$

$$\overline{H}_y^{(I,II)} = D_x^{(I)} \rho_I V_I T_I \overline{B}_{xy}^{(I,II)}, \qquad (4.15)$$

$$\overline{H}_z^{(I,II)} = -D_x^{(I)} \rho_I V_I T_I \overline{B}_{xz}^{(I,II)}. \qquad (4.16)$$

4.2 CASE NOT STUDIED BY BOHR AND ROSENFELD

Now [9], let us consider two space-time regions with volumes $V_I = L_I^3$ and $V_{II} = L_{II}^3$ and durations T_I and T_{II}, and let us ask for the electromagnetic field which is produced at a point $P_2 = (x_2, y_2, z_2, t_2)$ of region II by a measurement of the H_x's average over the region I. Thus, we assume that in volume V_I there are originally two distributions of current $+j_z^{(I)}$ and $-j_z^{(I)}$, that of the test body, and that of its neutralizing body. In the interval from $t_I^{'}$ to $t_I^{'} + \Delta t_I^{'}$ the first current distribution experiences a simple non-uniform translation in the *y*-direction through a distance $D_y^{(I)}$ with uncertainty $\Delta y_I^{'}$; in the interval from $t_I^{'} + \Delta t_I^{'}$ to $t_I^{''}$ it remains at rest at the displaced position; finally, in the interval from $t_I^{''}$ to $t_I^{''} + \Delta t_I^{''}$ ($\Delta t_I^{'} \approx \Delta t_I^{''}$) it moves non-uniformly parallel to the *y*-axis back to its original position, which coincides with that of the neutralization distribution of current. We assume according to Section 3.1 that $\Delta t_I^{'} << T_I = t_I^{''} - t_I^{'}$ and $\Delta y_I^{'} << D_y^{(I)} < L_I$.

The neutralizing current is fixed to the reference system by means of screws, and remains there during the complete measurement process.

The test body itself, before the first measurement of the linear momentum is taken, is fixed to the reference system by a mechanical arm set in such a way that it fits between a bolt secured with screws to the reference system, and another bolt fixed to the test body. All of these devices are neutral of charge and current, including the device which measures the linear momentum of the test body, and make no contribution to any electromagnetic field which could interfere with the field to be measured. The mechanical arm is controlled by a clock whose design allows it to secure or release both bolts, when required.

In the interval between $t_I^{'}$ to $t_I^{'} + \Delta t_I^{'}$ the test body is detached from the reference frame, experiencing a simple non-uniform displacement caused by the first linear momentum measurement in the *y*-direction. In the time from $t_I^{'} + \Delta t_I^{'}$ to $t_I^{''}$, the test body remains at rest in the displaced position.

The test body is chosen as a heavy test body which has to satisfy the uncertainty relation $D_y^{(I)} \Delta p_y^{(I)}{'} \approx \hbar$, where $\Delta p_y^{(I)}{'}$ is the uncertainty in the first linear momentum measurement $p_y^{(I)}{'}$.

Furthermore, if we measure the momentum at the time $t_I^{'}$ within a short interval $\Delta t_I^{'}$ ($\Delta t_I^{'} << T_I$) a certain velocity $v_y^{(I)}{'}$ will be transferred to the test body according to the well-known relation (H′=energy) $\Delta H'\Delta t_I^{'} = v_y^{(I)}{'}\Delta p_y^{(I)}{'}\Delta t_I^{'} \approx \hbar$. Just then $v_y^{(I)}{'}$ is the velocity which is necessary to displace the test body by $D_y^{(I)}$ within the time $\Delta t_I^{'}$, that is, $v_y^{(I)}{'} \approx \dfrac{\hbar}{\Delta p_y^{(I)}{'}\Delta t_I^{'}} \approx \dfrac{D_y^{(I)}}{\Delta t_I^{'}}$. In contrast to the uncertainties $D_y^{(I)}$ and $\Delta p_y^{(I)}{'}$ the velocity $v_y^{(I)}{'}$ is a known quantity since the intervals $D_y^{(I)}$ and $\Delta t_I^{'}$ can be chosen (disregarding an uncertainty $\Delta\, v_y^{(I)}{'}$ which is small if the mass of the test body is large). In a similar way we measure the momentum at $t_I^{''}$ within a short interval $\Delta t_I^{''}$ and then bring the test body back to its original position so that it again covers the region V_I exactly.

For the body to stop a counterstroke is given during time $\Delta t_I^{'}$ by means of a lever triggered by a spring (whose elastic constant it is not necessary to specify here); it is also connected to the clock, which records measurement times. The stroke is effected on the test body on a rigid bar embedded in it, that is, on the inverted U–shaped protrusion on top of the test body.

In the time interval from $t_I^{''}$ to $t_I^{''} + \Delta t_I^{''}$ the second linear momentum measurement $p_y^{(I)}{''}$ is carried out with uncertainty $\Delta p_y^{(I)}{''}$. The test body experiences another simple non-uniform displacement $D_y^{(I)}$ (unknown but constant) with uncertainty $\Delta y_I^{''} \approx \Delta y_I^{'}$ so, a certain velocity $v_y^{(I)}{''} \approx -\dfrac{D_y^{(I)}}{\Delta t_I^{''}}$ will be transferred to it. A second counterstroke is given during time $\Delta t_I^{''}$ to the test body for returning it to its original position.

Finally, the source of the reaction field is a magnetization $\vec{M}$ which causes a magnetic moment $\vec{m} = \dfrac{1}{2c}\int_V \left[\vec{r}x\vec{J}(\vec{r})\right]dv$, from which $\vec{M} = \dfrac{d\vec{m}}{dv} = \dfrac{1}{2c}\vec{r}x\vec{J}(\vec{r})$. Taking in the case under examination the following approximation valid only in the measurement region, that $\vec{r} = D_y^{(I)}\hat{j}$ and $\vec{J}(\vec{r}) = j_z^{(I)}\hat{k}$ we have

$$\vec{m} = \frac{1}{2c} D_y^{(I)} j_z^{(I)} V_I \hat{i} \;, \qquad (4.17)$$

from which

$$\vec{M} = \frac{1}{2c} D_y^{(I)} j_z^{(I)} \hat{i} \;. \qquad (4.18)$$

The expression (4.18) can also be written as

$$\vec{M} = \frac{1}{2c} D_y^{(I)} j_z^{(I)} \hat{i} \int_{t_I^{'}}^{t_I^{''}} \delta\ (t - t_1) dt_1 \;. \qquad (4.19)$$

The vectorial potential obtained is

$$\vec{\Psi}^{(I)}(P_2) = \int_{V_I} c \vec{\nabla}_1 x \left(\frac{\vec{M}}{r} \right) dv_1 \,, \qquad (4.20)$$

with $r = |\vec{r}_2 - \vec{r}_1|$, $\vec{r}_2 = (x_2, y_2, z_2)$, $\vec{r}_1 = (x_1, y_1, z_1)$ and $\vec{M} = (M_x^{(I)}, 0, 0)$.

The measurement situation under consideration does not produce scalar potential $\phi^{(I)}(P_2)$, thus possessing the following vectorial potential components

$$\Psi_x^{(I)}(P_2) = 0 \,, \qquad (4.21)$$

$$\Psi_y^{(I)}(P_2) = \int_{V_I} c \frac{\partial}{\partial z_1} \left(\frac{M_x^{(I)}}{r} \right) dv_1 \,, \qquad (4.22)$$

$$\Psi_z^{(I)}(P_2) = -\int_{V_I} c \frac{\partial}{\partial y_1} \left(\frac{M_x^{(I)}}{r} \right) dv_1 \,. \qquad (4.23)$$

Based on the relations $\vec{E}^{(I)} = -\frac{\partial \vec{\Psi}^{(I)}}{c \partial t_2}$ and $\vec{H}^{(I)} = \vec{\nabla}_2 x \vec{\Psi}^{(I)}$ the following components are obtained

$$E_x^{(I)}(P_2) = 0 \,, \qquad (4.24)$$

$$E_y^{(I)}(P_2) = \frac{1}{2} D_y^{(I)} j_z^{(I)} \int_{T_I} dt_1 \int_{V_I} dv_1 B_{xy}^{(2,1)} \,, \qquad (4.25)$$

$$E_z^{(I)}(P_2) = -\frac{1}{2} D_y^{(I)} j_z^{(I)} \int_{T_I} dt_1 \int_{V_I} dv_1 B_{xz}^{(2,1)} \,, \qquad (4.26)$$

$$H_x^{(I)}(P_2) = -\frac{1}{2} D_y^{(I)} j_z^{(I)} \int_{T_I} dt_1 \int_{V_I} dv_1 G_{xx}^{(2,1)} \,, \qquad (4.27)$$

$$H_y^{(I)}(P_2) = -\frac{1}{2} D_y^{(I)} j_z^{(I)} \int_{T_I} dt_1 \int_{V_I} dv_1 A_{xy}^{(2,1)} \,, \tag{4.28}$$

$$H_z^{(I)}(P_2) = -\frac{1}{2} D_y^{(I)} j_z^{(I)} \int_{T_I} dt_1 \int_{V_I} dv_1 A_{xz}^{(2,1)} \,, \tag{4.29}$$

where $G_{xx}^{(2,1)} = (\frac{\partial^2}{\partial y_2 \partial y_1} + \frac{\partial^2}{\partial z_2 \partial z_1})\left\{\frac{1}{r}\delta\left(t_2 - t_1 - \frac{r}{c}\right)\right\}$ is a new function which does not appear in Bohr and Rosenfeld's analysis.

Writing $-\frac{D_y^{(I)}}{2} = D_y^{(I)}$ as a new variable and averaging the above components over the space-time region II, it is possible to obtain

$$\overline{E}_x^{(I,II)} = 0 \,, \tag{4.30}$$

$$\overline{E}_y^{(I,II)} = -D_y^{(I)} j_z^{(I)} V_I T_I \overline{B}_{xy}^{(II,I)} \,, \tag{4.31}$$

$$\overline{E}_z^{(I,II)} = D_y^{(I)} j_z^{(I)} V_I T_I \overline{B}_{xz}^{(II,I)} \,, \tag{4.32}$$

$$\overline{H}_x^{(I,II)} = D_y^{(I)} j_z^{(I)} V_I T_I \overline{G}_{xx}^{(II,I)} \,, \tag{4.33}$$

$$\overline{H}_y^{(I,II)} = D_y^{(I)} j_z^{(I)} V_I T_I \overline{A}_{xy}^{(II,I)} \,, \tag{4.34}$$

$$\overline{H}_z^{(I,II)} = D_y^{(I)} j_z^{(I)} V_I T_I \overline{A}_{xz}^{(II,I)} \,. \tag{4.35}$$

5. MEASURABILITY OF ELECTROMAGNETIC FIELD QUANTITIES

5.1 CASES CONSIDERED BY BOHR AND ROSENFELD

5.1.1 MEASUREMENT OF $\overline{E}_x^{(I)}$

Considering the measurement of the average value of E_x in a space-time region I and following to Bohr and Rosenfeld [(1)], the linear momentum balance will therefore be

$$p_x^{''(I)} - p_x^{'(I)} = \rho_I V_I T_I (\overline{E}_x^{(I)} + \overline{E}_x^{(I,I)}) \,, \tag{5.1}$$

$\overline{E}_x^{(I)}$ being the average of E_x in the space-time region I if no measurement were made on the test body at the instant $t_I^{'}$ where the initial linear momentum $p_x^{'(I)}$ is measured. At $t_I^{''}$ we similarly measure the final linear momentum $p_x^{''(I)}$, and $\overline{E}_x^{(I,I)}$ is the field average which arises from these measurements as described above in Section 4.1 according to formula (4.11) when the space-time region I is equal to the space-time region II.

The average $\overline{E}_x^{(I)}$ can be determined with arbitrary precision by choosing the value of ρ_I sufficiently large. However, this makes $\overline{E}_x^{(I,I)}$ very large also, and then the achievable precision in the measurement of $\overline{E}_x^{(I)}$ has the value

$$\Delta\overline{E}_x^{(I)} \approx \frac{\Delta p_x^{(I)}}{\rho_I V_I T_I} + \Delta\overline{E}_x^{(I,I)}, \qquad (5.2)$$

which, assuming that the displacement of test body $D_x^{(I)}$ is of order Δx_I, has the following value, limited by Heisenberg's uncertainty principle

$$\Delta\overline{E}_x^{(I)} \approx \frac{\hbar}{\rho_I \Delta x_I V_I T_I} + \rho_I \Delta x_I V_I T_I \left|\overline{A}_{xx}^{(I,I)}\right|. \qquad (5.3)$$

Finding a minimum value of (5.3) with a variation of ρ_I, we obtain the following critical value

$$\rho_I = \frac{1}{\Delta x_I V_I T_I}\sqrt{\frac{\hbar}{\left|\overline{A}_{xx}^{(I,I)}\right|}}, \qquad (5.4)$$

which, on replacing it in (5.3) allows us to find the minimum uncertainty

$$\Delta\overline{E}_{x\ \min}^{(I)} \approx \sqrt{\hbar\left|\overline{A}_{xx}^{(I,I)}\right|}. \qquad (5.5)$$

If (5.5) were an inevitable limit in the precision of the measurement of $\overline{E}_x^{(I)}$, we would reach the conclusion that it is not possible to compensate for the effect of the field of the test body and the measurement would only make sense when $\hbar$ was neglected, that is to say, at the classical limit [12], [13], [14], [15].

Nevertheless, the coefficient of the displacement $D_x^{(I)}$ in $\overline{E}_x^{(I,I)}$ only depends on geometrical relations, enabling it to organize things, so that the effects of $\overline{E}_x^{(I,I)}$ would be completely compensated. The measuring device is modified, fixing the test body, previously completely free during time T_I, to a rigid system by means of a nut, and with a spring whose tension will be proportional to the displacement $D_x^{(I)}$. So, we would have

$$p_x^{''\,(I)} - p_x^{'\,(I)} = \rho_I V_I T_I (\overline{E}_x^{(I)} + \overline{E}_x^{(I,I)}) - k_I D_x^{(I)} T_I \,, \tag{5.6}$$

which suggests that

$$k_I = \rho_I^2 V_I^2 T_I \left| \overline{A}_{xx}^{(I,I)} \right| , \tag{5.7}$$

is the constant of the spring.

The spring will have no problems with its application because, by having the test body a large mass m, the test body's oscillation period $T \approx 2\pi \sqrt{\dfrac{m}{k_I}}$ is much greater than T_I and its displacement in this period of time will be small compared with $D_x^{(I)}$ and with $D_x^{(I)}$'s own powers from order two onwards.

Observe that the mass and charge of the test body are parameters independent of each other in their adjustment.

Finally, we can write that

$$\Delta \overline{E}_x^{(I)} \approx \frac{\hbar}{\rho_I \Delta x_I V_I T_I} \,, \tag{5.8}$$

which may be substantially reduced, by choosing ρ_I sufficiently large, in complete agreement with the prediction of the formalism [10] (see Figures 1, 2 and 3).

Apart from detailed description of Figures 1, 2 and 3 given above, we can say that there is not a special reason why the clocks on our devices not are synchronized.

The absolute value of $\overline{A}_{xx}^{(I,I)}$ has been written in (5.7) because according to Hnizdo [4] "the geometric factor $\overline{A}_{xx}^{(I,I)}$ with fully coinciding space-time regions can turns out to have negative values. He noted that Bohr and Rosenfeld did not consider it necessary to make a comment on this rather unusual specification that their measurement procedure would place on the spring mechanism. In any case, despite its inherent instability, a spring mechanism with negative spring constant should present no difficulty of principle for a Bohr and Rosenfeld measurement procedure because a Bohr and Rosenfeld spring, together with the test body to which it is attached; is supposed to be release only for the exact duration of the field measurement, and the spring force is designed so that its effect is compensated by the test body's self-force.

A hint that Bohr and Rosenfeld were aware of the possibility of the geometric factor $\overline{A}_{xx}^{(I,I)}$ being negative is given by their careful writing of its square root in equations (5.4) and (5.5)".

5.1.2 MEASUREMENT OF $\overline{E}_x^{(I)}$ AND $\overline{E}_x^{(II)}$

In this case for the momentum balance of the two test bodies we have [1]

$$p_x^{''(I)} - p_x^{'(I)} = \rho_I V_I T_I (\overline{E}_x^{(I)} + \overline{E}_x^{(I,I)} + \overline{E}_x^{(II,I)}), \quad (5.9)$$

$$p_x^{''(II)} - p_x^{'(II)} = \rho_{II} V_{II} T_{II} (\overline{E}_x^{(II)} + \overline{E}_x^{(II,II)} + \overline{E}_x^{(I,II)}), \quad (5.10)$$

where $\overline{E}_x^{(I,II)}$ is defined by expression (4.11), and $\overline{E}_x^{(II,I)}$ is obtained from this equation by simple interchange of the indices I and II. It would seem from the previous section that the appearance of fields $\overline{E}_x^{(I,I)}$ and $\overline{E}_x^{(II,II)}$ implies in itself limited precision in the measurements of $\overline{E}_x^{(I)}$ and $\overline{E}_x^{(II)}$. The reactions $\rho_I V_I T_I \overline{E}_x^{(I,I)}$ and $\rho_{II} V_{II} T_{II} \overline{E}_x^{(II,II)}$ can be cancelled by means of springs with elastic constants k_I given by (5.7) and k_{II} given by

$$k_{II} = \rho_{II}^2 V_{II}^2 T_{II} \left| \overline{A}_{xx}^{(II,II)} \right|. \quad (5.11)$$

Therefore we obtain

$$p_x^{''(I)} - p_x^{'(I)} = \rho_I V_I T_I (\overline{E}_x^{(I)} + \overline{E}_x^{(II,I)}), \quad (5.12)$$

$$p_x^{''(II)} - p_x^{'(II)} = \rho_{II} V_{II} T_{II} (\overline{E}_x^{(II)} + \overline{E}_x^{(I,II)}), \quad (5.13)$$

and bearing in mind that $D_x^{(I)}$ and $D_x^{(II)}$ are respectively known in values Δx_I and Δx_{II}, these uncertainties arise

$$\Delta\overline{E}_x^{(I)} \approx \frac{\hbar}{\rho_I \Delta x_I V_I T_I} + \Delta\overline{E}_x^{(II,I)}, \tag{5.14}$$

$$\Delta\overline{E}_x^{(II)} \approx \frac{\hbar}{\rho_{II} \Delta x_{II} V_{II} T_{II}} + \Delta\overline{E}_x^{(I,II)}. \tag{5.15}$$

By replacing in (5.14) and (5.15) the expressions corresponding to (4.11) and correctly choosing $\rho_I \Delta x_I$ and $\rho_{II} \Delta x_{II}$ either one of the quantities $\Delta\overline{E}_x^{(I)}$ or $\Delta\overline{E}_x^{(II)}$ can obviously be arbitrarily diminished, but only at the expense of an increase of the other.

By multiplying (5.14) by (5.15) and finding its minimum by a variation of $\rho = \rho_I \rho_{II}$, we obtain the following critical value

$$\rho = \rho_I \rho_{II} = \frac{\hbar\sqrt{\left|\overline{A}_{xx}^{(I,II)}\right|\left|\overline{A}_{xx}^{(II,I)}\right|}}{\Delta x_I \Delta x_{II} V_I V_{II} T_I T_{II} \left|\overline{A}_{xx}^{(I,II)}\right|\left|\overline{A}_{xx}^{(II,I)}\right|}. \tag{5.16}$$

Replacing (5.16) in the above product, we get the minimum value

$$\Delta\overline{E}_x^{(I)} \Delta\overline{E}_x^{(II)} \approx \hbar\left(\left|\overline{A}_{xx}^{(I,II)}\right| + \left|\overline{A}_{xx}^{(II,I)}\right|\right) \geq \hbar\left|\overline{A}_{xx}^{(I,II)} + \overline{A}_{xx}^{(II,I)}\right|. \tag{5.17}$$

In spite of the great similarity of relation (5.17) to the uncertainty relation (2.1) required by the formalism, there is, nevertheless, a fundamental difference in that the latter contains not the sum of the magnitudes of the quantities $\overline{A}_{xx}^{(I,II)}$ and $\overline{A}_{xx}^{(II,I)}$ but their algebraic difference.

The equation (5.17) is in full contradiction with the previous study of measurement of a simple field component. The minus sign (-) instead of plus sign (+) in the formula (2.1) for example, when regions I and II overlap completely, causes the cancellation of the product of the uncertainties, a situation which does not occur, according to (5.17).

In general, the two expressions (5.17) and (2.1) agree exactly only when at least one of the quantities $\overline{A}_{xx}^{(I,II)}$ or $\overline{A}_{xx}^{(II,I)}$ vanishes which in general requires that one of the expressions $t_1 - t_2 - \frac{r}{c}$ or $t_2 - t_1 - \frac{r}{c}$, appearing as argument of the Dirac δ-function in the integrals (2.8), remain different from zero for every pair of points (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) of regions I and II respectively.

It is true that there appears here, in comparison with the compensation procedure needed already for measuring a single field quantity, the further complication that the displacements of the two test bodies not only must remain unknown but are also completely independent of each other. A somewhat more complicated procedure is necessary in order to compensate as much as possible the influence of the relative displacement of the test bodies on the field measurements. We select two bodies ε_I and ε_{II}, one from each test body system I and II, for which the expression $r - c(t_1 - t_2)$ vanishes for two times t_I^* and t_{II}^* lying in the time intervals T_I and T_{II}, respectively; that is, the two bodies are light-like related. To establish the necessary correlation between the test bodies one might at first think of a spring which should connect the bodies ε_I and ε_{II} directly with each other; however, due to the retardation of the forces one would thereby run into difficulties. But we can manage with a short spring much smaller than cT_I, if we add to the second test body system a neutral component body ε_{III} which is situated in the immediate vicinity of ε_I, and connected with it by a spring whose elastic constant is *k*.

The body ε_{III} is initially to be bound to the rigid frame by means of a device previously described and after removing this device, at time $t_I^{'}$, its momentum is to be measured with the same accuracy as that of the test body system II. It thereby undergoes an unknown displacement $D_x^{(III)}$ in the *x*-direction which is of the same order of magnitude as Δx_{II} (see Figure 4).

The force exerted by the spring on the first test body system is therefore $F = -k\left(D_x^{(III)} - D_x^{(I)}\right)$ and so the linear momentum *P* transferred from ε_{III} to ε_I during T_I is $P = FT_I$, from which

$$P = \frac{1}{2}\rho_I \rho_{II} V_I V_{II} T_I T_{II}\left(\overline{A}_{xx}^{(I,II)} + \overline{A}_{xx}^{(II,I)}\right)\left(D_x^{(I)} - D_x^{(III)}\right), \tag{5.18}$$

where

$$k = \frac{1}{2}\rho_I \rho_{II} V_I V_{II} T_{II}\left(\overline{A}_{xx}^{(I,II)} + \overline{A}_{xx}^{(II,I)}\right). \tag{5.19}$$

Similarly, ε_{III} undergoes a change of linear momentum $-P$ during the same period of time. At time $t_I^{"}$ the linear momentum of ε_{III} is measured again with the same accuracy. However, before this measurement, and in fact at time t_{II}^{*} a short light signal is to be sent from ε_{II} to ε_{III}, by which the relative displacement $\left(D_x^{(III)} - D_x^{(II)}\right)$ of these bodies can be measured with arbitrary accuracy by means of a suitable device described in detail by Bohr and Rosenfeld (1). At the emission and absorption of the signal the two test bodies undergo momentum changes which indeed remain completely unknown, but cancel each other exactly in the sum of the momentum changes measured on the bodies. The above implies that $ct_{II}^{*} = D_x^{(III)} - D_x^{(II)}$ and for doing this, body ε_{III} carries a small mirror on itself and body ε_{II} a light-emitting device, which can rotate on a fixed support and therefore point in any direction in which ε_{III} is found with respect to ε_{II}. The whole purpose of the compensating spring is to compensate automatically the effects that are proportional to the displacements $D_x^{(I)}$ and $D_x^{(III)}$ without any need of measuring the latter.

In the Figure 4 the mirror on the neutral third body seems to be facing the wrong way. It appears to be impossible for it to be struck by a beam from the light-emitting device, and all the more, because it is apparently hidden behind a corner of the laboratory but, we can suppose that the above does not happen.

In addition to the above, it is necessary to clarify that we are not allowed to measure the displacements $D_x^{(III)}$ and $D_x^{(II)}$, but the relative distance $\left(D_x^{(III)} - D_x^{(II)}\right)$, and thus it might seems that having the light-emitting device know the position of ε_{III} might disturb the momentum measurement that is essential for the determination of the field values.

Thus the position of ε_{III} presumably is a potential position if it were to be measured, or the degree to which our classical concept of position fails to apply to the body.

Also, the spring connecting bodies ε_I and ε_{III} in the Figure 4 really is along the *x*-direction, even though in the Figure 4 it apparently is along the *y*-direction so that the test body ε_{III} after its release can move in the *x*-direction too. Alternatively, a rigid rod could be extended along the *y*-direction from one of the bodies, and the spring could be connected in the *x*-direction between the free end of the rod and the other body.

As you can see from the Figure 4, the clocks are not synchronized but, there is not a special reason for that. Also, the division in the laboratory floor is indicating a potentially large distance falls between bodies ε_I and ε_{III}, but these two bodies are supposed to be very near each other. The gap should instead presumably lie between body ε_{II} and body ε_{III} but, remember that bodies ε_{II} and ε_{III} belong to the space-time region II and body ε_I belongs to the space-time region I.

Instants $t_I^{'}$ and $t_I^{''}$ are measurement times for the linear momentum of body ε_{III} at the beginning and at the end respectively in region I and similarly for ε_{II} in times $t_{II}^{'}$ and $t_{II}^{''}$ in region II.

Note (see Figure 4) that the times of measurements of body ε_{III} are recorded with the same clock that controls the measurements of ε_I. Also, the lever which is connected to this clock to counterstroke ε_I and ε_{III} simultaneously, now has two springs which drive it, whose elastic constants is not necessary to specify here provided that the stroke is considerable on the two bodies. The lever with two spring connected to it can deliver the correct counterstrokes to bodies ε_I and ε_{III} on the inverted U-shaped protrusions on the top of the test bodies.

The relevant point is that after the momentum measurements the bodies are returned approximately to rest, that is, any momentum imparted to the test bodies by the fields to be measured is removed by the counterstrokes.

The measurement process in space-time region II is governed by the clock which is located inside it and which works in the manner previously described. There are also some rulers which are fixed by screws to the reference system, which could serve respectively to compare the lengths of the experimental set-ups and the spatial regions of measurement.

Thus, for the momentum balance of the two test body systems during the measurement we have, if we include body ε_{III} in system II is

$$p_x^{''(I)} - p_x^{'(I)} = \rho_I V_I T_I (\overline{E}_x^{(I)} + \overline{E}_x^{(II,I)}) + P\,, \qquad (5.20)$$

$$p_x^{''(II)} - p_x^{'(II)} + p_x^{''(III)} - p_x^{'(III)} = \rho_{II} V_{II} T_{II} (\overline{E}_x^{(II)} + \overline{E}_x^{(I,II)}) - P\,. \qquad (5.21)$$

Transforming the formula (5.20) the following is obtained

$$p_x^{''(I)} - p_x^{'(I)} = \rho_I V_I T_I \overline{E}_x^{(I)} + \frac{1}{2}\rho_I \rho_{II} V_I V_{II} T_I T_{II} \overline{A}_{xx}^{(II,I)} D_x^{(II)} + \frac{1}{2}\rho_I \rho_{II} V_I V_{II} T_I T_{II} \overline{A}_{xx}^{(II,I)} D_x^{(II)}$$

$$+\frac{1}{2}\rho_I \rho_{II} V_I V_{II} T_I T_{II} \overline{A}_{xx}^{(I,II)} D_x^{(I)} + \frac{1}{2}\rho_I \rho_{II} V_I V_{II} T_I T_{II} \overline{A}_{xx}^{(II,I)} D_x^{(I)}$$

$$-\frac{1}{2}\rho_I \rho_{II} V_I V_{II} T_I T_{II} \overline{A}_{xx}^{(I,II)} D_x^{(III)} - \frac{1}{2}\rho_I \rho_{II} V_I V_{II} T_I T_{II} \overline{A}_{xx}^{(II,I)} D_x^{(III)}$$

$+\frac{1}{2}\rho_I \rho_{II} V_I V_{II} T_I T_{II} \overline{A}_{xx}^{(I,II)} D_x^{(II)} - \frac{1}{2}\rho_I \rho_{II} V_I V_{II} T_I T_{II} \overline{A}_{xx}^{(I,II)} D_x^{(II)}$, where, reorganizing the elements, we obtain

$$p_x^{''(I)} - p_x^{'(I)} = \rho_I V_I T_I \overline{E}_x^{(I)} + \frac{1}{2}\rho_I \rho_{II} V_I V_{II} T_I T_{II} \{-D_x^{(II)} (\overline{A}_{xx}^{(I,II)} -$$

$$\overline{A}_{xx}^{(II,I)}) + (D_x^{(II)} - D_x^{(III)})(\overline{A}_{xx}^{(I,II)} + \overline{A}_{xx}^{(II,I)}) + D_x^{(I)} (\overline{A}_{xx}^{(I,II)} + \overline{A}_{xx}^{(II,I)})\}\,. \qquad (5.22)$$

Similarly, transforming the equation (5.21) the following is obtained

$$p_x^{''(II)} - p_x^{'(II)} + p_x^{''(III)} - p_x^{'(III)} = \rho_{II} V_{II} T_{II}\ \overline{E}_x^{(II)} + \frac{1}{2}\rho_I \rho_{II} V_I V_{II}\ T_I T_{II} \overline{A}_{xx}^{(I,II)} D_x^{(I)}$$

$$+\frac{1}{2}\rho_I \rho_{II} V_I V_{II}\ T_I T_{II} \overline{A}_{xx}^{(I,II)} D_x^{(I)}\ - \frac{1}{2}\rho_I \rho_{II} V_I V_{II}\ T_I T_{II} \overline{A}_{xx}^{(I,II)} D_x^{(I)}$$

$$+\frac{1}{2}\rho_I \rho_{II} V_I V_{II}\ T_I T_{II} \overline{A}_{xx}^{(I,II)} D_x^{(III)}\ - \frac{1}{2}\rho_I \rho_{II} V_I V_{II}\ T_I T_{II} \overline{A}_{xx}^{(II,I)} D_x^{(I)}$$

$$+\frac{1}{2}\rho_I \rho_{II} V_I V_{II}\ T_I T_{II} \overline{A}_{xx}^{(II,I)} D_x^{(III)}\ + \frac{1}{2}\rho_I \rho_{II} V_I V_{II}\ T_I T_{II} \overline{A}_{xx}^{(II,I)} D_x^{(II)}$$

$$-\frac{1}{2}\rho_I \rho_{II} V_I V_{II}\ T_I T_{II} \overline{A}_{xx}^{(II,I)} D_x^{(II)}\ - \frac{1}{2}\rho_I \rho_{II} V_I V_{II}\ T_I T_{II} \overline{A}_{xx}^{(I,II)} D_x^{(II)}$$

$$+\frac{1}{2}\rho_I\rho_{II}V_IV_{II}\ T_IT_{II}\overline{A}_{xx}^{(I,II)}D_x^{(II)}\ ,$$

where, reorganizing the elements, we obtain

$$p_x^{''(II)}-p_x^{'(II)}+p_x^{''(III)}-p_x^{'(III)}=\rho_{II}V_{II}T_{II}\ \overline{E}_x^{(II)}+\frac{1}{2}\rho_I\rho_{II}V_IV_{II}\ T_IT_{II}\ \{D_x^{(I)}(\overline{A}_{xx}^{(I,II)}-$$

$$\overline{A}_{xx}^{(II,I)})-(D_x^{(II)}-D_x^{(III)})(\overline{A}_{xx}^{(I,II)}+\overline{A}_{xx}^{(II,I)})+D_x^{(II)}(\overline{A}_{xx}^{(I,II)}+\overline{A}_{xx}^{(II,I)})\}. \qquad (5.23)$$

In equations (5.22) and (5.23) the quantities $D_x^{(I)}$ and $D_x^{(II)}$ multiply terms in the curly brackets which, exactly as the simple reactions of each test body on itself, be cancelled by means of a suitable spring connections with the rigid frame and the test bodies by screws and nuts respectively.

The elastic constant of the spring for body ε_I, would be

$$k_{I,II}=\rho_I^2V_I^2T_I\left|\overline{A}_{xx}^{(I,I)}\right|+\frac{1}{2}\rho_I\rho_{II}V_IV_{II}T_{II}(\overline{A}_{xx}^{(I,II)}+\overline{A}_{xx}^{(II,I)})\,, \qquad (5.24)$$

acting on the elimination of the effect of the test body field in region I, $\overline{E}_x^{(I,I)}$, and of one part or factor which, after the transformation of the formula (5.20) in (5.22), depends on the displacement $D_x^{(I)}$. In analog form for body ε_{II} it can be determined that

$$k_{II,I}=\rho_{II}^2V_{II}^2T_{II}\left|\overline{A}_{xx}^{(II,II)}\right|+\frac{1}{2}\rho_I\rho_{II}V_IV_{II}T_I(\overline{A}_{xx}^{(I,II)}+\overline{A}_{xx}^{(II,I)})\,, \qquad (5.25)$$

with a similar function to that of $k_{I,II}$.

Furthermore, as was described by Bohr and Rosenfeld [1], the terms proportional to the relative displacement $\left(D_x^{(III)}-D_x^{(II)}\right)$ are known with arbitrary accuracy and can therefore easily be taken into account in the field measurements.

Based on equations (5.22) and (5.23), and bearing in mind that $\Delta p_x^{(I)}\approx\frac{\hbar}{\Delta x_I}$ and $\Delta p_x^{(II)}\approx\Delta p_x^{(III)}\approx\frac{\hbar}{\Delta x_{II}}$, the following uncertainties can be obtained

$$\Delta\overline{E}_x^{(I)}\approx\frac{\hbar}{\rho_I\Delta x_IV_IT_I}+\frac{1}{2}\rho_{II}V_{II}\Delta x_{II}T_{II}\left|\overline{A}_{xx}^{(I,II)}-\overline{A}_{xx}^{(II,I)}\right|, \qquad (5.26)$$

$$\Delta\overline{E}_x^{(II)} \approx \frac{\hbar}{\rho_{II}\Delta x_{II}V_{II}T_{II}} + \frac{1}{2}\rho_I V_I \Delta x_I T_I \left|\overline{A}_{xx}^{(I,II)} - \overline{A}_{xx}^{(II,I)}\right|. \qquad (5.27)$$

Taking the product of the uncertainties (5.26) and (5.27) and finding their minimum value by variation of $\rho = \rho_I \rho_{II}$, we obtain the critical value

$$\rho = \rho_I \rho_{II} = \frac{2\hbar}{\Delta x_I \Delta x_{II} V_I V_{II} T_I T_{II}\left|\overline{A}_{xx}^{(I,II)} - \overline{A}_{xx}^{(II,I)}\right|}, \qquad (5.28)$$

which, when replacing it in the product above mentioned allows the following uncertainty to be found

$$\Delta\overline{E}_x^{(I)}\Delta\overline{E}_x^{(II)} \approx 2\hbar\left|\overline{A}_{xx}^{(I,II)} - \overline{A}_{xx}^{(II,I)}\right| \approx \hbar\left|\overline{A}_{xx}^{(I,II)} - \overline{A}_{xx}^{(II,I)}\right|, \qquad (5.29)$$

in agreement with the formalism.

5.1.3 MEASUREMENT OF $\overline{E}_x^{(I)}$ AND $\overline{E}_y^{(II)}$

Bohr and Rosenfeld said that they had measured $\overline{E}_x^{(I)}$ and $\overline{E}_y^{(II)}$ and, $\overline{E}_x^{(I)}$ and $\overline{H}_y^{(II)}$ but really, they did not consider the last case.

Following their study of the first case we have

$$p_x^{''(I)} - p_x^{'(I)} = \rho_I V_I T_I (\overline{E}_x^{(I)} + \overline{E}_x^{(I,I)} + \overline{R}_y^{(II,I)}), \qquad (5.30)$$

$$p_y^{''(II)} - p_y^{'(II)} = \rho_{II} V_{II} T_{II} (\overline{E}_y^{(II)} + \overline{R}_y^{(II,II)} + \overline{R}_y^{(I,II)}), \qquad (5.31)$$

where $\overline{R}_y^{(II,I)} = D_y^{(II)}\rho_{II}V_{II}T_{II}\overline{A}_{xy}^{(II,I)}$, $\overline{R}_y^{(II,II)} = D_y^{(II)}\rho_{II}V_{II}T_{II}\overline{A}_{xy}^{(II,II)}$ and $\overline{R}_y^{(I,II)} = D_x^{(I)}\rho_I V_I T_I \overline{A}_{xy}^{(I,II)}$ (see Figure 5).

After compensating the effects of $\overline{E}_x^{(I,I)}$ and $\overline{R}_y^{(II,II)}$ in equations (5.30) and (5.31) respectively, we obtain

$$p_x^{''(I)} - p_x^{'(I)} = \rho_I V_I T_I (\overline{E}_x^{(I)} + \overline{R}_y^{(II,I)}), \qquad (5.32)$$

$$p_y^{''(II)} - p_y^{'(II)} = \rho_{II} V_{II} T_{II} (\overline{E}_y^{(II)} + \overline{R}_y^{(I,II)}). \qquad (5.33)$$

These expressions give uncertainties

$$\Delta\overline{E}_x^{(I)} \approx \frac{\hbar}{\rho_I \Delta x_I V_I T_I} + \rho_{II}\Delta y_{II} V_{II} T_{II}\left|\overline{A}_{xy}^{(II,I)}\right|, \qquad (5.34)$$

$$\Delta\overline{E}_y^{(II)} \approx \frac{\hbar}{\rho_{II}\Delta y_{II} V_{II} T_{II}} + \rho_I \Delta x_I V_I T_I \left|\overline{A}_{xy}^{(I,II)}\right|. \tag{5.35}$$

With the product of (5.34) and (5.35) and finding its minimum value by variation of $\alpha = \rho_I \rho_{II}$ the following critical value is obtained

$$\alpha = \rho_I \rho_{II} = \frac{\hbar\sqrt{\left|\overline{A}_{xy}^{(I,II)}\right|\left|\overline{A}_{xy}^{(II,I)}\right|}}{\Delta x_I \Delta y_{II} V_I V_{II} T_I T_{II} \left|\overline{A}_{xy}^{(I,II)}\right|\left|\overline{A}_{xy}^{(II,I)}\right|}, \tag{5.36}$$

which, replacing it in the previous product gives

$$\Delta\overline{E}_x^{(I)}\Delta\overline{E}_y^{(II)} \approx \hbar\left(\left|\overline{A}_{xy}^{(I,II)}\right| + \left|\overline{A}_{xy}^{(II,I)}\right|\right) \geq \hbar\left|\overline{A}_{xy}^{(I,II)} + \overline{A}_{xy}^{(II,I)}\right|, \tag{5.37}$$

which, like (5.17), does not generally, but only in certain cases, represent an agreement between measurability and free quantum electromagnetic formalism.

We select two bodies ε_I and ε_{II}, one from each test body system I and II, for which the expression $r - c(t_1 - t_2)$ vanishes for two times t_I^* and t_{II}^* lying in the time intervals T_I and T_{II}, respectively; that is, the two bodies are light-like related. To establish the necessary correlation between the test bodies one might at first think of a spring which should connect the bodies ε_I and ε_{II} directly with each other; however, due to the retardation of the forces one would thereby run into difficulties. We add to the second test body system a neutral component body ε_{III} which is situated in the immediate vicinity of ε_I, whose momentum in the *y*-direction is measured at the times $t_I^{'}$ and $t_I^{''}$; the relative displacement $\left(D_y^{(III)} - D_y^{(II)}\right)$ of the bodies ε_{III} and ε_{II} are again determined by means of a light signal, as a result of which both bodies undergo equal and opposite momentum changes.

In Figure 5, bodies ε_I and ε_{III} are controlled by the clock which is located in space-time region I. This clock's counterstroke mechanism acts simultaneously on ε_I and ε_{III}, stroking with the two arms at the same time on the inverted U-shaped protrusions on the top of the test bodies, and synchronizing the stroke given to each of them by the clock's internal mechanism. Also, ε_I and ε_{III} are initially fixed to the reference system by mechanical arms which belong to the clock, and which fit just between the bolts of the test bodies, and the bolts which are screwed to the reference system. The neutralizing bodies of ε_I and ε_{II} stay fixed by screws to the reference frame throughout the whole measuring process. It is not necessary to specify the elastic constants of the springs which move the mechanical arm of the clock in region I, together with the clock's spring in region II. Body ε_{III} has a mirror which reflects light signals sent from ε_{II} using a device for this effect, which can rotate on an axis, in such a way that it can move in whatever direction that ε_{III} does, and in this way it can send the light signal to any position that ε_{III} has at a given moment. The three bodies mentioned have a rigid bar embedded in them with an inverted U-geometry, and on these the counterstroke is made, which counteracts the changes in velocity. The change in the spatial arrangement of the clocks in space-time regions I and II, with respect to what they have in Figure 4, is considerable.

To relate bodies ε_I and ε_{III} two arms are used which were originally parallel to directions *x* and *y*, which can rotate on a pivot. As you can see from Figure 5, the clocks are not synchronized but, there is not a special reason for that. Also, the division in the laboratory floor is indicating a potentially large distance falls between bodies ε_I and ε_{III}, but these two bodies are supposed to be very near each other. The gap should instead presumably lie between body ε_{III} and body ε_{II} but, remember that bodies ε_{II} and ε_{III} belong to the space-time region II and body ε_I belongs to the space-time region I.

Note (see Figure 5) that the times of measurements of body ε_{III} are recorded with the same clock that controls the measurements of ε_I. The measurement process in space-time region II is governed by the clock which is located inside it.

There are also some rulers which are fixed by screws to the reference frame, which could serve respectively to compare the lengths of the experimental set-ups and the spatial regions of measurement.

A spring parallel to the *y*-axis is mounted on the first arm and ε_{III}, and a spring parallel to the *x*-axis acts between the second arm and ε_I. The elastic constant of the two springs is chosen as

$$k = \frac{1}{2}\rho_I \rho_{II} V_I V_{II} T_{II} \left(\overline{A}_{xy}^{(I,II)} + \overline{A}_{xy}^{(II,I)}\right), \tag{5.38}$$

this taking place during T_I.

Force F transferred from ε_{III} to the arm parallel to the x-axis is $F = -k\left(D_y^{(III)} - D_x^{(I)}\right)$ and the linear momentum $P = FT_I$ is

$$P = \frac{1}{2}\rho_I \rho_{II} V_I V_{II} T_I T_{II} \left(\overline{A}_{xy}^{(I,II)} + \overline{A}_{xy}^{(II,I)}\right)\left(D_x^{(I)} - D_y^{(III)}\right). \tag{5.39}$$

The balancing equations of the linear momentum are therefore

$$p_x^{''(I)} - p_x^{'(I)} = \rho_I V_I T_I (\overline{E}_x^{(I)} + D_y^{(II)} \rho_{II} V_{II} T_{II} \overline{A}_{xy}^{(II,I)}) + P\,, \tag{5.40}$$

$$p_y^{''(II)} - p_y^{'(II)} + p_y^{''(III)} - p_y^{'(III)} = \rho_{II} V_{II} T_{II} (\overline{E}_y^{(II)} + D_x^{(I)} \rho_I V_I T_I \overline{A}_{xy}^{(I,II)}) - P\,. \tag{5.41}$$

Replacing the value of P given by (5.39) in the equations (5.40) and (5.41) and reorganizing the resulting equations, we have

$$p_x^{''(I)} - p_x^{'(I)} = \rho_I V_I T_I \overline{E}_x^{(I)} + \frac{1}{2}\rho_I \rho_{II} V_I V_{II} T_I T_{II} \{-D_y^{(II)} (\overline{A}_{xy}^{(I,II)} - \overline{A}_{xy}^{(II,I)}) + (D_y^{(II)} - D_y^{(III)})(\overline{A}_{xy}^{(I,II)} + \overline{A}_{xy}^{(II,I)}) + D_x^{(I)} (\overline{A}_{xy}^{(I,II)} + \overline{A}_{xy}^{(II,I)})\}, \tag{5.42}$$

$$p_y^{''(II)} - p_y^{'(II)} + p_y^{''(III)} - p_y^{'(III)} = \rho_{II} V_{II} T_{II}\ \overline{E}_y^{(II)} + \frac{1}{2}\rho_I \rho_{II} V_I V_{II}\ T_I T_{II}\ \{D_x^{(I)} (\overline{A}_{xy}^{(I,II)} - \overline{A}_{xy}^{(II,I)}) - (D_y^{(II)} - D_y^{(III)})(\overline{A}_{xy}^{(I,II)} + \overline{A}_{xy}^{(II,I)}) + D_y^{(II)} (\overline{A}_{xy}^{(I,II)} + \overline{A}_{xy}^{(II,I)})\}\,. \tag{5.43}$$

Based on equations (5.42) and (5.43) after the compensation of the above terms in the parenthesis with springs of constants [10] (see Figure 5)

$$k_I = \rho_I^2 V_I^2 T_I \left|\overline{A}_{xx}^{(I,I)}\right| + \frac{1}{2}\rho_I \rho_{II} V_I V_{II} T_{II} (\overline{A}_{xy}^{(I,II)} + \overline{A}_{xy}^{(II,I)})\,, \tag{5.44}$$

between ε_I and the reference system and

$$k_{II} = \rho_{II}^2 V_{II}^2 T_{II} \left|\overline{A}_{xy}^{(II,II)}\right| + \frac{1}{2}\rho_I \rho_{II} V_I V_{II} T_I (\overline{A}_{xy}^{(I,II)} + \overline{A}_{xy}^{(II,I)})\,, \tag{5.45}$$

between ε_{II} and the reference frame, we obtain

$$\Delta \overline{E}_x^{(I)} \approx \frac{\hbar}{\rho_I \Delta x_I V_I T_I} + \frac{1}{2}\rho_{II} V_{II} \Delta y_{II} T_{II} \left|\overline{A}_{xy}^{(I,II)} - \overline{A}_{xy}^{(II,I)}\right|, \tag{5.46}$$

$$\Delta\overline{E}_y^{(II)} \approx \frac{\hbar}{\rho_{II}\Delta y_{II}V_{II}T_{II}} + \frac{1}{2}\rho_I V_I \Delta x_I T_I \left|\overline{A}_{xy}^{(I,II)} - \overline{A}_{xy}^{(II,I)}\right|. \qquad (5.47)$$

Taking the product of the uncertainties (5.46) and (5.47) and finding their minimum value by variation of $\alpha = \rho_I \rho_{II}$, we obtain the critical value

$$\alpha = \rho_I \rho_{II} = \frac{2\hbar}{\Delta x_I \Delta y_{II} V_I V_{II} T_I T_{II} \left|\overline{A}_{xy}^{(I,II)} - \overline{A}_{xy}^{(II,I)}\right|}, \qquad (5.48)$$

which, when replacing it in the product above mentioned allows us to obtain the uncertainty relation

$$\Delta\overline{E}_x^{(I)}\Delta\overline{E}_y^{(II)} \approx 2\hbar\left|\overline{A}_{xy}^{(I,II)} - \overline{A}_{xy}^{(II,I)}\right| \approx \hbar\left|\overline{A}_{xy}^{(I,II)} - \overline{A}_{xy}^{(II,I)}\right|, \qquad (5.49)$$

in agreement with the free quantum electromagnetic formalism.

5.1.4 MEASUREMENT OF $\overline{E}_x^{(I)}$ AND $\overline{H}_x^{(II)}$

According to Bohr and Rosenfeld [(1), (10)] the complete commutativity and independent measurability of averages of parallel dissimilar components required by the free quantum electromagnetic formalism finds its direct interpretation in the identical vanishing of the component $H_x^{(I)}$ of the field produced by the measurement of $\overline{E}_x^{(I)}$, as shown by (4.8), and it is not necessary to compensate the effects of $\overline{H}_x^{(II,II)}$ and $\overline{H}_x^{(I,II)}$.

Therefore we conclude that

$$\Delta\overline{E}_x^{(I)}\Delta\overline{H}_x^{(II)} = 0, \qquad (5.50)$$

in full agreement with the formalism.

5.2 CASES NOT CONSIDERED BY BOHR AND ROSENFELD

5.2.1 MEASUREMENT OF $\overline{H}_x^{(I)}$

Considering the measurement of $\overline{H}_x^{(I)}$ the linear momentum balance will be [(9)]

$$p_y^{''(I)} - p_y^{'(I)} = j_z^{(I)} V_I T_I (\overline{H}_x^{(I)} + \overline{H}_x^{(I,I)}), \quad (5.51)$$

$\overline{H}_x^{(I)}$ being the average of H_x in the space-time region I if no measurement were made on the test body at the instant $t_I^{'}$ where the initial linear momentum $p_y^{'(I)}$ is measured. At $t_I^{''}$ we similarly measure the final linear momentum $p_y^{''(I)}$, and $\overline{H}_x^{(I,I)}$ is the field average which arises from these measurements as described above in Section 4.2 according to formula (4.33) when the space-time region I is equal to the space-time region II.

The average $\overline{H}_x^{(I)}$ can be determined with arbitrary precision by choosing the value of $j_z^{(I)}$ sufficiently large. However, this makes $\overline{H}_x^{(I,I)}$ very large also, and then the achievable precision in the measurement of $\overline{H}_x^{(I)}$ has the value

$$\Delta\overline{H}_x^{(I)} \approx \frac{\Delta p_y^{(I)}}{j_z^{(I)} V_I T_I} + \Delta\overline{H}_x^{(I,I)}, \quad (5.52)$$

which, assuming that the displacement of test body $D_y^{(I)}$ is of order Δy_I, has the following value, limited by Heisenberg's uncertainty principle

$$\Delta\overline{H}_x^{(I)} \approx \frac{\hbar}{j_z^{(I)} \Delta y_I V_I T_I} + j_z^{(I)} \Delta y_I V_I T_I \left|\overline{G}_{xx}^{(I,I)}\right|. \quad (5.53)$$

Finding a minimum value of (5.53) with a variation of $j_z^{(I)}$, we obtain the following critical value

$$j_z^{(I)} = \frac{1}{\Delta y_I V_I T_I} \sqrt{\frac{\hbar}{\left|\overline{G}_{xx}^{(I,I)}\right|}}, \quad (5.54)$$

which, on replacing it in (5.53) allows us to find the minimum uncertainty

$$\Delta\overline{H}_{x\ \min}^{(I)} \approx \sqrt{\hbar\left|\overline{G}_{xx}^{(I,I)}\right|}. \quad (5.55)$$

If (5.55) were an inevitable limit in the precision of the measurement of $\overline{H}_x^{(I)}$, we would reach the conclusion that it is not possible to compensate for the effect of the field of the test body and the measurement would only make sense when $\hbar$ was neglected, that is to say, at the classical limit [(12), (13), (14), (15)].

Nevertheless, the coefficient of the displacement $D_y^{(I)}$ in $\overline{H}_x^{(I,I)}$ only depends on geometrical relations, enabling it to organize things, so that the effects of $\overline{H}_x^{(I,I)}$ would be completely compensated. The measuring device is modified, fixing the test body, previously completely free during time T_I, to a rigid system by means of a nut, and with a spring whose tension will be proportional to the displacement $D_y^{(I)}$. So, we would have

$$p_y^{''(I)} - p_y^{'(I)} = j_z^{(I)} V_I T_I (\overline{H}_x^{(I)} + \overline{H}_x^{(I,I)}) - s_I D_y^{(I)} T_I \, , \tag{5.56}$$

which suggests that

$$s_I = j_z^{(I)\,2} V_I^2 T_I \left| \overline{G}_{xx}^{(I,I)} \right| , \tag{5.57}$$

is the constant of the spring.

The spring will have no problems with its application because, by having the test body a large mass m, the test body's oscillation period $T \approx 2\pi \sqrt{\frac{m}{s_I}}$ is much greater than T_I and its displacement in this period of time will be small compared with $D_y^{(I)}$ and with $D_y^{(I)}$'s own powers from order two onwards.

Observe that the mass and current of the test body are parameters independent of each other in their adjustment.

Finally, we can write that

$$\Delta \overline{H}_x^{(I)} \approx \frac{\hbar}{j_z^{(I)} \Delta y_I V_I T_I} , \tag{5.58}$$

which may be substantially reduced, by choosing $j_z^{(I)}$ sufficiently large, in complete agreement with the prediction of the formalism (10).

5.2.2 MEASUREMENT OF $\overline{H}_x^{(I)}$ AND $\overline{H}_x^{(II)}$

In this case for the momentum balance of the two test bodies we have (9)

$$p_y^{''(I)} - p_y^{'(I)} = j_z^{(I)} V_I T_I (\overline{H}_x^{(I)} + \overline{H}_x^{(I,I)} + \overline{H}_x^{(II,I)}) , \tag{5.59}$$

$$p_y^{''(II)} - p_y^{'(II)} = j_z^{(II)} V_{II} T_{II} (\overline{H}_x^{(II)} + \overline{H}_x^{(II,II)} + \overline{H}_x^{(I,II)}), \qquad (5.60)$$

It would seem from the previous section that the appearance of fields $\overline{H}_x^{(I,I)}$ and $\overline{H}_x^{(II,II)}$ implies in itself limited precision in the measurements of $\overline{H}_x^{(I)}$ and $\overline{H}_x^{(II)}$. The reactions $j_z^{(I)} V_I T_I \overline{H}_x^{(I,I)}$ and $j_z^{(II)} V_{II} T_{II} \overline{H}_x^{(II,II)}$ can be cancelled by means of springs with elastic constants s_I given by (5.57) and s_{II} given by

$$s_{II} = j_z^{(II)\ 2} V_{II}^2 T_{II} \left|\overline{G}_{xx}^{(II,II)}\right|. \qquad (5.61)$$

Therefore we obtain

$$p_y^{''(I)} - p_y^{'(I)} = j_z^{(I)} V_I T_I (\overline{H}_x^{(I)} + \overline{H}_x^{(II,I)}), \qquad (5.62)$$

$$p_y^{''(II)} - p_y^{'(II)} = j_z^{(II)} V_{II} T_{II} (\overline{H}_x^{(II)} + \overline{H}_x^{(I,II)}), \qquad (5.63)$$

from which

$$\Delta\overline{H}_x^{(I)} \approx \frac{\hbar}{j_z^{(I)} \Delta y_I V_I T_I} + \Delta\overline{H}_x^{(II,I)}, \qquad (5.64)$$

$$\Delta\overline{H}_x^{(II)} \approx \frac{\hbar}{j_z^{(II)} \Delta y_{II} V_{II} T_{II}} + \Delta\overline{H}_x^{(I,II)}. \qquad (5.65)$$

By replacing in (5.64) and (5.65) the expressions corresponding to (4.33) and correctly choosing $j_z^{(I)}\Delta y_I$ and $j_z^{(II)}\Delta y_{II}$ either one of the quantities $\Delta\overline{H}_x^{(I)}$ or $\Delta\overline{H}_x^{(II)}$ can obviously be arbitrarily diminished, but only at the expense of an increase of the other.

By multiplying (5.64) by (5.65) and finding its minimum by a variation of $j_z = j_z^{(I)} j_z^{(II)}$, we obtain the following critical value

$$j_z = j_z^{(I)} j_z^{(II)} = \frac{\hbar\sqrt{\left|\overline{G}_{xx}^{(I,II)}\right|\left|\overline{G}_{xx}^{(II,I)}\right|}}{\Delta y_I \Delta y_{II} V_I V_{II} T_I T_{II} \left|\overline{G}_{xx}^{(I,II)}\right|\left|\overline{G}_{xx}^{(II,I)}\right|}. \qquad (5.66)$$

Replacing (5.66) in the above product, we get the minimum value

$$\Delta\overline{H}_x^{(I)} \Delta\overline{H}_x^{(II)} \approx \hbar\left(\left|\overline{G}_{xx}^{(I,II)}\right| + \left|\overline{G}_{xx}^{(II,I)}\right|\right) \geq \hbar\left|\overline{G}_{xx}^{(I,II)} + \overline{G}_{xx}^{(II,I)}\right|. \qquad (5.67)$$

In spite of the great similarity of relation (5.67) to the uncertainty relation (2.1) required by the formalism, there is, nevertheless, a fundamental difference in that the latter contains not the sum of the magnitudes of the quantities $\overline{G}_{xx}^{(I,II)}$ and $\overline{G}_{xx}^{(II,I)}$ but their algebraic difference.

The equation (5.67) is in full contradiction with the previous study of measurement of a simple field component. The minus sign (-) instead of plus sign (+) in the formula (2.1) for example, when regions I and II overlap completely, causes the cancellation of the product of the uncertainties, a situation which does not occur, according to (5.67).

In general, the two expressions (5.67) and (2.1) agree exactly only when at least one of the quantities $\overline{G}_{xx}^{(I,II)}$ or $\overline{G}_{xx}^{(II,I)}$ vanishes which in general requires that one of the expressions $t_1 - t_2 - \frac{r}{c}$ or $t_2 - t_1 - \frac{r}{c}$ in the Dirac δ-function's arguments remain different from zero for every pair of points (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) of regions I and II respectively.

We select two bodies b_I and b_{II}, one from each test body system I and II, for which the expression $r - c(t_1 - t_2)$ vanishes for two times t_I^* and t_{II}^* lying in the time intervals T_I and T_{II}, respectively; that is, the two bodies are light-like related. To establish the necessary correlation between the test bodies one might at first think of a spring which should connect the bodies b_I and b_{II} directly with each other; however, due to the retardation of the forces one would thereby run into difficulties. But we can manage with a short spring much smaller than cT_I, if we add to the second test body system a neutral component body b_{III} which is situated in the immediate vicinity of b_I, and connected with it by a spring whose elastic constant is s.

The body b_{III} is initially to be bound to the rigid frame by means of a device previously described and after removing this device, at time $t_I^{'}$, its momentum is to be measured with the same accuracy as that of the test body system II. It thereby undergoes an unknown displacement $D_y^{(III)}$ in the y-direction which is of the same order of magnitude as Δy_{II}.

The force exerted by the spring on the first test body system is therefore $F = -s\left(D_y^{(III)} - D_y^{(I)}\right)$ and so the linear momentum Y transferred from b_{III} to b_I during T_I is $Y = FT_I$, from which

$$Y = \frac{1}{2} j_z^{(I)} j_z^{(II)} V_I V_{II} T_I T_{II} \left(\overline{G}_{xx}^{(I,II)} + \overline{G}_{xx}^{(II,I)}\right)\left(D_y^{(I)} - D_y^{(III)}\right), \tag{5.68}$$

where

$$s = \frac{1}{2} j_z^{(I)} j_z^{(II)} V_I V_{II} T_{II} \left(\overline{G}_{xx}^{(I,II)} + \overline{G}_{xx}^{(II,I)}\right). \tag{5.69}$$

Similarly, b_{III} undergoes a change of linear momentum $-Y$ during the same period of time. At time $t_I^{''}$ the linear momentum of b_{III} is measured again with the same accuracy. However, before this measurement, and in fact at time t_{II}^{*} a short light signal is to be sent from b_{II} to b_{III}, by which the relative displacement $\left(D_y^{(III)} - D_y^{(II)}\right)$ of these bodies can be measured with arbitrary accuracy by means of a suitable device described in detail by Bohr and Rosenfeld (1). At the emission and absorption of the signal the two test bodies undergo momentum changes which indeed remain completely unknown, but cancel each other exactly in the sum of the momentum changes measured on the bodies. The above implies that $ct_{II}^{*} = D_y^{(III)} - D_y^{(II)}$ and for doing this, body b_{III} carries a small mirror on itself and body b_{II} a light-emitting device, which can rotate on a fixed support and therefore point in any direction in which b_{III} is found with respect to b_{II}.

Instants $t_I^{'}$ and $t_I^{''}$ are measurement times for the linear momentum of body b_{III} at the beginning and at the end respectively in region I and similarly for b_{II} in times $t_{II}^{'}$ and $t_{II}^{''}$ in region II.

The times of measurements of body b_{III} are recorded with the same clock that controls the measurements of b_I. After the momentum measurements the bodies are returned approximately to rest, that is, any momentum imparted to the test bodies by the fields to be measured is removed by the counterstrokes. The measurement process in space-time region II is governed by a clock which is located inside it.

Thus, for the momentum balance of the two test body systems during the measurement we have, if we include body b_{III} in system II is

$$p_y^{''(I)} - p_y^{'(I)} = j_z^{(I)} V_I T_I (\overline{H}_x^{(I)} + \overline{H}_x^{(II,I)}) + Y \, , \qquad (5.70)$$

$$p_y^{''(II)} - p_y^{'(II)} + p_y^{''(III)} - p_y^{'(III)} = j_z^{(II)} V_{II} T_{II} (\overline{H}_x^{(II)} + \overline{H}_x^{(I,II)}) - Y \, . \qquad (5.71)$$

Transforming and reorganizing the formula (5.70) the following is obtained

$$p_y^{''(I)} - p_y^{'(I)} = j_z^{(I)} V_I T_I \overline{H}_x^{(I)} + \frac{1}{2} j_z^{(I)} j_z^{(II)} V_I V_{II} T_I T_{II} \{ -D_y^{(II)} (\overline{G}_{xx}^{(I,II)} -$$

$$\overline{G}_{xx}^{(II,I)}) + (D_y^{(II)} - D_y^{(III)})(\overline{G}_{xx}^{(I,II)} + \overline{G}_{xx}^{(II,I)}) + D_y^{(I)} (\overline{G}_{xx}^{(I,II)} + \overline{G}_{xx}^{(II,I)}) \} \, . \qquad (5.72)$$

Similarly, transforming and reorganizing the equation (5.71) the following is obtained

$$p_y^{''(II)} - p_y^{'(II)} + p_y^{''(III)} - p_y^{'(III)} = j_z^{(II)} V_{II} T_{II} \; \overline{H}_x^{(II)} + \frac{1}{2} j_z^{(I)} j_z^{(II)} V_I V_{II} \; T_I T_{II} \; \{ D_y^{(I)} (\overline{G}_{xx}^{(II,I)} -$$

$$\overline{G}_{xx}^{(I,II)}) - (D_y^{(II)} - D_y^{(III)})(\overline{G}_{xx}^{(I,II)} + \overline{G}_{xx}^{(II,I)}) + D_y^{(II)} (\overline{G}_{xx}^{(I,II)} + \overline{G}_{xx}^{(II,I)}) \} \, . \qquad (5.73)$$

In equations (5.72) and (5.73) the quantities $D_y^{(I)}$ and $D_y^{(II)}$ multiply terms in the brackets which, exactly as the simple reactions of each test body on itself, be cancelled by means of a suitable spring connections with the rigid frame and the test bodies by screws and nuts respectively.

The elastic constant of the spring for body b_I, would be

$$s_{I,II} = j_z^{(I)\ 2} V_I^2 T_I \left| \overline{G}_{xx}^{(I,I)} \right| + \frac{1}{2} j_z^{(I)} j_z^{(II)} V_I V_{II} T_{II} (\overline{G}_{xx}^{(I,II)} + \overline{G}_{xx}^{(II,I)}) \, , \qquad (5.74)$$

acting on the elimination of the effect of the test body field in region I, $\overline{H}_x^{(I,I)}$, and of one part or factor which, after the transformation of the formula (5.70) in (5.72), depends on the displacement $D_y^{(I)}$. In analog form for body b_{II} it can be determined that

$$s_{II,I} = j_z^{(II)\ 2} V_{II}^2 T_{II} \left| \overline{G}_{xx}^{(II,II)} \right| + \frac{1}{2} j_z^{(I)} j_z^{(II)} V_I V_{II} T_I (\overline{G}_{xx}^{(I,II)} + \overline{G}_{xx}^{(II,I)}) \, , \qquad (5.75)$$

with a similar function to that of $s_{I,II}$.

Furthermore, as was described by Bohr and Rosenfeld [1], the terms proportional to the relative displacement $\left(D_y^{(III)} - D_y^{(II)} \right)$ are known with arbitrary accuracy and can therefore easily be taken into account in the field measurements.

Now, the following uncertainties can be obtained taking into account the Heisenberg's uncertainty principle

$$\Delta\overline{H}_x^{(I)} \approx \frac{\hbar}{j_z^{(I)}\Delta y_I V_I T_I} + \frac{1}{2} j_z^{(II)} V_{II} \Delta y_{II} T_{II} \left| \overline{G}_{xx}^{(I,II)} - \overline{G}_{xx}^{(II,I)} \right|, \qquad (5.76)$$

$$\Delta\overline{H}_x^{(II)} \approx \frac{\hbar}{j_z^{(II)}\Delta y_{II} V_{II} T_{II}} + \frac{1}{2} j_z^{(I)} V_I \Delta y_I T_I \left| \overline{G}_{xx}^{(I,II)} - \overline{G}_{xx}^{(II,I)} \right|. \qquad (5.77)$$

Taking the product of the uncertainties (5.76) and (5.77) and finding their minimum value by variation of $j_z = j_z^{(I)} j_z^{(II)}$, we obtain the critical value

$$j_z = j_z^{(I)} j_z^{(II)} = \frac{2\hbar}{\Delta y_I \Delta y_{II} V_I V_{II} T_I T_{II} \left| \overline{G}_{xx}^{(I,II)} - \overline{G}_{xx}^{(II,I)} \right|}, \qquad (5.78)$$

which, when replacing it in the product above mentioned allows the following uncertainty to be found

$$\Delta\overline{H}_x^{(I)} \Delta\overline{H}_x^{(II)} \approx 2\hbar \left| \overline{G}_{xx}^{(I,II)} - \overline{G}_{xx}^{(II,I)} \right| \approx \hbar \left| \overline{G}_{xx}^{(I,II)} - \overline{G}_{xx}^{(II,I)} \right|. \qquad (5.79)$$

The function $\Delta(r,t) = \frac{1}{4\pi\, r}\{\delta\,(r-ct) - \delta\,(r+ct)\}$, with $r = |\vec{r}_2 - \vec{r}_1|$, $\vec{r}_2 = (x_2, y_2, z_2)$ and $\vec{r}_1 = (x_1, y_1, z_1)$, satisfies the equality $\aleph\Delta = 0$, where $\aleph = \nabla^2 - \frac{\partial^2}{c^2\partial t^2}$ is the d'Alambertian operator. But $\delta\,(r-ct) = \frac{\delta\,(t-\frac{r}{c})}{c}$ and $\delta\,(r+ct) = \frac{\delta\,(t+\frac{r}{c})}{c}$, with which $\Delta(r,t) = \frac{1}{4\pi\, c}\left\{\frac{1}{r}\delta\,(t-\frac{r}{c}) - \frac{1}{r}\delta\,(t+\frac{r}{c})\right\}$ and as $\delta\,(a) = \delta\,(-a)$ therefore $\delta\,(t+\frac{r}{c}) = \delta\,(-\,t-\frac{r}{c})$, that is, $\Delta(r,t_2-t_1) = \frac{1}{4\pi\, c}\left\{\frac{1}{r}\delta\,(t_2-t_1-\frac{r}{c}) - \frac{1}{r}\delta\,(t_1-t_2-\frac{r}{c})\right\}$. It can be seen that $\left[\vec{\nabla}_1.\vec{\nabla}_2 - \frac{\partial^2}{c^2\partial t_1\partial t_2}\right]\Delta = 0$, that is $\left[\frac{\partial^2}{\partial y_1\partial y_2} + \frac{\partial^2}{\partial z_1\partial z_2}\right]\Delta = -\left[\frac{\partial^2}{\partial x_1\partial x_2} - \frac{\partial^2}{c^2\partial t_1\partial t_2}\right]\Delta$. Therefore, based on the definitions of

$G_{xx}^{(2,1)}$ and $A_{xx}^{(1,2)}$ it is possible to obtain $G_{xx}^{(2,1)} - G_{xx}^{(1,2)} = A_{xx}^{(1,2)} - A_{xx}^{(2,1)}$ and concluding with the equality $\left|\overline{G}_{xx}^{(I,II)} - \overline{G}_{xx}^{(II,I)}\right| = \left|\overline{A}_{xx}^{(I,II)} - \overline{A}_{xx}^{(II,I)}\right|$, we arrive at

$$\Delta\overline{H}_x^{(I)}\Delta\overline{H}_x^{(II)} \approx \hbar\left|\overline{A}_{xx}^{(I,II)} - \overline{A}_{xx}^{(II,I)}\right|, \qquad (5.80)$$

in complete agreement with the free quantum electromagnetic formalism and taking into account that in this case $\left[\frac{\partial^2}{\partial y_1 \partial y_2} + \frac{\partial^2}{\partial z_1 \partial z_2}\right]\left\{\frac{1}{r}\delta(t_2 - t_1 - \frac{r}{c}) - \frac{1}{r}\delta(t_1 - t_2 - \frac{r}{c})\right\} =$ $\left[\frac{\partial^2}{\partial y_2 \partial y_1} + \frac{\partial^2}{\partial z_2 \partial z_1}\right]\left\{\frac{1}{r}\delta(t_2 - t_1 - \frac{r}{c}) - \frac{1}{r}\delta(t_1 - t_2 - \frac{r}{c})\right\}$.

5.2.3 MEASUREMENT OF $\overline{H}_x^{(I)}$ AND $\overline{H}_y^{(II)}$

In this case we would have for the linear momentum balance before the compensation of the test body fields that

$$p_y^{''(I)} - p_y^{'(I)} = j_z^{(I)} V_I T_I (\overline{H}_x^{(I)} + \overline{H}_x^{(I,I)} + \overline{H}_x^{(II,I)}), \qquad (5.81)$$

$$p_z^{''(II)} - p_z^{'(II)} = j_x^{(II)} V_{II} T_{II} (\overline{H}_y^{(II)} + \overline{H}_y^{(II,II)} + \overline{H}_y^{(I,II)}). \qquad (5.82)$$

After compensating the effects of $\overline{H}_x^{(I,I)}$ and $\overline{H}_y^{(II,II)}$ in equations (5.81) and (5.82) respectively with springs whose elastic constants are $s_I = j_z^{(I)\,2} V_I^2 T_I \left|\overline{G}_{xx}^{(I,I)}\right|$ and $s_{II} = j_x^{(II)\,2} V_{II}^2 T_{II} \left|\overline{B}_{xy}^{(II,II)}\right|$, the linear momentum balance will then be

$$p_y^{''(I)} - p_y^{'(I)} = j_z^{(I)} V_I T_I (\overline{H}_x^{(I)} + D_z^{(II)} j_x^{(II)} V_{II} T_{II} \overline{B}_{xy}^{(II,I)}), \qquad (5.83)$$

$$p_z^{''(II)} - p_z^{'(II)} = j_x^{(II)} V_{II} T_{II} (\overline{H}_y^{(II)} + D_y^{(I)} j_z^{(I)} V_I T_I \overline{A}_{xy}^{(I,II)}). \qquad (5.84)$$

These uncertainties are obtained from the above equations

$$\Delta\overline{H}_x^{(I)} \approx \frac{\hbar}{j_z^{(I)} V_I T_I \Delta y_I} + j_x^{(II)} \Delta z_{II} V_{II} T_{II} \left|\overline{B}_{xy}^{(II,I)}\right|, \qquad (5.85)$$

$$\Delta\overline{H}_y^{(II)} \approx \frac{\hbar}{j_x^{(II)} V_{II} T_{II} \Delta z_{II}} + j_z^{(I)} \Delta y_I V_I T_I \left|\overline{A}_{xy}^{(I,II)}\right|. \qquad (5.86)$$

As the calculation being made is restricted only in order of magnitude, we can assume that $\left|\overline{B}_{xy}^{(II,I)}\right| \approx \left|\overline{A}_{xy}^{(II,I)}\right|$. So, the product of the uncertainties (5.85) and (5.86) has a critical value

$$\theta \;=\; j_z^{(I)} j_x^{(II)} = \frac{\hbar\sqrt{\left|\overline{A}_{xy}^{(I,II)}\right|\left|\overline{A}_{xy}^{(II,I)}\right|}}{\Delta y_I \Delta z_{II} V_I V_{II} T_I T_{II} \left|\overline{A}_{xy}^{(I,II)}\right|\left|\overline{A}_{xy}^{(II,I)}\right|}, \qquad (5.87)$$

which, on replacing it in the above product gives

$$\Delta\overline{H}_x^{(I)}\Delta\overline{H}_y^{(II)} \approx \hbar\left(\left|\overline{A}_{xy}^{(I,II)}\right| + \left|\overline{A}_{xy}^{(II,I)}\right|\right) \geq \hbar\left|\overline{A}_{xy}^{(I,II)} + \overline{A}_{xy}^{(II,I)}\right|. \qquad (5.88)$$

Comparing the inequalities (5.37) and (5.88) we conclude that doing a completely similar treatment of measurement as it has been made in the measurement case of $\overline{E}_x^{(I)}$ and $\overline{E}_y^{(II)}$, the following equation could be obtained

$$\Delta\overline{H}_x^{(I)}\Delta\overline{H}_y^{(II)} \approx 2\hbar\left|\overline{A}_{xy}^{(I,II)} - \overline{A}_{xy}^{(II,I)}\right| \approx \hbar\left|\overline{A}_{xy}^{(I,II)} - \overline{A}_{xy}^{(II,I)}\right|, \qquad (5.89)$$

in complete agreement with the formalism.

5.2.4 MEASUREMENT OF $\overline{H}_x^{(I)}$ AND $\overline{E}_x^{(II)}$

The complete commutativity and independent measurability of averages of parallel dissimilar components required by the free quantum electromagnetic formalism finds its direct interpretation in the identical vanishing of the component $E_x^{(I)}$ of the field produced by the measurement of $\overline{H}_x^{(I)}$, as shown by (4.24), and it is not necessary to compensate the effects of $\overline{E}_x^{(II,II)}$ and $\overline{E}_x^{(I,II)}$.

Therefore we conclude that

$$\Delta\overline{H}_x^{(I)}\Delta\overline{E}_x^{(II)} = 0, \qquad (5.90)$$

in full agreement with the formalism.

5.2.5 MEASUREMENT OF $\overline{E}_x^{(I)}$ AND $\overline{H}_y^{(II)}$

In the present case we have for the linear momentum balance before the compensation of the test body fields that

$$p_x^{''(I)} - p_x^{'(I)} = \rho_I V_I T_I (\overline{E}_x^{(I)} + \overline{E}_x^{(I,I)} + D_z^{(II)} j_x^{(II)} V_{II} T_{II} \overline{B}_{xy}^{(II,I)}), \quad (5.91)$$

$$p_z^{''(II)} - p_z^{'(II)} = j_x^{(II)} V_{II} T_{II} (\overline{H}_y^{(II)} + D_z^{(II)} j_x^{(II)} V_{II} T_{II} \overline{B}_{xy}^{(II,II)} + D_x^{(I)} \rho_I V_I T_I \overline{B}_{xy}^{(I,II)}). \quad (5.92)$$

After compensating the effects of the fields $\overline{E}_x^{(I,I)}$ and $D_z^{(II)} j_x^{(II)} V_{II} T_{II} \overline{B}_{xy}^{(II,II)}$ in equations (5.91) and (5.92) respectively with suitable springs, the linear momentum balance will be

$$p_x^{''(I)} - p_x^{'(I)} = \rho_I V_I T_I (\overline{E}_x^{(I)} + D_z^{(II)} j_x^{(II)} V_{II} T_{II} \overline{B}_{xy}^{(II,I)}), \quad (5.93)$$

$$p_z^{''(II)} - p_z^{'(II)} = j_x^{(II)} V_{II} T_{II} (\overline{H}_y^{(II)} + D_x^{(I)} \rho_I V_I T_I \overline{B}_{xy}^{(I,II)}). \quad (5.94)$$

These uncertainties are obtained from the above equations

$$\Delta \overline{E}_x^{(I)} \approx \frac{\hbar}{\rho_I \Delta x_I V_I T_I} + j_x^{(II)} V_{II} \Delta z_{II} T_{II} \left| \overline{B}_{xy}^{(II,I)} \right|, \quad (5.95)$$

$$\Delta \overline{H}_y^{(II)} \approx \frac{\hbar}{j_x^{(II)} V_{II} T_{II} \Delta z_{II}} + \rho_I \Delta x_I V_I T_I \left| \overline{B}_{xy}^{(I,II)} \right|. \quad (5.96)$$

The product of (5.95) and (5.96) has a critical value

$$\pi = \rho_I j_x^{(II)} = \frac{\hbar \sqrt{\left| \overline{B}_{xy}^{(I,II)} \right| \left| \overline{B}_{xy}^{(II,I)} \right|}}{\Delta x_I \Delta z_{II} V_I V_{II} T_I T_{II} \left| \overline{B}_{xy}^{(I,II)} \right| \left| \overline{B}_{xy}^{(II,I)} \right|}, \quad (5.97)$$

which, on replacing it in the above product gives

$$\Delta \overline{E}_x^{(I)} \Delta \overline{H}_y^{(II)} \approx \hbar \left(\left| \overline{B}_{xy}^{(I,II)} \right| + \left| \overline{B}_{xy}^{(II,I)} \right| \right) \geq \hbar \left| \overline{B}_{xy}^{(I,II)} + \overline{B}_{xy}^{(II,I)} \right|. \quad (5.98)$$

Through a completely similar treatment of measurement as it has been made in previous sections, the following equation could be obtained

$$\Delta \overline{E}_x^{(I)} \Delta \overline{H}_y^{(II)} \approx \hbar \left| \overline{B}_{xy}^{(I,II)} - \overline{B}_{xy}^{(II,I)} \right|, \quad (5.99)$$

in complete agreement with the formalism.

5.2.6 MEASUREMENT OF $\overline{H}_x^{(I)}$ AND $\overline{E}_y^{(II)}$

In this case we have for the linear momentum balance before the compensation of the test body fields that

$$p_y^{''(I)} - p_y^{'(I)} = j_z^{(I)} V_I T_I (\overline{H}_x^{(I)} + \overline{H}_x^{(I,I)} + \overline{H}_x^{(II,I)}), \quad (5.100)$$

$$p_y^{''(II)} - p_y^{'(II)} = \rho_{II} V_{II} T_{II} (\overline{E}_y^{(II)} + \overline{E}_y^{(II,II)} + \overline{E}_y^{(I,II)}), \quad (5.101)$$

where $\overline{H}_x^{(I,I)} = D_y^{(I)} j_z^{(I)} V_I T_I \overline{G}_{xx}^{(I,I)}$, $\overline{E}_y^{(I,II)} = -D_y^{(I)} j_z^{(I)} V_I T_I \overline{B}_{xy}^{(II,I)}$, $\overline{E}_y^{(II,II)} = D_y^{(II)} \rho_{II} V_{II} T_{II} \overline{L}_{xx}^{(II,II)}$ and $\overline{H}_x^{(II,I)} = D_y^{(II)} \rho_{II} V_{II} T_{II} \overline{B}_{xy}^{(I,II)}$.

The quantity $\overline{L}_{xx}^{(II,II)}$ which not is in the Bohr and Rosenfeld's analysis arises from the expression $L_{xx}^{(2,1)} = \left(\frac{\partial^2}{c^2 \partial t_2 \partial t_1} - \frac{\partial^2}{\partial y_2 \partial y_1} \right) \left\{ \frac{1}{r} \delta(t_1 - t_2 - \frac{r}{c}) \right\}$ by simple integration.

After compensating the effects of the fields $\overline{H}_x^{(I,I)}$ and $\overline{E}_y^{(II,II)}$ in equations (5.100) and (5.101) respectively with suitable springs, the linear momentum balance will be

$$p_y^{''(I)} - p_y^{'(I)} = j_z^{(I)} V_I T_I (\overline{H}_x^{(I)} + \overline{H}_x^{(II,I)}), \quad (5.102)$$

$$p_y^{''(II)} - p_y^{'(II)} = \rho_{II} V_{II} T_{II} (\overline{E}_y^{(II)} + \overline{E}_y^{(I,II)}). \quad (5.103)$$

From the equations (5.102) and (5.103) respectively we can obtain the following uncertainties

$$\Delta \overline{H}_x^{(I)} \approx \frac{\hbar}{j_z^{(I)} V_I T_I \Delta y_I} + \rho_{II} \Delta y_{II} V_{II} T_{II} \left| \overline{B}_{xy}^{(I,II)} \right|, \quad (5.104)$$

$$\Delta \overline{E}_y^{(II)} \approx \frac{\hbar}{\rho_{II} \Delta y_{II} V_{II} T_{II}} + j_z^{(I)} V_I \Delta y_I T_I \left| \overline{B}_{xy}^{(II,I)} \right|. \quad (5.105)$$

The product of (5.104) and (5.105) has a critical value

$$\beta = \rho_{II} j_z^{(I)} = \frac{\hbar \sqrt{\left| \overline{B}_{xy}^{(I,II)} \right| \left| \overline{B}_{xy}^{(II,I)} \right|}}{\Delta y_I \Delta y_{II} V_I V_{II} T_I T_{II} \left| \overline{B}_{xy}^{(I,II)} \right| \left| \overline{B}_{xy}^{(II,I)} \right|}, \quad (5.106)$$

which, on replacing it in the above product gives

$$\Delta \overline{H}_x^{(I)} \Delta \overline{E}_y^{(II)} \approx \hbar \left(\left| \overline{B}_{xy}^{(I,II)} \right| + \left| \overline{B}_{xy}^{(II,I)} \right| \right) \geq \hbar \left| \overline{B}_{xy}^{(I,II)} + \overline{B}_{xy}^{(II,I)} \right|. \quad (5.107)$$

Through a completely similar treatment of measurement as it has been made in previous sections, the following equation could be obtained

$$\Delta\overline{H}_x^{(I)}\Delta\overline{E}_y^{(II)} \approx \hbar\left|\overline{B}_{xy}^{(I,II)} - \overline{B}_{xy}^{(II,I)}\right|, \qquad (5.108)$$

in full agreement with the formalism.

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