Quantum Measurement Approach to a Non-Markovian Master Equation*

Alireza Shabani[1] and Daniel Lidar[2]

[1]Department of Physics, University of Toronto

[2]Chemical Physics Theory Group, University of Toronto

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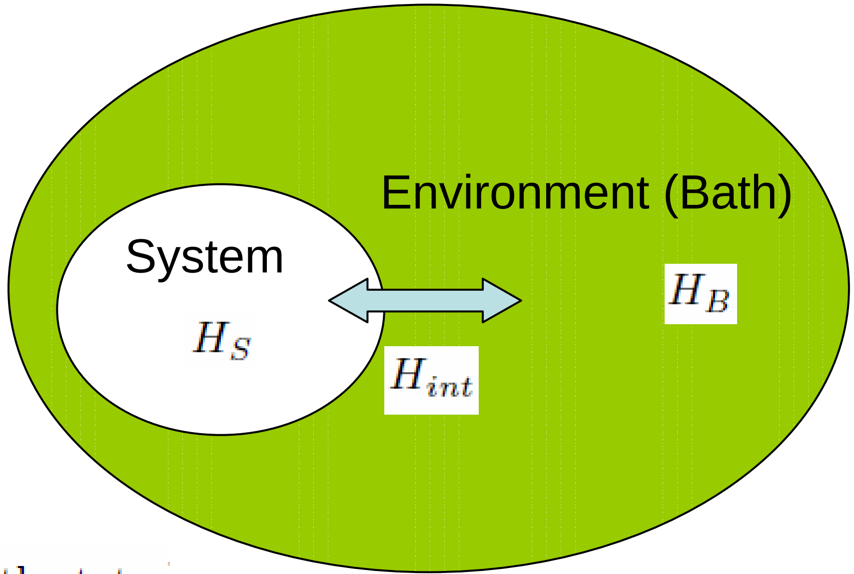
Open System Dynamics



Exact Solution (Kraus Rep.):

$$\rho_S(t) = \sum_\alpha E_\alpha(t)\rho_S(0)E_\alpha^\dagger(t),$$

$$\sum_\alpha E_\alpha^\dagger(t)E_\alpha(t) = I.$$

$$E_\alpha(t) = \sqrt{\nu}\langle\mu|U(t)|\nu\rangle \; , \; |\nu\rangle \text{ and } \mu\rangle \text{ are bath states}.$$

Markovian Regime (Lindblad Eq.):

$$\frac{\partial\rho_S}{\partial t} = \mathcal{L}\rho_S = -\frac{1}{2}\sum_\alpha a_\alpha([F_\alpha, \rho_S F_\alpha^\dagger] + [F_\alpha\rho_S, F_\alpha^\dagger]) \; , \; \rho_S(t) = e^{\mathcal{L}t}\rho_S(0).$$

Dynamics in Two Limits

	Exact Solution	*Markovian Approximation*
Advantages	No approximation	1. Closed form of dynamical map. 2. Effective numerical solution.
Disadvantages	Analytically solvable only for simple models.	Inadequate description for a bath with a significant memory effect.

Goal of the presented work:

“ To develop a dynamical master equation beyond the Markovian regime

that is analytically solvable and the resulting map is completely positive.”

Measurement Theory Picture of Dynamics

Kraus Sum Representation: $\rho_{out} = \sum_k M_k \rho_{in} M_k^\dagger$, $\sum_k M_k^\dagger M_k = I$.

Non-Selective Generalized Measurement (GM): $\rho_{out} = \sum_k p_k \rho_k$, k'th outcome: $\rho_k = \dfrac{M_k \rho_{in} M_k^\dagger}{\mathrm{Tr}(M_k^\dagger M_k \rho_{in})}$

probability: $p_k = \mathrm{Tr}(M_k^\dagger M_k \rho_{in})$

Exact Solution: $\rho_S(t) = \sum_\alpha E_\alpha(t) \rho_S(0) E_\alpha^\dagger(t)$. GM operators: $\{E_\alpha\}$

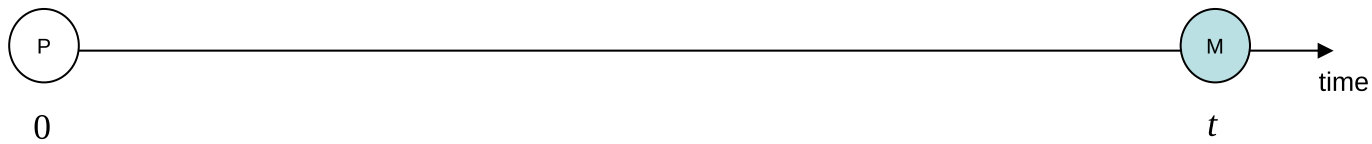
Lindblad (Quantum Jump):

$$\tau \ll ||\mathcal{L}||^{-1}\ , \quad \rho_S(t+\tau) \approx (I - \frac{\tau}{2}\sum_\alpha F_\alpha^\dagger F_\alpha)\rho_S(t)(I - \frac{\tau}{2}\sum_\alpha F_\alpha^\dagger F_\alpha) + \tau \sum_\alpha F_\alpha \rho_S(t) F_\alpha^\dagger\ .$$

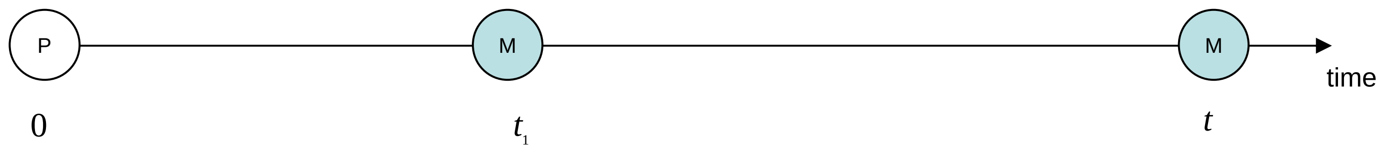
GM operators: $\{\sqrt{\tau} F_\alpha, I - \frac{\tau}{2}\sum_\beta F_\beta^\dagger F_\beta\}$

Single-Shot Measurement Process

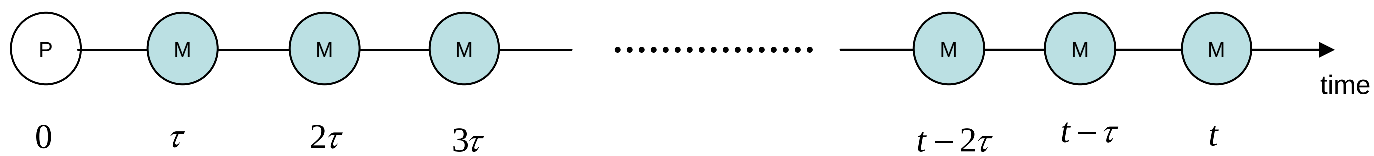
Exact Solution:

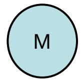


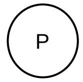
Single-Shot Measurement:



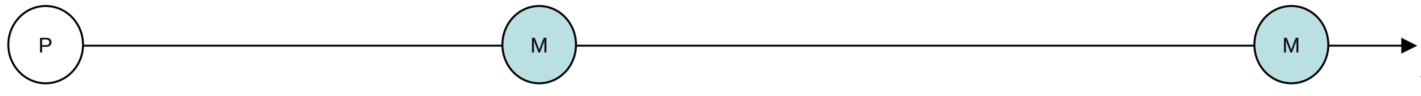
Markovian Approximation (Quantum Trajectories):



Measurement: 

Preparation: 

Non-Markovian Master Equation



$\rho_S(0)$ $\rho_S(t_1)$ $\rho_S(t) = \Lambda(t - t_1)\rho_S(t_1)$

Probabilistic Procedure:

Probability of an extra measurement at time t_1 : $w(t_1)$,

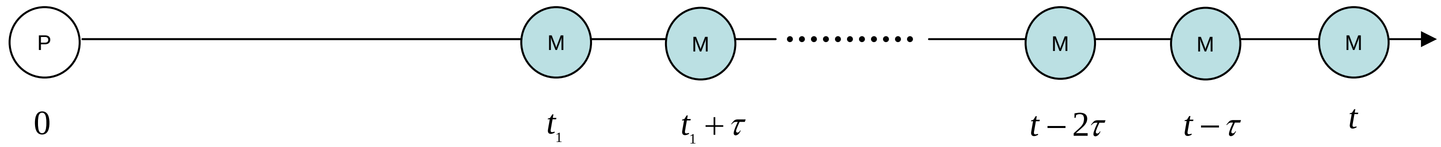
$$\rho_S(t = N\epsilon) = \sum_{m=1}^{N-1} w(m\epsilon)\Lambda(m\epsilon)\rho_S(t_1 = (N - m)\epsilon)$$

Non-Markovian Master Equation (Newton Iteration Method):

$$\frac{\partial \rho_S}{\partial t} = \int_0^t dt' k(t')\Lambda(t')\dot{\Lambda}(t')\Lambda^{-1}(t')\rho_S(t - t') \ .$$

Post-Markovian Master Equation

$\Lambda(t) = e^{\mathcal{L}t}$:



Post-Markovian Master Equation:

$$\frac{\partial \rho_S}{\partial t} = \mathcal{L} \int_0^t dt' k(t') e^{\mathcal{L}t'} \rho_S(t - t')$$

— Markovian approximation can be recovered by choosing: $k(t) = \delta(t)$,

— The intuitively [1,2] addressed memory function could also be retrieved in the limit of $||\mathcal{L}|| \ll \mathrm{T}^{-1}$

$$\frac{\partial \rho_S}{\partial t} = \mathcal{L} \int_0^t dt' k(t') \rho_S(t - t')$$

1) S.Daffer *et al.* , quant-ph/309081.
2) S.M.Barnett *et al.* , PRA, 2001, 64, 33808.

$Example$: $Single\ Qubit\ Dephasing$

Spin-Boson Hamiltonian: $H_{SB} = \sum_k \sigma_z \otimes (\lambda_k \mathbf{b} + \lambda_k^* \mathbf{b}^\dagger)$

$$\rho(t) = \frac{1}{2}(I + f(t)\alpha_x\sigma_x + f(t)\alpha_y\sigma_y + \alpha_z\sigma_z)$$

Exact Solution	Markovian Regime
$f(t) = exp[-\sum_k \lvert\lambda_k\rvert^2 \frac{sin^2(\omega_k t)}{\omega_k^2} coth \frac{\hbar\omega_k}{2k_B T}]$	$f(t) = exp[-\frac{t}{\tau}\sum_k \lvert\lambda_k\rvert^2 \frac{sin^2(\omega_k \tau)}{\omega_k^2} coth \frac{\hbar\omega_k}{2k_B T}] \triangleq e^{-at}$

Post-Markovian Equation Result:

Memory Function: $k(t) = (1-\theta)\frac{1}{\gamma}e^{-\gamma t} + \theta\delta(t)$

$$f(t) \ = \ (1 \ - \ \theta)e^{-(\gamma/2+a)t} \cos(\sqrt{2a\gamma - (\gamma/2+a)^2}t \ + \ \varphi) \ + \ \theta e^{-at}$$

Quantum Dynamical Map

Laplace Transformation: $s\widetilde{\rho}_S(s) - \rho_S(0) = [\widetilde{k}(s) * \frac{\mathcal{L}}{s-\mathcal{L}}]\widetilde{\rho}_S(s)$

Eigenvalue, right and left eigenoperators of the superoperator $\mathcal{L} : \{\lambda_i, R_i, L_i\}$.

$$\begin{cases} \mathcal{L}\rho_i = \lambda_i \rho_i \\ \rho_S(t) = \sum_i \mu_i(t) R_i \end{cases} \Longrightarrow \quad s\widetilde{\mu}_i(s) - \mu_i(0) = \lambda_i \widetilde{k}(s-\lambda_i)\widetilde{\mu}_i(s)$$

Dynamical Map:

$$\Phi(t) : \rho \longmapsto \sum_i \xi_i(t) \mathrm{Tr}[L_i \rho] R_i \quad , \quad \xi_i(t) = \mathrm{Lap}^{-1}[\frac{1}{s - \lambda_i \widetilde{k}(s-\lambda_i)}]$$

Complete Positivity:

$$\Phi((|i\rangle\langle j|))_{1\leq i,j\leq n} \geq 0 \quad \Longrightarrow \quad \sum_k \xi_k(t) L_k^T \otimes R_k \geq 0\ .$$

Experimental Determination of the Kernel Function:

Quantum state tomography result: $\rho(t)$.

Kernel function:

$$k(t) = \mathrm{Lap}^{-1}[(s - 1/\mathrm{Lap}[\xi_i(t)])]e^{-\lambda_i t}/\lambda_i$$

$$\xi_i(t) = \mathrm{Tr}[L_i\rho(t)]/\mathrm{Tr}[L_i\rho(0)]$$

Conclusion and Possible Extensions:

We have introduced:

— Phenomenological picture of a non-Markovian master equation in the measurement theory.

— A post-Markovian master equation which can be analytically solved by applying the Laplace transform.

— A condition on the memory function to preserve the complete positivity of the corresponding dynamical map.

We like to present in the future:

— Improving the introduced non-Markovian equation by going to higher steps of Newton iteration method.

— Exploring the memory function for a set of performed experiment results.