On the regularity problem for the nonlinear Boltzmann equation in one space dimension

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Abstract: The nonlinear Boltzmann equation is a fundamental model of classical statistical physics. It describes the time evolution of the mass density (in the (x, v) space: x is the position of a particle, v is the molecular velocity) for a system of interacting particles (say, a monoatomic rarefied gas). Theorems about solutions of the Boltzmann equation are based either on classical perturbation techniques (strong solutions near the global equilibrium), or on the monotonicity properties (local in time strong solutions, global solutions for small data in the whole space), or on the weak convergence and the regularity coming from the kinetic averaging lemmas (the DiPerna-Lions theory). It remains an important open problem to show whether strong (say, L^∞) solutions can exist globally in a nonperturbative setting. This is also closely related to the problem of uniqueness (open for "mild" solutions provided by the DiPerna-Lions theory), since it is known that mild solutions are unique if they are strong.

In one space dimension a little more is known about "large data" solutions. If one imposes a certain truncation on the collisions of particles with small relative speed, one can prove that the "mild" solutions satisfy the equation in the sense of distributions (this is unknown in general dimension), and one can also prove uniqueness. The ideas that work in the one-dimensional case originated in the works of Tartar, Bony and Toscani in the context of discrete-velocity models, and were pushed by Arkeryd and Cercignani to the Boltzmann equation. Further progress has been limited by the lack of estimates on the local regularity of solutions. Indeed, the "natural" physical apriori estimates imply that the solutions are nonnegative, integrable and have bounded entropy, which translates into bounds in the Hardy $\mathscr{H}^1$ space for appropriately localized solutions. However, to make sense of all terms in the equation one would like to have higher regularity (say, L^2) of the solutions.

I will discuss some ideas on how I think one could obtain more regular solutions in one dimension. One observation (that has its roots in the work of Carleman in the 1930's) is that if the density in the physical space $\rho(x, t)$ is in L^∞ on a certain time interval, then the solution of the full equation is in L^∞. Another idea is that the "gain" term provides some "mixing" of the x and v variables and is generally more regular than the "loss" term. As a simple consequence of this fact we obtain that (for a suitably truncated equation) if the density $\rho(\cdot, t)$ is in L^p, for any $p > 1$, uniformly on a certain time interval, then the solutions are in L^∞. An interesting question that remains without the answer is whether the $\mathscr{H}^1$ regularity of the solution could be used to obtain a regularizing effect.