Pseudodifferential Methods for Boundary Value Problems
The Fields Institute

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Introduction

These notes provide an outline for lectures delivered by the author at the Fields Institute on December 13, 2006. The topic of the lectures is the application of pseudodifferential operator techniques to solve boundary value problems for first order differential operators. These techniques came to the fore in the analysis of

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boundary value problems for the Dirac operator on a manifold with boundary, see [10, 1, 2, 3, 4]. The boundary conditions we consider are defined by pseudodifferential operators, frequently specialized to pseudodifferential projections. We assume a familiarity with the basics of L^2-Sobolev space theory and pseudodifferential operators.

1 Manifolds with Boundary

Let Ω be a closed, n-dimensional manifold with boundary. As local models we have

$$\mathbb{B}_1 = \{x \in \mathbb{R}^n : \|x\| < 1\} \text{ and } \mathbb{B}_1^+ = \{x \in \mathbb{R}^n : \|x\| < 1 \text{ and } x_n \geq 0\}. \tag{1}$$

The interior of Ω has a cover by open sets $\{U_j\}$ and the boundary has a cover by open sets $\{V_k\}$ such that, for each j there is a homeomorphism $\varphi_j : U_j \to \mathbb{B}_1 \subset \mathbb{R}^n$, and for each k there is a homeomorphism $\varphi_k : V_k \to \mathbb{B}_1^+ \subset \mathbb{R}^n_+$. In the later case $\varphi_k(V_k \cap b\Omega) \subset b\mathbb{B}_1^+$. The pairs (U_j, φ_j) are called interior coordinate charts and (V_k, φ_k) are boundary coordinate charts. On the nontrivial intersections of the coordinate charts we require that the induced maps from subsets of $\mathbb{R}^n$ to itself be diffeomorphisms, e.g. If $U_j \cap U_{j'} \neq \emptyset$, then

$$\varphi_j \circ \varphi_{j'}^{-1} : \varphi_{j'}(U_j \cap U_{j'}) \longrightarrow \varphi_j(U_j \cap U_{j'}), \tag{2}$$

is a diffeomorphism.

A function, r, which is non-negative (or non-positive) in the interior of Ω and vanishes to order one ($dr \neq 0$) along the boundary is called a defining function for the boundary of Ω. The normal bundle to the boundary is the line bundle along the boundary

$$Nb\Omega = T\Omega \restriction_{b\Omega} / Tb\Omega. \tag{3}$$

The dual bundle, the co-normal bundle, $N^*b\Omega$, is the sub-bundle of $T^*\Omega \restriction_{b\Omega}$ consisting of 1-forms that annihilate $Tb\Omega$. It is spanned at every point, x by dr_x. The geometry of Ω near to the boundary is described by the tubular neighborhood theorem:

Theorem 1 (The tubular neighborhood theorem). *If Ω is a manifold with boundary, then there is a neighborhood U of $b\Omega$ that is diffeomorphic to $b\Omega \times [0, 1)$. It can be realized as a one sided neighborhood of the zero section within $Nb\Omega$.*

Using the identification of U with a neighborhood of the zero section, it is easy to show that Ω can be embedded as a subset of the smooth manifold without boundary: $\widetilde{\Omega} \simeq \Omega \amalg_{b\Omega} \Omega$. The interior of Ω is an open subset of $\widetilde{\Omega}$. If Ω is a compact manifold with boundary, then $\widetilde{\Omega}$ is a compact manifold without boundary. If we fix an orientation on Ω, then $\widetilde{\Omega} \simeq \Omega \amalg_{b\Omega} [-\Omega]$, where $[-\Omega]$ denotes Ω with the opposite orientation, is also an oriented manifold.

We use $\mathscr{C}^\infty(\Omega), \mathscr{C}^k(\Omega)$, etc. to denote smooth, respectively $\mathscr{C}^k$-functions on the interior of Ω, and $\mathscr{C}^\infty(\overline{\Omega}), \mathscr{C}^k(\overline{\Omega})$, these classes of functions on the closure. If $F \to \Omega$ is a vector bundle, then $\mathscr{C}^\infty(\Omega; F), \mathscr{C}^k(\Omega; F)$ are the sections of F, that are smooth, resp. $\mathscr{C}^k$, up to the boundary. If it is clear from the context, we often omit explicit mention of the bundle from the notation. When doing analysis on a manifold with boundary it is very useful to be able to extend functions from $\overline{\Omega}$ to $\widetilde{\Omega}$. Seeley proved a very general such result:

Theorem 2 (Seeley Extension Theorem). *If Ω is a manifold with boundary, then there is a continuous linear map*

$$E : \mathscr{C}^\infty(\overline{\Omega}) \longrightarrow \mathscr{C}^\infty(\widetilde{\Omega}). \tag{4}$$

E also extends to define a continuous linear map $\mathscr{C}^k(\overline{\Omega}) \to \mathscr{C}^k(\widetilde{\Omega})$.

Recall that, for $s \in \mathbb{R}$, the L^2-Sobolev space $H^s(\mathbb{R}^n)$ is defined as those tempered distributions $u \in \mathscr{S}'(\mathbb{R}^n)$ whose Fourier transform $\hat{u}$ is a function, which satisfies:

$$\|u\|_s^2 = \int\limits_{\mathbb{R}^n} |\hat{u}(\xi)|^2(1+|\xi|^2)^s d\xi < \infty. \tag{5}$$

Let X be a compact manifold without boundary, having coordinate cover (U_j, φ_j). Let $\{\psi_j\}$ be a partition of unity subordinate to this cover. A distribution $u \in \mathscr{C}^{-\infty}(X)$ belongs to $H^s(X)$, if for every j, the compactly supported distribution $\psi_j u \circ \varphi_j^{-1}$, on $\mathbb{R}^n$ belongs to $H^s(\mathbb{R}^n)$. It is a well known result that the Sobolev spaces are invariant under such changes of coordinate and therefore, the space $H^s(X)$ is well defined as a topological vector space. A norm, which defines this topology is given by

$$\|u\|^2_{H^s(X)} = \sum_j \|\psi_j u \circ \varphi_j^{-1}\|^2_{H^s(\mathbb{R}^n)}. \tag{6}$$

Defining function spaces on manifolds with boundary is a bit more involved, we return to this question in Section 3

Good references for the material in this section are [9] and [11].

2 The Basic Example

Before going on, we consider, in detail, the simplest case, which reveals the main ideas we encounter in the general case. We let $\Omega = D_1$ the unit disk in the plane.

The operator we study is the $\bar{\partial}$-operator,

$$\bar{\partial}u = \frac{1}{2}(\partial_x + i\partial_y)u. \tag{7}$$

The Cauchy-Pompieu formula states that, if $u \in \mathscr{C}^1(\overline{\Omega})$, then

$$u(z) = \frac{1}{2\pi}\int\limits_{D_1} \frac{\bar{\partial}u(w,\bar{w})dxdy}{w-z} + \frac{1}{2\pi i}\int\limits_{bD_1} \frac{u(w,\bar{w})dw}{z-w}. \tag{8}$$

From the perspective of pseudodifferential operators, this follows from the fact that $[2\pi(w-z)]^{-1}$ is a fundamental solution for the $\bar{\partial}$-operator,

$$\bar{\partial}\frac{1}{2\pi(w-z)} = \delta(w-z). \tag{9}$$

As we shall see, the first term in (8) defines a bounded map from $H^s(D_1) \to H^{s+1}(D_1)$, for every $s \in \mathbb{R}$. The second term in formula (8) defines a holomorphic function in D_1, an element of the nullspace of $\bar{\partial}$. The main task before us is to understand the behavior of this second term as $z \to bD_1$.

Using the Fourier representation

$$u(r,\theta) = \sum_{n=-\infty} u_n(r)e^{in\theta}, \tag{10}$$

we see that

$$\|u\|^2_{L^2} = 2\pi\sum_{n=-\infty}^{\infty}\int\limits_0^1 |u_n(r)|^2 r\,dr, \tag{11}$$

and, after integrating by parts, we find that

$$\|\bar{\partial}u\|^2_{L^2} = \frac{\pi}{2}\left[\sum_{n=-\infty}^{\infty}\left(r|a_n'(r)|^2 + \frac{n^2|a_n(r)|^2}{r}\right)dr - \sum_{n=-\infty}^{\infty} n|a_n(1)|^2\right]. \tag{12}$$

Our goal is to find boundary conditions for the $\bar{\partial}$-operator, so that resultant unbounded operator on $L^2(D_1)$ is Fredholm and has a compact resolvent. For non-negative integers define $H^k(D_1)$ to be the closure of $\mathscr{C}^\infty(D_1)$ with respect to the norm:

$$\|u\|^2_{H^k} = \sum_{m+n=k}\|\partial_x^m\partial_y^n u\|^2_{L^2(D_1)}. \tag{13}$$

For real $s \geq 0$, define $H^s(D_1)$ by interpolation. For real s, a distribution u in $\mathscr{C}^{-\infty}(bD_1)$ belongs to $H^s(bD_1)$ provided:

$$\|u\|^2_{H^s(bD_1)} = \sum_{n=-\infty}^{\infty} |\hat{u}(n)|^2(1+n^2)^s < \infty, \tag{14}$$

where $\hat{u}(n) = \langle u, e^{in\theta}\rangle$.

A boundary condition for $\bar{\partial}$ defines a Fredholm operator (with compact resolvent) provided that functions in the domain of the operator satisfy an estimate of the form

$$\|u\|_{H^s(D_1)} \leq C[\|\bar{\partial}u\|_{L^2(D_1)} + \|u\|_{L^2(D_1)}], \tag{15}$$

for an $s > 0$. Equation (12) shows that the difficulty in proving this is produced precisely by the values $a_n(1)$ for $n > 0$, as all other terms on the right hand side of (12) are positive. Indeed if $\bar{\partial}u = 0$ then

$$u(r,\theta) = \sum_{n=0}^{\infty} u_n r^n e^{in\theta}. \tag{16}$$

In this case the negative boundary term in (12) exactly balances the other two positive terms.

While it is not immediate from (12), an L^2-function such that $f = \bar{\partial}u \in L^2(D_1)$, satisfies an important estimate, and has an important "global" regularity property. Standard interior estimates imply that $u \in H^1_{\text{loc}}(D_1)$, and hence has a well defined restriction to bD_r, for each $r < 1$. Suppose that $\varphi \in \mathscr{C}^\infty(\overline{D_1})$, then a simple integration by parts shows that, for $r < 1$, we have:

$$\int_{bD_r} u(r,e^{i\theta})\varphi(r,e^{i\theta})dz = -2i\left[\int_{D_r} f\varphi dxdy + \int_{D_r} u\bar{\partial}\varphi dxdy\right]. \tag{17}$$

The limit as $r \to 1$ certainly exists on the right hand side and therefore, the left hand side also has a well defined limit.

Clearly, the limiting pairing on the left hand side of (17) only depends on $\varphi \restriction_{bD_1}$, hence we can set

$$\varphi = \sum_{n=0}^{\infty} a_n \bar{z}^n. \tag{18}$$

The Cauchy-Schwarz inequality then shows that

$$\left|\sum_{n=1}^{\infty} u_n(1)a_{n+1}\right| \leq \|f\|_{L^2}\sqrt{\sum_{n=0}^{\infty}\frac{|a_n|^2}{2(n+1)}} + \|u\|_{L^2}\sqrt{\sum_{n=0}^{\infty}\frac{n|a_n|^2}{2}}. \tag{19}$$

This estimate proves the following basic result:

Theorem 3. *Suppose that u and $\bar{\partial}u$ are in $L^2(D_1)$, then $r \mapsto u(r,\cdot)$, is continuous as a map from $(0,1]$ to $H^{-\frac{1}{2}}(bD_r)$. More explicitly,*

$$\sum_{n=-\infty}^{\infty} \frac{|u_n(r)|^2}{\sqrt{1+n^2}} \tag{20}$$

is uniformly bounded for $r \in (0,1]$, and

$$\lim_{r\to 1^-} \sum_{n=-\infty}^{\infty} \frac{|u_n(r)-u_n(1)|^2}{\sqrt{1+n^2}} = 0 \tag{21}$$

In other words u has distributional boundary values in a negative Sobolev space. As a corollary we can also use the Cauchy-Pompieu formula for data of this type. This leads naturally to the question: in what sense does the limit

$$\lim_{z\to bD_1} \frac{1}{2\pi i} \int\limits_{bD_1} \frac{u(1,e^{i\theta})de^{i\theta}}{z-e^{i\theta}} \tag{22}$$

exist? For the case at hand this question can be answered by a direct calculation. For $z \in D_1$, the Cauchy kernel can expanded to give

$$\frac{1}{e^{i\theta}-z} = e^{-i\theta} \sum_{n=0}^{\infty} (e^{-i\theta}z)^n. \tag{23}$$

Using the expansion in equation (23) we deduce that

$$\lim_{r\to 1^-} \int\limits_{bD_1} \frac{u(1,e^{i\theta})d(e^{i\theta})}{e^{i\theta}-re^{i\phi}} = \sum_{n=0}^{\infty} u_n(1)e^{in\phi}. \tag{24}$$

Indeed, if $u(1,\cdot) \in H^s(bD_1)$ for any $s \in \mathbb{R}$, then this limit exists in $H^s(bD_1)$. We denote the projection operator defined on the right hand side of (24) by Π_+. This operator is a pseudodifferential operator of degree zero. It has the following principal symbol:

$$\sigma_0(\Pi_+)(e^{i\theta},\xi) = \begin{cases} 1 \text{ if } \xi > 0 \\ 0 \text{ if } \xi < 0. \end{cases} \tag{25}$$

To see this, we use oscillatory testing: choose ϕ, ψ smooth with compact support, so that $\psi(x) = 1$, and $d\phi(x) = \xi$, then

$$\sigma_0(Q)(x,\xi) = \lim_{\lambda\to\infty} e^{-i\lambda\phi} Q(\psi e^{i\lambda\phi})(x). \tag{26}$$

For the case at hand, let $\phi_\pm = \pm\theta$, and choose ψ with $\psi(e^{i\theta}) = 1$, then

$$\lim_{n\to\infty} e^{-in\phi_\pm}\Pi_+(\psi e^{in\phi_\pm})(e^{i\theta}) = \begin{cases} \lim_{n\to\infty}\sum_{j=-n}^{\infty}\psi_j e^{ij\theta} = \psi(e^{i\theta}) \; (+) \\ \lim_{n\to\infty}\sum_{j=n}^{\infty}\psi_j e^{ij\theta} = 0 \qquad (-) \end{cases} \tag{27}$$

In the case at hand Π_+ is usually called the Szegő projector, though it agrees with what is, more generally, called the Calderon projector for $\bar\partial$.

We now define boundary value problems for the $\bar\partial$-operator on D_1. Let $\mathcal{R}$ denote a pseudodifferential projection acting distributions defined on the boundary. We define an operator $(\bar\partial, \mathcal{R})$ as the unbounded operator on $L^2(D_1)$ with the domain

$$\mathrm{Dom}(\bar\partial, \mathcal{R}) = \{u \in L^2(D_1) : \bar\partial u \in L^2(D_1) \text{ and } \mathcal{R}(u \restriction_{bD_1}) = 0\}. \tag{28}$$

Theorem 3 and the fact that $\mathcal{R}$ is a pseudodifferential operator show that the boundary condition makes sense. It is elementary to prove that this is a closed operator. We now compute the formal adjoint of this operator. A function v is in the domain of the L^2-adjoint if and only if there exists an $f \in L^2(D_1)$ so that, for every $u \in \mathrm{Dom}(\bar\partial, \mathcal{R})$ we have:

$$\langle \bar\partial u, v\rangle = \langle u, f\rangle \tag{29}$$

Taking $v \in \mathcal{C}^\infty(\overline{D_1})$ and integrating by parts we see that

$$\langle \bar\partial u, v\rangle - \langle u, \bar\partial^* v\rangle = \langle u, e^{-i\theta} v\rangle_{bD_1}. \tag{30}$$

For $u \restriction_{bD_1}$ we can take any function of the form $\mathcal{R} f$. Since the boundary term must vanish, for all u, we see that $(\mathrm{Id} - \mathcal{R}^*)e^{-i\theta} v \restriction_{bD_1} = 0$ is necessary as well. Hence the adjoint boundary condition is that defined by the projector $\mathrm{Id} - \mathcal{R}^*$. We suppose that $\mathcal{R}$ is self adjoint, so that is the same as the boundary condition defined by $\mathrm{Id} - \mathcal{R}$.

We now give a condition that implies that this is a Fredholm operator with a compact resolvent. Our condition is expressed in terms of the *comparison operator*

$$\mathcal{T} = \mathcal{R}\Pi_+ + (\mathrm{Id} - \mathcal{R})(\mathrm{Id} - \Pi_+). \tag{31}$$

Theorem 4. *The operator $(\bar\partial, \mathcal{R})$ is a Fredholm operator with a compact resolvent provided that $\mathcal{T}$ is an elliptic pseudodifferential operator.*

Proof. First suppose that u lies in the nullspace of $(\bar\partial, \mathcal{R})$. In this case $\bar\partial u = 0$ and therefore $\mathcal{T}(u \restriction_{bD_1}) = \mathcal{R}(u \restriction_{bD_1}) = 0$. As $\mathcal{T}$ is elliptic this shows that u belongs to a finite dimensional space of smooth functions. Thus the nullspace of $(\bar\partial, \mathcal{R})$ is finite dimensional and contained in $\mathcal{C}^\infty(\overline{D_1})$.

The key to proving the theorem is to show that the range of the operator has finite codimension and that, for data in the domain, we have an estimate like that in (15). If we let $\mathcal{C}$ denote the operator defined by the Cauchy kernel, then we need two basic estimates: for $s \in \mathbb{R}$, the following operators are bounded

$$\begin{aligned} u \in H^s(D_1) &\mapsto \mathcal{C}u \in H^{s+1}(D_1) \\ f \in H^s(bD_1) &\mapsto \mathcal{C}(f\delta_\nu) \in H^{s+\frac{1}{2}}(D_1). \end{aligned} \tag{32}$$

Here δ_ν is the δ-measure normal to bD_1. The map from $H^s(bD_1)$ to $H^{s+\frac{1}{2}}(D_1)$ is denoted $\mathcal{K}$, and called the Poisson operator. The hypothesis of the theorem implies that there is a pseudodifferential operator, $\mathcal{U}$ of degree 0 so that

$$\mathcal{T}\mathcal{U} = \mathrm{Id} - K_1, \quad \mathcal{U}\mathcal{T} = \mathrm{Id} - K_2, \tag{33}$$

where $K_1, K_2 \in \Psi^{-\infty}(bD_1)$, and have finite rank.

Let $v \in L^2(D_1)$ and set

$$u_1 = \mathcal{C}v \text{ and } u_0 = -\mathcal{K}\mathcal{U}\mathcal{R}(u_1 \restriction_{bD_1}). \tag{34}$$

From the Cauchy-Pompieu formula it follows that, in the sense of distributions, $\bar{\partial}(u_0 + u_1) = v$. Moreover, the fact that $u \to u \restriction_{bD_1}$ is bounded from $H^1(D_1) \to H^{\frac{1}{2}}(bD_1)$ and (32) imply that both u_0, and u_1 belong to $H^1(D_1)$; there is a constant C so that

$$\|u_0 + u_1\|_{H^1(D_1)} \leq C\|v\|_{L^2}. \tag{35}$$

What remains is to check the boundary condition. To that end we state a simple but fundamental lemma.

Lemma 1. *If* $\mathcal{T}f \in \operatorname{Im}\mathcal{R}$, *then*

$$\mathcal{T}\Pi_+ f = \mathcal{T}f. \tag{36}$$

.

We see that the boundary value of u_0 is $-\Pi_+\mathcal{U}\mathcal{R}(u_1 \restriction_{bD_1})$, and

$$\mathcal{T}\mathcal{U}\mathcal{R}(u_1 \restriction_{bD_1}) = (\mathrm{Id} - K_1)\mathcal{R}(u_1 \restriction_{bD_1}). \tag{37}$$

Assume that v is chosen so that $K_1\mathcal{R}(u_1 \restriction_{bD_1}) = 0$; this amounts to imposing finitely many, bounded linear conditions. With this assumption we see that

$$\mathcal{T}\mathcal{U}\mathcal{R}(u_1 \restriction_{bD_1}) = \mathcal{R}(u_1 \restriction_{bD_1}) \in \operatorname{Im}\mathcal{R}, \tag{38}$$

hence the lemma implies that

$$\mathcal{T}\Pi_+\mathcal{U}\mathcal{R}(u_1 \restriction_{bD_1}) = \mathcal{T}\mathcal{U}\mathcal{R}(u_1 \restriction_{bD_1}) = \mathcal{R}(u_1 \restriction_{bD_1}). \tag{39}$$

Putting the pieces together, we have shown that, if $v \in L^2(D_1)$ satisfies finitely many linear conditions, then there is a solution $u \in \operatorname{Dom}(\bar\partial, \mathcal{R})$ to the equation

$$\bar\partial u = v, \tag{40}$$

which satisfies $\|u\|_{H^1(D_1)} \leq C\|v\|_{L^2(D_1)}$. Hence the range of the operator contains a closed subspace of finite codimension; it is therefore also of finite codimension and closed. The nullspace is also finite dimensional and this suffices to show that the operator is Fredholm.

To show that $\operatorname{Dom}(\bar\partial, \mathcal{R}) \subset H^1(D_1)$, we suppose that $\bar\partial u = f$, $\mathcal{R}(u \restriction_{bD_1}) = 0$. Let $u_1 = \mathcal{C} f \in H^1(D_1)$. Then $u_0 = u - u_1$ satisfies,

$$\bar\partial u_0 = 0 \text{ and } \mathcal{R}(u_0 \restriction_{bD_1}) = -\mathcal{R}(u_1 \restriction_{bD_1}) \in H^{\frac{1}{2}}(bD_1). \tag{41}$$

Since $\bar\partial u_0 = 0$, we see that

$$-\mathcal{R}(u_1 \restriction_{bD_1}) = \mathcal{T}(u_0 \restriction_{bD_1}) \tag{42}$$

and therefore

$$(\operatorname{Id} - K_2)u_0 \restriction_{bD_1} = -\mathcal{U}\mathcal{R}(u_1 \restriction_{bD_1}) \in H^{\frac{1}{2}}(bD_1). \tag{43}$$

As K_2 is a smoothing operator, we see that there is a constant C_1, such that if $u \in \operatorname{Dom}(\bar\partial, \mathcal{R})$, then

$$\|u\|_{H^1(D_1)} \leq C_1[\|\bar\partial u\|_{L^2(D_1)} + \|u\|_{L^2(D_1)}]. \tag{44}$$

This estimate implies that $\operatorname{Dom}(\bar\partial, \mathcal{R}) \subset H^1(D_1)$, which implies that the operator has a compact resolvent. $\square$

As a corollary of this theorem we can identify the L^2-adjoint of $(\bar\partial, \mathcal{R})$ with $(\bar\partial^*, e^{i\theta}(\operatorname{Id} - \mathcal{R})e^{-i\theta})$.

In fact much more is true: for each $s \in [0, \infty)$, there is a C_s, so that if $\bar\partial u = f \in H^s(D_1)$, and $\mathcal{R}(u \restriction_{bD_1}) = 0$, then $u \in H^{s+1}$ and

$$\|u\|_{H^{s+1}(D_1)} \leq C_s[\|f\|_{H^s(D_1)} + \|u\|_{L^2(D_1)}]. \tag{45}$$

The condition that $\mathcal{T}$ be an elliptic pseudodifferential operator, coupled with the fact that $\mathcal{R}$ is a projection implies that

$$\sigma_0(\mathcal{R})(e^{i\theta}, \xi) = \begin{cases} 1 \text{ if } \xi > 0 \\ 0 \text{ if } \xi < 0. \end{cases} \tag{46}$$

There are many possible projections satisfying this condition.

It is clear that the main conclusions of the theorem remain true if there is an $\mu < 1$ so that the operator $\mathcal{U} : H^s(bD_1) \to H^{s-\mu}(bD_1)$, for all $s \geq -\frac{1}{2}$. In the 1-dimensional such examples are not naturally occuring, though in higher dimensions they are quite important.

A similar discussion applies to study higher order elliptic equations as well. For example if $P = \Delta = (\partial_x^2 + \partial_y^2)$, then $G(x, y) = [2\pi]^{-1} \log |z - w|$ is a fundamental solution. Green's formula states that, if $u \in \mathcal{C}^2(\overline{D}_1)$, then

$$u(z) = \int\limits_{D_1} \Delta u(w) G(z, w) dA_w + \int\limits_{bD_1} [u(w)\partial_{\nu_w} G(z, w) - \partial_{\nu_w} u(w) G(z, w)] ds_w, \tag{47}$$

here ν is the outward unit normal vector to bD_1. If $\Delta u = 0$, then u is determined by its Cauchy data $(u, \partial_\nu u) \restriction_{bD_1}$. The Green's function satisfies estimates much like those satisfied by the Cauchy kernel. The Calderon projector, $\mathcal{P}$, takes a pair of functions defined on the boundary (f, g) to the pair $(u, \partial_\nu u) \restriction_{bD_1}$, where u is the element of $\ker \Delta$, given by

$$u(z) = \int\limits_{bD_1} [f(w)\partial_{\nu_w} G(z, w) - g(w) G(z, w)] ds_w. \tag{48}$$

Boundary conditions are now defined by pseudodifferential projections $\mathcal{R}$ acting on the pair $(u, \partial_\nu u) \restriction_{bD_1}$. The BVP is elliptic if the comparison operator $\mathcal{T} = \mathcal{R}\mathcal{P} + (\mathrm{Id} - \mathcal{R})(\mathrm{Id} - \mathcal{P})$ is elliptic. For simplicity we will largely stick to the case of first order systems in the sequel.

3 Functional Spaces on Manifolds with Boundary

To extend the results of the previous section to the case of a general manifold with boundary we first need to introduce function spaces that are adapted to the study of boundary value problems. We let Ω denote a compact manifold with boundary, which we often think of as a subset of the double, $\widetilde{\Omega}$, which is a compact manifold without boundary. There is a certain amount of subtlety involved in the definitions of spaces of distributions on a manifold with boundary, which, in the end, has to do with what one means by regularity up to the boundary. We usually think of Ω as a closed subset of $\widetilde{\Omega}$, but in this section we often emphasize that point by writing $\overline{\Omega}$.

The main distinction derives from whether one wishes to consider a function to be smooth on Ω if the function and all its derivatives extend smoothly to $b\Omega$, or one wishes to consider a function to be smooth on Ω if the function and all its derivatives vanish along $b\Omega$. In the latter case, its extenstion by zero to all of $\widetilde{\Omega}$ is smooth. We denote the former space of functions by $\mathscr{C}^\infty(\overline{\Omega})$ and the later by $\dot{\mathscr{C}}^\infty(\Omega)$. The elements of the dual space of $\mathscr{C}^\infty(\overline{\Omega})$ are called supported distributions and are denoted by $\dot{\mathscr{C}}^{-\infty}(\Omega)$. The elements of the dual space of $\dot{\mathscr{C}}^\infty(\Omega)$ are called extendible distributions, and are denoted by $\mathscr{C}^{-\infty}(\Omega)$.

An important difference between these two spaces concerns the action of differential operators. As usual this is defined by duality: if P is any differential operator then P^t maps both spaces of smooth functions to themselves, and therefore we can define an action of P on either $\dot{\mathscr{C}}^{-\infty}(\Omega)$ or $\mathscr{C}^{-\infty}(\Omega)$ by duality:

$$\langle Pu, \varphi\rangle \overset{d}{=} \langle u, P^t\varphi\rangle. \tag{49}$$

If $u \in \mathscr{C}^{-\infty}(\Omega)$, then we take $\varphi \in \dot{\mathscr{C}}^\infty(\Omega)$ in equation (49), while if $u \in \dot{\mathscr{C}}^{-\infty}(\Omega)$, then we take $\varphi \in \mathscr{C}^\infty(\overline{\Omega})$. Of course $\mathscr{C}^\infty(\overline{\Omega})$ is a subset of both $\dot{\mathscr{C}}^{-\infty}(\Omega)$ and $\mathscr{C}^{-\infty}(\Omega)$. If $u \in \mathscr{C}^\infty(\overline{\Omega})$, then the meaning of Pu depends on whether we think of it as an extendible or a supported distribution. The difference in the two definitions is a distribution with support on $b\Omega$. For example, if $u \in \mathscr{C}^\infty(\overline{D_1})$ and $P = \bar\partial$ then

$$\bar\partial_{\text{ext}}u - \bar\partial_{\text{supp}}u = \delta(r-1)\frac{u(1, e^{i\theta})e^{i\theta}d\theta}{2}. \tag{50}$$

A distribution $u \in \dot{\mathscr{C}}^{-\infty}(\Omega)$ if and only if there is an element $U \in \mathscr{C}^{-\infty}(\widetilde{\Omega})$ such that $\operatorname{supp} U \subset \overline{\Omega}$, which defines u. In this case u is defined on an element $\varphi \in \mathscr{C}^\infty(\overline{\Omega})$ by

$$u(\varphi) = U(\widetilde{\varphi}), \tag{51}$$

where $\widetilde{\varphi}$ is any extension of φ to an element of $\mathscr{C}^\infty(\widetilde{\Omega})$, for example the Seeley extension $E\varphi$. Because $\operatorname{supp} U \subset \overline{\Omega}$, the value of $U(\widetilde{\varphi})$ is independent of which extension is used. The H^s-norm is defined on supported distributions by setting

$$\|u\|_s = \|U\|_{H^s(\widetilde{\Omega})}. \tag{52}$$

The subspace of $\dot{\mathscr{C}}^{-\infty}(\Omega)$ for which this norm is finite is denoted by $\dot{H}^s(\Omega)$. The important thing to note about this space is that in order for u to be smooth in this sense, that is belonging to $\dot{H}^s(\Omega)$, for a large value of s, it must have many derivatives in Ω, which *vanish* at the boundary. This is because $\operatorname{supp} U \subset \Omega$. The space $\dot{\mathscr{C}}^\infty(\Omega)$ is a dense subset of $\dot{\mathscr{C}}^{-\infty}(\Omega)$.

On the other hand $\dot{\mathscr{C}}^\infty(\Omega)$ is a closed subspace of $\mathscr{C}^\infty(\widetilde{\Omega})$ and therefore the Hahn-Banach theorem implies that if $u \in \mathscr{C}^{-\infty}(\Omega)$, then there is $U \in \mathscr{C}^{-\infty}(\widetilde{\Omega})$ that extends u. We define the H^s-norm for this space of distributions by

$$\|u\|_s = \inf_{U \text{ extending } u} \|U\|_{H^s}. \tag{53}$$

The subspace of $\mathscr{C}^{-\infty}(\Omega)$ for which this norm is finite is denoted by $H^s(\Omega)$. From the definition of the norm, it is again clear that a distribution u is smooth in this sense if it has many derivatives with *smooth extensions* to $b\Omega$, rather than having to vanish to high order along $b\Omega$. The space $\mathscr{C}^\infty(\overline{\Omega})$ is dense in $\mathscr{C}^{-\infty}(\Omega)$. It is clear that for every $s \in \mathbb{R}$, we have a natural map: $\dot{H}^s(\Omega) \to H^s(\Omega)$. This map turns out to be injective if $s \geq -\frac{1}{2}$ and surjective if $s \leq \frac{1}{2}$. The L^2-pairing on Ω between $\mathscr{C}^\infty(\overline{\Omega})$ and $\dot{\mathscr{C}}^\infty(\Omega)$ can be extended to show that, for all $s \in \mathbb{R}$, we have the isomorphisms

$$[H^s(\Omega)]' \simeq \dot{H}^{-s}(\Omega) \text{ and } [\dot{H}^s(\Omega)]' \simeq H^{-s}(\Omega). \tag{54}$$

If $s > \frac{1}{2}$, then restriction to the boundary extends to define a continuous *trace* map:

$$\tau : H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(b\Omega). \tag{55}$$

Because this map is not defined for $s = \frac{1}{2}$, it is convenient to work with spaces that treat regularity in the tangential and normal directions slightly differently. These spaces greatly facilitate the analysis of operators defined on $L^2(\Omega)$. We first define these spaces for the half space $\mathbb{R}^n_+$. Let $x' = (x_1, \ldots, x_{n-1}), \xi' = (\xi_1, \ldots, \xi_{n-1})$, and define the tangential Fourier transform to be

$$\widetilde{u}(\xi', x_n) = \int\limits_{\mathbb{R}^{n-1}} u(x', x_n)e^{-ix'\cdot\xi'}dx'. \tag{56}$$

For m a non-negative integer and $s \in \mathbb{R}$ we define

$$\|u\|^2_{(m,s)} = \sum_{j=0}^{m} \int\limits_{\mathbb{R}^{n-1}} \int\limits_0^\infty |\partial^j_{x_n}\widetilde{u}(\xi', x_n)|^2(1 + |\xi'|^2)^{s-j} d\xi' dx_n. \tag{57}$$

The space $H_{(m,s)}(\mathbb{R}^n_+)$ is the closure of $\mathscr{C}^\infty_c(\overline{\mathbb{R}}^n_+)$ with respect to this norm. It consists of all distributions in $\mathscr{C}^{-\infty}(\overline{\mathbb{R}^n_+})$ such that $\partial^j_{x_n}\widetilde{u}$ is a function for $0 \leq j \leq m$, and the norm in (57) is finite. The corresponding space of supported distributions, $\dot{H}_{(m,s)}(\mathbb{R}^n_+)$, is defined as the closure of $\mathscr{C}^\infty_c(\operatorname{int}\mathbb{R}^n_+)$ with respect to this norm.

These spaces are useful for two reasons:

Theorem 5. *If m is a positive integer and $0 \leq j < m$, then the map*

$$\mathscr{C}^\infty(\mathbb{R}^n_+) \ni u \to \partial^j_{x_n} u(\cdot, x_n),$$

for $x_n \geq 0$, extends as a continuous map from $H_{(m,s)}(\mathbb{R}^n_+) \to H^{s+m-j-\frac{1}{2}}(\mathbb{R}^{n-1})$. Moreover, $x_n \to \partial^j_{x_n} u(\cdot, x_n)$ is continuous from $[0, 1)$ to $H^{s+m-j-\frac{1}{2}}(\mathbb{R}^{n-1})$.

Of particular note is the fact that $H^{\frac{1}{2}}(\mathbb{R}^n_+) \supset H_{(1,-\frac{1}{2})}(\mathbb{R}^n_+)$. While the restriction to the boundary is not defined for $u \in H^{\frac{1}{2}}(\mathbb{R}^n_+)$, it is defined, as an element of $L^2(b\mathbb{R}^n_+)$, for $u \in H_{(1,-\frac{1}{2})}(\mathbb{R}^n_+)$.

Because they behave well under localization and change of coordinate, these spaces can be transferred to a manifold with boundary. For Ω a compact manifold with boundary we let $H_{(m,s)}(\Omega)$, $\dot{H}_{(m,s)}(\Omega)$ denote the corresponding function spaces. Suppose that (V, φ) is either a boundary or interior coordinate chart, and $\psi \in \mathscr{C}^\infty_c(V)$. A distribution u, defined on Ω, belongs to one of these spaces if $(\psi u) \circ \varphi^{-1}$ belongs to the corresponding space in $\mathbb{R}^n_+$. Using the tubular neighborhood theorem, Theorem 5 extends to this situation:

Theorem 6. *Let Ω be a compact manifold with boundary, r a defining function for $b\Omega$, $\mathring{\Omega} = \{r > 0\}$. If m is a positive integer and $0 \leq j < m$, then the map*

$$\mathscr{C}^\infty(\Omega) \ni u \to \partial_\nu^j u(\cdot, r),$$

for $r \geq 0$, extends as a continuous map from $H_{(m,s)}(\Omega) \to H^{s+m-j-\frac{1}{2}}(b\Omega)$. Moreover, $r \to \partial_\nu^j u(\cdot, r)$ is continuous from $[0, 1)$ to $H^{s+m-j-\frac{1}{2}}(b\Omega)$.

The connection with the analysis of boundary value problems for differential operators is provided by the following weak, but extremely useful regularity theorem. In the situation described in Theorem 6, a differential operator, P of degree m is called *transversely elliptic* if $\sigma_m(P)(x, dr)$ is invertible for all $x \in b\Omega$. In other words, the boundary of Ω is non-characteristic for P.

Theorem 7. *Suppose that Ω is a compact manifold with boundary and P is a transversely elliptic operator or order m. Suppose that $u \in L^2(\Omega) = H_{(0,0)}(\Omega)$, and $Pu \in L^(\Omega)$, then $u \in H_{m,-m}(\Omega)$.*

As indicated by the identification, $L^2(\Omega) = H_{(0,0)}(\Omega)$ we interpret u as an extendible distribution when defining Pu. The theorem has a very useful corollary, which is a generalization of Theorem 3.

Corollary 1. *If u, Pu both belong to $L^2(\Omega)$, then, for $0 \leq j < m$ the maps $r \mapsto \partial_\nu^j u(r, \cdot)$ are continuous from $[0, 1)$ to $H^{-(\frac{1}{2}+j)}(b\Omega)$. In particular,*

$$\Gamma u = (u(r, \cdot), \partial_\nu u(r, \cdot), \ldots, \partial_\nu^{m-1} u(r, \cdot)) \restriction_{r=0}$$

is well defined as a distribution on the boundary.

The range of Γ consists of distributional sections of a vector bundle $E \to b\Omega$. Suppose that $\mathcal{R}$ is a pseudodifferential operator defined on $b\Omega$, which acts on sections of E. As is well known, pseudodifferential operators act continuously on distributions. Thus we can define an unbounded operator on $L^2(\Omega)$, with domain

$$\mathrm{Dom}(P, \mathcal{R}) = \{u \in L^2 : \; Pu \in L^2 \text{ and } \mathcal{R}\Gamma u = 0\}. \tag{58}$$

It is not difficult to show that these operators are closed. The question of principal interest is to know when these operators are Fredholm.

Good references for the material in this section are [9] and [11].

4 Estimates for Operators Satisfying the Transmission Condition

In the sequel we let Ω be a compact with boundary, $\widetilde{\Omega}$, its double and and E, F complex vector bundles over $\widetilde{\Omega}$. We suppose that P is a first order elliptic, differential operator from sections of E to sections of F. In general we are rather sloppy about which bundle is which, largely leaving them out of the notation, except when absolutely necessary.

The ellipticity of P means that for each non-zero $\xi \in T_x^*\widetilde{\Omega}$, the principal symbol, $p_0(x, \xi)$ is an invertible element of $\mathrm{Hom}(E_x, F_x)$. This in turn is well known to imply that there is a parametrix for P, that is an operator $Q \in \Psi^{-1}(\widetilde{\Omega}; F, E)$ so that

$$PQ = \mathrm{Id}_F - K_1 \quad QP = \mathrm{Id}_E - K_2 \tag{59}$$

with K_1, K_2 smoothing operators of finite rank. (The smoothing operators are those with Schwartz kernels in $\mathscr{C}^\infty(\widetilde{\Omega} \times \widetilde{\Omega})$ tensored with the appropriate vector bundle.) The symbol of the operator Q has an asymptotic expansion:

$$\sigma(Q) \sim \sum_{j \geq 0} q_j \tag{60}$$

For each x, $q_j(x, \xi)$ is a *rational function* of ξ of degree $-1 - j$. Indeed, the denominator of q_j is just a power of $\det p_0(x, \xi)$. This implies that Q is an operator satisfying the transmission condition.

Definition 1. A classical pseudodifferential operator in $Q \in \Psi^*(\widetilde{\Omega})$ satisfies the transmission condition, if whenever $u \in \mathscr{C}^\infty(\overline{\Omega})$ and we denote by u_0 the extension of u, by zero, to all of $\widetilde{\Omega}$, then $Qu_0 \restriction_{\mathrm{int}\,\Omega}$ extends to define an element of $\mathscr{C}^\infty(\overline{\Omega})$.

There is a simple symbolic criterion for a classical pseudodifferential operator to satisfy the transmission condition. It is a local condition; we introduce coordinates, $x = (x', x_n)$ in a neighborhood, U of a point $p \in bY$ so that $p \leftrightarrow x = 0$,

$U \cap bY = \{x_n = 0\}$ and $x_n > 0$ in the interior of $Y \cap U$. Assume that Q is a classical pseudodifferential operator of order m such that (complete) symbol of Q has an asymptotic expansion:

$$\sigma(Q)(x,\xi) = q(x,\xi) \sim \sum_{j=-\infty}^{0} q_j(x,\xi), \tag{61}$$

where

$$q_j(x,\lambda\xi) = \lambda^{m-j} q_j(x,\xi) \text{ for } \lambda > 0. \tag{62}$$

The operator satisfies the transmission condition with respect to Y, provided

$$q_j(x', x_n, \xi', \xi_n) - e^{-\pi i(m+j)} q_j(x', x_n, -\xi', -\xi_n) \tag{63}$$

vanishes to infinite order along the inward pointing conormal bundle to bY, i.e., where $x_n = 0, \xi' = 0$ and $\xi_n > 0$. As shown in [9], this is a coordinate invariant condition and so can be used to check the transmission condition for pseudodifferential operators defined on manifolds.

In our applications the terms in the asymptotic expansion of $\sigma(Q)$ are homogeneous, rational functions of ξ, which therefore satisfy the following condition:

$$q_j(x,\lambda\xi) = \lambda^{m-j} q_j(x,\xi), \text{ for all } \lambda \in \mathbb{C}^*. \tag{64}$$

We call these properties the *strengthened transmission condition*. In the arguments which follow we often use this stronger condition as it simplifies the exposition.

To understand the symbolic properties underlying the transmission condition we consider a function $u \in \mathscr{C}_c^\infty(\overline{\mathbb{R}}_+^n)$. If

$$a(x', \xi_n) = \int_0^\infty u(x', x_n) e^{-ix_n\xi_n} dx_n, \tag{65}$$

then $a(x', \xi_n)$ has an asymptotic expansion

$$a(x', \xi_n) \sim \sum_{j=1}^{\infty} \frac{\partial_{x_n}^j u(x', 0)}{(i\xi_n)^j} = \sum_{j=1}^{\infty} a_j(x', \xi_n). \tag{66}$$

Let $\Gamma^+ \subset \mathbb{C}$ be the contour $(-\infty, R] \cup \{Re^{i\theta} : \theta \in [\pi, 0]\} \cup [R, \infty)$. The function a_j satisfies

$$a_j(x', \xi_n) = a_j(x', 1)\xi_n^{-j}. \tag{67}$$

For such a function, the oscillatory integral

$$\int_{\Gamma^+} a_j(x', \xi_n) e^{ix_n\xi_n} d\xi_n, \tag{68}$$

is well defined. In fact, if $x_n > 0$, then a simple contour deformation argument shows that this integral vanishes. As an oscillatory integral, this remains true for a function of the form $a(x')\xi_n^j$, for any $j \in \mathbb{Z}$.

Now suppose that v is a compactly supported distribution with a representation, as an oscillatory integral, of the form:

$$v(x', x_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} b(x', \xi_n) e^{ix_n\xi_n} d\xi_n, \tag{69}$$

where b has an asymptotic expansion

$$b(x', \xi_n) \sim \sum_{j=-\infty}^{m} b_j(x', \xi_n), \tag{70}$$

where $b_j(x', \xi_n) = b_j(x', 1)\xi_n^j$. For $x_n > 0$, and $N > 0$ we observe that

$$\begin{aligned} v(x', x_n) &= v(x', x_n) - \frac{1}{2\pi} \sum_{j=1}^{N} \int_{\Gamma^+} b_j(x', \xi_n) e^{ix_n\xi_n} d\xi_n \\ &= \frac{1}{2\pi} \Bigg[\int_{|\xi_n|>R} [b(x', \xi_n) - \sum_{j=1}^{N} b_j(x', \xi_n)] e^{ix_n\xi_n} d\xi_n + \\ & \int_{-R}^{R} b(x', \xi_n) e^{ix_n\xi_n} d\xi_n - \int_{\pi}^{0} \sum_{j=1}^{N} b_j(x', Re^{i\theta}) e^{ix_n Re^{i\theta}} R de^{i\theta} \Bigg] \end{aligned} \tag{71}$$

The compactly supported terms are smooth functions and the integral over $|\xi_n| > R$ is a $C^{N-1}(\overline{\mathbb{R}}^n_+)$ function. As N is arbitrary, the restriction of v to $\operatorname{int}\mathbb{R}^n_+$ extends to $\mathscr{C}^\infty(\overline{\mathbb{R}}^n_+)$. This simple analytic continuation argument explains the essence of the transmission condition. In this section we use this sort of contour deformation to establish mapping properties for Q acting on $H^s(\Omega)$ as well as its effect on distributions supported on $b\Omega$ itself. The result we obtain is

Theorem 8. *Suppose that Q is a classical pseudodifferential operator of order m, on $\widetilde{\Omega}$, satisfying the strengthened transmission condition with respect to Ω. For $s \in \mathbb{R}$, $Q : H^s(\Omega) \to H^{s-m}(\Omega)$.*

To prove this theorem we use the following local result.

Proposition 1. *Let Q be an classical pseudodifferential operator of integral degree m on $\mathbb{R}^n$ satisfying the strengthened transmission condition with respect to $\mathbb{R}^n_+$. If $f \in H^s_{comp}(\mathbb{R}^n_-)$, then, for any $k \in \mathbb{N}_0$, we have:*

$$Qf \restriction_{\mathbb{R}^n_+} \in H_{(k,s-m-k),loc}(\mathbb{R}^n_+) \tag{72}$$

Proof. Because pseudodifferential operators are pseudolocal, it follows that $Qf \restriction_{\operatorname{int}\mathbb{R}^n_+}$ is smooth. As $\mathscr{C}^\infty(\mathbb{R}^n_-)$ is dense in $H^s(\mathbb{R}^n_-)$, it suffices to show that that, for every s, there is a constant C, such that for $f \in \mathscr{C}^\infty(\mathbb{R}^n_-)$, and $\varphi \in \mathscr{C}^\infty_c(\overline{\mathbb{R}}^n_+)$, we have

$$\|\varphi Q(f)\|_{j,s-m+\frac{1}{2}-j} \leq C\|f\|_{H^s(\mathbb{R}^n_-)}. \tag{73}$$

Let $q \sim \sum q_j$, where $q_j(x,\xi)$, is a homogeneous rational function in ξ, of degree $m-j$.

Remark 1. In the following argument, which is modeled on the proof of Theorem 18.2.17 in [9], we proceed somewhat formally. Let $\phi \in \mathscr{C}^\infty_c(\mathbb{R})$, with support in $[-1,1]$, and total integral 1. For each $\epsilon > 0$, we let $\phi_\epsilon(x) = \epsilon^{-1}\phi(\epsilon^{-1}x)$. To be entirely rigorous, we should work with the regularized functions $f_\epsilon = f *_{x_n} \phi_\epsilon$, which belong to $\mathscr{C}^\infty_c(\mathbb{R}^n)$, derive the formulæ below, with $\epsilon > 0$, and allow ϵ to tend to zero. This argument is quite standard and we leave it to the interested reader.

We begin with a lemma. Let $\psi(\xi')$ be a smooth function, with $\psi(\xi') = 0$, if $\|\xi'\| < 1$, and $\psi(\xi') = 1$, for $\|\xi'\| > 2$.

Lemma 2. *If $f \in H^s_{comp}(\mathbb{R}^n_-)$, then*

$$Q_0(f) = \left[\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} q(x,\xi)(1-\psi(\xi'))\hat{f}(\xi)e^{ix\cdot\xi}\right]_{\mathbb{R}^n_+} \tag{74}$$

belongs to $\mathscr{C}^\infty(\overline{\mathbb{R}}^n_+)$.

Proof of the Lemma. For each N, there is an R so that, if $\|\xi'\| \leq 2$, then the poles of $\{q_j(x,\xi',\xi_n) : \ j = 0,\dots,N\}$ lie inside $D_R(0)$. Because f is supported in the lower half space, its Fourier transform extends to be a holomorphic function of ξ_n is the upper half space. Let Γ_+ denote the arc, in the ξ_n-plane, $\{\xi_n = Re^{i\theta}, \theta \in [\pi, 0]\}$. Using the analyticity properties of $\hat{f}$ and the q_j, we can therefore argue as in equation (71), that for $x_n > 0$, we have

$$Q_0(f)(x', x_n) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^{n-1}} \int\limits_{|\xi_n|>R} \left(q(x,\xi) - \sum_{j=0}^{N} q_j(x,\xi) \right) e^{ix_n\xi_n}(1-\psi(\xi'))\times$$
$$\hat{f}(\xi) d\xi_n e^{ix\cdot\xi'} d\xi'$$
$$+\frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^{n-1}} \int\limits_{|\xi_n|\leq R} q(x,\xi) e^{ix_n\xi_n}(1-\psi(\xi'))\hat{f}(\xi) d\xi_n e^{ix\cdot\xi'} d\xi'$$
$$-\frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^{n-1}} \int\limits_{\Gamma_+} \sum_{j=0}^{N} q_j(x,\xi) e^{ix_n\xi_n}(1-\psi(\xi'))\hat{f}(\xi) d\xi_n e^{ix\cdot\xi'} d\xi' \tag{75}$$

By taking N large, we can make the difference appearing in the first integral vanish as rapidly as we like, thereby making the first integral as smooth as we wish. The other two terms are integrals over compact sets, which therefore define $\mathscr{C}^\infty$-functions in $\{x_n \geq 0\}$. The existence of an estimate, as above follows from the closed graph theorem. $\square$

From the lemma it suffices to consider

$$Q_1(f)(x) = \left[\frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} q(x,\xi)\psi(\xi')\hat{f}(\xi) e^{ix\cdot\xi} d\xi \right]_{\mathbb{R}^n_+}, \tag{76}$$

for $f \in \mathscr{C}^\infty_c(\overline{\mathbb{R}}^n_-)$. For each $j \in \mathbb{N}_0$, define the pseudodifferential operator:

$$Q_{j1}(f)(x) = \left[\frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} q_j(x,\xi)\psi(\xi')\hat{f}(\xi) e^{ix\cdot\xi} d\xi \right]_{\mathbb{R}^n_+}. \tag{77}$$

For $N \in \mathbb{N}$, the difference $Q - \sum_{j<N} Q_{j1}$ is a pseudodifferential operator of order $-N$, and therefore it suffices to prove estimates for $Q_{j1}(f),\ j = 0, \dots$

To prove these estimates, we take the tangential Fourier transform of $Q_{j1}(f)$. We let

$$\widetilde{q}_j(\eta', x_n, \xi) = \int\limits_{\mathbb{R}^{n-1}} q_j(x', x_n, \xi) e^{-ix'\cdot\eta'} dx'. \tag{78}$$

From the symbolic estimates, it follows that, for each $M \in \mathbb{N}$, there is a constant, C_M, so that

$$\widetilde{q}_j(\eta', x_n, \xi) \leq C_M \frac{\|\xi\|^{m-j}}{(1+\|\eta'\|)^M}. \tag{79}$$

There is a universal constant, C' so that if $f \in H^s(\mathbb{R}^n_-)$, then

$$\int\limits_{\mathbb{R}^{n-1}} \int\limits_{-\infty}^{0} |\tilde{f}(\xi', x_n)|^2 (1 + \|\xi'\|)^{2s} dx_n d\xi' \leq C' \|f\|_{H^s(\mathbb{R}^n_-)}. \tag{80}$$

Moreover, $\hat{f}(\xi', \xi_n)$ analytically extends to $\{\operatorname{Im} \xi_n > 0\}$; for $\beta > 0$, the Cauchy-Schwarz inequality implies the estimate:

$$\begin{aligned} |\hat{f}(\xi', \alpha + i\beta)|^2 &= \left| \int\limits_{-\infty}^{0} \tilde{f}(\xi', x_n) e^{-ix_n(\alpha+i\beta)} dx_n \right|^2 \\ &\leq \frac{\int\limits_{-\infty}^{0} |\tilde{f}(\xi', x_n)|^2 dx_n}{2\beta}. \end{aligned} \tag{81}$$

As $q_j(x, \xi', \xi_n)$ is homogeneous in ξ, its poles, as a function of ξ_n, in the upper half plane, are of the form $\{\|\xi'\| w_l(\omega') : \ j = 1, \ldots, L\}$; we let

$$w_l(\omega') = \alpha_l(\omega') + i\beta_l(\omega'). \tag{82}$$

Here $\|\xi'\|\omega' = \xi'$. We can use contour integration to evaluate the ξ_n-integral. Assuming, for the moment, that all the poles of q_j are simple, we obtain that

$$\begin{aligned} Q_{j1}(f)(x', x_n) = \sum_{l=1}^{L} \frac{i}{(2\pi)^{n-1}} \int\limits_{\mathbb{R}^{n-1}} & q_j^{(l)}(x', x_n, \xi', \|\xi'\| w_l(\omega')) \psi(\xi') \\ & \hat{f}(\xi', \|\xi'\| w_l(\omega')) e^{ix_n \|\xi'\| w_l(\omega')} e^{ix' \cdot \xi'} d\xi', \end{aligned} \tag{83}$$

where

$$q_j^{(l)}(x, \xi', \xi_n) = (\xi_n - \|\xi'\| w_l(\omega')) q_j(x, \xi', \xi_n). \tag{84}$$

Away from $\xi' = 0$, these are homogeneous symbols of degree $m - j + 1$. Clearly it suffices to separately estimate each term in (83). For each M, there is a constant C_M such that the tangential Fourier transform of $q_j^{(l)}$ satisfies the estimate:

$$\widetilde{q}_j^{(l)}(\eta', x_n, \xi) \leq C_M \frac{\|\xi\|^{m-j+1}}{(1 + \|\eta'\|)^M}. \tag{85}$$

This shows that the tangential Fourier transform of each term in the sum satisfies the estimate:

$$|\widetilde{Q}_{j1}^{(l)} f(\eta', x_n)| \leq C \int\limits_{\mathbb{R}^{n-1}} \frac{\|\xi'\|^{m-j+1} \psi(\xi') |\hat{f}(\xi', \|\xi'\| w_l(\omega'))| e^{-x_n \beta_l(\omega') \|\xi'\|} d\xi'}{(1 + \|\xi' - \eta'\|)^M} \tag{86}$$

We apply the Cauchy-Schwarz inequality to the right hand side of (86) to obtain:

$$|\widetilde{Q}_{j1}^{(l)} f(\eta', x_n)|^2 \leq C \int_{\mathbb{R}^{n-1}} \frac{\|\xi'\|^{2(m-j+1)} \psi(\xi') |\hat{f}(\xi', \|\xi'\| w_l(\omega'))|^2 e^{-2x_n \beta_l(\omega')\|\xi'\|} d\xi'}{(1 + \|\xi' - \eta'\|)^M} \times \int_{\mathbb{R}^{n-1}} \frac{d\xi'}{(1 + \|\xi' - \eta'\|)^M} \tag{87}$$

We choose M sufficiently large that the second integral converges. By ellipticity and compactness, the the imaginary part of the exponent $\beta_l(\omega') \geq \beta > 0$, as ω' varies over the unit sphere. Using this estimate, and the estimate in (81) we see that

$$\int_{\mathbb{R}^{n-1}} \int_0^\infty |\widetilde{Q}_{j1}^{(l)} f(\eta', x_n)|^2 (1 + \|\eta'\|)^{2(s+j-m)} dx_n d\eta' \leq \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{\psi(\xi') \|\xi'\|^{2(m-j+1)} (1 + \|\eta'\|)^{2(s+j-m)}}{(1 + \|\xi' - \eta'\|)^M \beta^2 \|\xi'\|^2} \times \int_{-\infty}^0 |\tilde{f}(\xi', y_n)|^2 dy_n d\xi' d\eta'. \tag{88}$$

One power of $\|\xi'\|$ in the denominator results from performing the x_n-integral, and the other comes from (81). To complete the proof we use the following elementary lemma:

Lemma 3. *If $t \in \mathbb{R}$ and $M > 2t + n$, then there is a constant C so that:*

$$\int_{\mathbb{R}^{n-1}} \frac{(1 + \|\eta'\|)^{2t}}{(1 + \|\xi' - \eta'\|)^M} \leq C(1 + \|\xi'\|)^{2t}. \tag{89}$$

The proof is left to the reader.

Interchanging the order of the η' and ξ' integrations in (88), we apply the lemma to obtain that

$$\int_{\mathbb{R}^{n-1}} \int_0^\infty |\widetilde{Q}_{j1}^{(l)} f(\eta', x_n)|^2 (1 + \|\eta'\|)^{2(s+j-m)} dx_n d\eta' \leq C \int_{\mathbb{R}^{n-1}} (1 + \|\xi'\|)^{2s} \int_{-\infty}^0 |\tilde{f}(\xi', y_n)|^2 dy_n d\xi'. \tag{90}$$

In light of equation (80), this proves the proposition, for $k = 0$, under the assumption that all poles of q_j are simple. The latter assumption is easily removed, by using Cauchy's formula

$$\frac{k!}{2\pi i}\int\limits_{\mathbb{R}} \frac{f(w)dw}{(z-w)^{k+1}} = \partial_z^k f(z), \tag{91}$$

the Leibniz formula, and symbolic estimates. It is seen to give the same result, as in the simple case, if we replace (81) with the estimate

$$|\partial_{\xi_n}^k \hat{f}(\xi', \alpha + i\beta)|^2 \leq C_k \frac{\int\limits_{-\infty}^{0} |\tilde{f}(\xi', x_n)|^2 dx_n}{\beta^{2k+1}} \tag{92}$$

To estimate derivatives in the x_n direction, we simply differentiate (83). Each derivative replaces the symbol, in ξ', with a symbol of one higher degree and the argument is otherwise the same. □

Proof of the Theorem. Let $f \in H^s(\Omega)$. Using the Seeley extension theorem we know that there is a constant C_s, and an extension f' of f to $\widetilde{\Omega}$, so that

$$\|f'\|_{H^s(\widetilde{\Omega})} \leq C_s \|f\|_{H^s(\Omega)}. \tag{93}$$

Because Q is a pseudodifferential operator of order m, it follows that there is a constant C_s' so that

$$\|Qf'\|_{H^{s-m}(\widetilde{\Omega})} \leq C_s' \|f'\|_{H^s(\widetilde{\Omega})}. \tag{94}$$

In light of the definition of the norm on $H^s(\Omega)$, this shows that $Qf' \restriction_\Omega \in H^{s-m}(\Omega)$. If we let

$$f_- = \begin{cases} f' \restriction_{\widetilde{\Omega} \setminus \Omega} \\ 0 \text{ in } \Omega, \end{cases} \tag{95}$$

then we need only show that $Qf_- \restriction_\Omega \in H^{s-m}(\Omega)$. To prove this we observe that it is enough to prove estimates in boundary coordinate charts. The needed estimates follow immediately from the proposition, and the well known relations amongst the spaces $H^{s-m}(\Omega)$ and $H_{(k,s-m-k)}(\Omega)$. This completes the proof of the Theorem. □

Using essentially the same argument we can treat the case of a single layer potential:

Theorem 9. *Suppose that Q is a classical pseudodifferential operator of order m, on $\widetilde{\Omega}$, satisfying the strengthened transmission condition with respect to Ω. If r is a defining function for $b\Omega$, and $f \in \mathscr{C}^\infty(b\Omega)$, then $Q(f \otimes \delta(r))$ extends to define a function in $\mathscr{C}^\infty(\Omega)$. If $f \in H^s(b\Omega)$, then, for $k \in \mathbb{N}_0$, we $Q(f \otimes \delta(r)) \in H_{(k,s-m-k-\frac{1}{2})}(\Omega)$.*

Remark 2. Similar results hold for multiple layer potentials, i.e. distributions of the form $Q(f \otimes \delta^{[j]}(r))$.

5 The Calderon Projection

We now let Q denote a parametrix for a first order differential operator, P acting between sections of a vector bundles E, and F :

$$P : \mathscr{C}^\infty(\widetilde{\Omega}; E) \longrightarrow \mathscr{C}^\infty(\widetilde{\Omega}; F), \tag{96}$$

A typical example is a Dirac operator. To simplify the discussion a little bit, we assume that P is actually invertible, so that Q can be taken to be a fundamental solution; that is the error terms in (59) actually vanish. For the case of a Dirac operator this can always be arranged.

The operator Q is a classical pseudodifferential operator. Indeed, its symbol has an asymptotic expansion:

$$\sigma(Q)(x,\xi) \sim \sum_{j=0}^{\infty} q_j(x,\xi), \tag{97}$$

with $q_j(x,\xi)$ a rational functional of ξ, homogeneous of degree $-1-j$. The denominator of q_j can be taken to be a power of $\det(p_0(x,\xi))$.

We suppose that a Riemannian metric is fixed on $\widetilde{\Omega}$, and Hermitian inner products on E, F, though this data is often suppressed in what follows. When needed $\langle\cdot,\cdot\rangle_E$, e.g. denotes the fiber inner product on E. If $\mathscr{H}$ is a Hilbert space, then $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ denotes the Hilbert space inner product. Fix a defining function r for $b\Omega$ in $\widetilde{\Omega}$, such that dr has unit length along $b\Omega$.

We let Ω_+ denote the subset of $\widetilde{\Omega}$ where $r \geq 0$, and Ω_- the subset where $r \leq 0$. We also let Y_ϵ denote the set $\{r = \epsilon\}$. As Q is a fundamental solution, it is clear that $u = Q(g \otimes \delta(r))$ belongs to the nullspace of P on $\widetilde{\Omega} \setminus b\Omega$. We denote the restrictions to the components of the complement of $b\Omega$ by $u_\pm$. It follows from Theorem 9 that if $g \in H^s(b\Omega; F\restriction_{b\Omega})$, then $u_\pm \in H_{(1,s-\frac{1}{2})}(\Omega_\pm; E)$. Let τ_ϵ denote restriction to $\{r = \epsilon\}$. From Theorem 6 it follows that $\tau_\epsilon u$ is well defined as an element of $H^s(Y_\epsilon)$, moreover the maps

$$\begin{aligned} [0,1] \ni \epsilon &\mapsto \tau_\epsilon u_+ \\ [-1,0] \ni \epsilon &\mapsto \tau_\epsilon u_- \end{aligned} \tag{98}$$

are continuous. Note, however, that generally $\tau_0 u_+ \neq \tau_0 u_-$.

We need to establish the properties of the maps

$$\mathcal{P}_{\pm} f = \lim_{\epsilon\to 0^{\pm}} \tau_{\epsilon} Q(\sigma_1(P, \pm dr)(f \otimes \delta(r))). \tag{99}$$

Here f is a distributional section of $E \restriction_{b\Omega}$, and $\sigma_1(P, dr)$ is the principal symbol of P in the co-normal direction dr. If $u_{\pm}$ belongs to the nullspace of P on $\Omega_{\pm}$, then it follows from Green's formula, and the fact that Q is a fundamental solution that

$$u_{\pm}(p) = Q\left(\sigma_1(P, \pm dr)[u_{\pm} \restriction_{b\Omega_{\pm}} \otimes \delta(r)]\right)(p) \text{ for } p \in \Omega_{\pm}. \tag{100}$$

Hence $\mathcal{P}_{\pm} u_{\pm} = u_{\pm} \restriction_{b\Omega_{\pm}}$. This shows that $\mathcal{P}_{\pm}$ are projection operators. These are the Calderon projectors for the operator P. Indeed, as Q is a fundamental solution,

$$PQ[\sigma_1(P, dr) f \otimes \delta(r)] = \sigma_1(P, dr) f \otimes \delta(r). \tag{101}$$

Hence, if f is a smooth section of E along $b\Omega$ and φ is a smooth section of F in $\widetilde{\Omega}$, then

$$\begin{aligned}
\int_{b\Omega} \langle \sigma_1(P, dr) f, \varphi\rangle_F &= \lim_{\epsilon\to 0^+} \int_{\{|r|>\epsilon} \langle Q(\sigma_1(P, dr) f \otimes \delta(r)), P^t\varphi\rangle_E \\
&= \lim_{\epsilon\to 0^+}\Bigg[\int_{\{r=\epsilon\}} \langle Q(\sigma_1(P, dr) f \otimes \delta(r)), \sigma(P^t, dr)\varphi\rangle_E - \\
&\quad \int_{\{r=-\epsilon\}} \langle Q(\sigma_1(P, dr) f \otimes \delta(r)), \sigma(P^t, dr)\varphi\rangle_E\Bigg] \\
&= \langle \sigma_1(P, dr)(\mathcal{P}_+ + \mathcal{P}_-) f, \varphi\rangle_{L^2(b\Omega;F)}.
\end{aligned} \tag{102}$$

As φ is an arbitrary smooth section of F and $\sigma_1(P, dr)$ is invertible, we see that

$$f = (\mathcal{P}_+ + \mathcal{P}_-) f. \tag{103}$$

Arguing as in the previous section we can use contour integration in the ξ_n-variable to obtain a formula for $Q(g \otimes \delta(r))$. Here g is a smooth section of $F \restriction_{b\Omega}$. As before, this is a local problem, we introduce coordinates (x', x_n), in a neighborhood of U of $p \in b\Omega_+$ so that

$$\Omega_{\pm} \cap U = \{\pm x_n \geq 0\}. \tag{104}$$

As before we let $\psi \in \mathcal{C}^{\infty}(\mathbb{R}^{n-1})$ be a function that is 0 in a neighborhood of 0 and 1 outside the ball of radius 2; we can again show that, for $x_n \neq 0$, the functions

$$Q(g \otimes \delta(r)) - \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} q(x', x_n, \xi', \xi_n)\hat{g}(\xi')\psi(\xi')e^{ix_n\xi_n} d\xi_n e^{ix'\cdot\xi'} d\xi' \tag{105}$$

extend smoothly to both $\overline{\Omega_\pm}$. We study the symbolic properties of $\mathcal{P}_\pm$ by evaluating the ξ_n integrals, for $x_n \neq 0$, in:

$$\frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^{n-1}} \int\limits_{-\infty}^{\infty} q_j(x', x_n, \xi', \xi_n)\hat{g}(\xi')\psi(\xi')e^{ix_n\xi_n} d\xi_n e^{ix'\cdot\xi'} d\xi' \tag{106}$$

using contour integration. If $x_n > 0$ then, for each ξ', we use a contour that includes a semi-circle in the upper half enclosing the poles of $q_j(x', x_n, \xi', \xi_n)$, whereas if $x_n < 0$, then we use a contour in the lower half plane enclosing the poles in the lower half plane. In fact, the locations of the poles of the q_j do not depend on j, but coincide with the zeros of det $p_0(x', x_n, \xi', \xi_n)$. Since

$$p_0(x', x_n, \xi', \xi_n) = \|\xi'\| p_0(x', x_n, \frac{\xi'}{\|\xi'\|}, \frac{\xi_n}{\|\xi'\|}), \tag{107}$$

the poles are also homogeneous of degree 1 in $\|\xi'\|$. As P is elliptic, $p_0(x, \omega', \xi_n)$ is invertible for ξ_n on the real axis, here $\omega' = \xi'/\|\xi'\|$. Hence (if $b\Omega$ is connected) the number of zeros in each half plane does not depend on (x', ω'). We let $\{\eta_l^\pm(\omega') : l = 1, \ldots, L_\pm\}$ denote the zeros of det $p_0(x', 0, \omega', \xi_n)$ in the upper (lower) half ξ_n-plane. The zeros may also depend on x', but we suppress that dependence for the time being. Evidently the sets

$$Z_\pm = \bigcup_{\omega' \in S^{n-1}} \{\eta_l^\pm(\omega') : \; l = 1, \ldots, L_\pm\} \tag{108}$$

have compact closures disjoint from the real axis.

Let $\Gamma_\pm$ be an interval on the real axis along with a semi-circle in $\pm \operatorname{Im} \xi_n > 0$, enclosing $\overline{Z_\pm}$. If $R > 0$, then $R\Gamma_\pm$ denotes the contour scaled by the factor R. As an oscillatory integral we see that, for $\pm x_n > 0$, we have

$$\frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^{n-1}} \int\limits_{-\infty}^{\infty} q_j(x', x_n, \xi', \xi_n)\hat{g}(\xi')\psi(\xi')e^{ix_n\xi_n} d\xi_n e^{ix'\cdot\xi'} d\xi' =$$

$$\frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^{n-1}} \left[\int\limits_{\|\xi'\|\Gamma_\pm} q_j(x', x_n, \xi', \xi_n) e^{ix_n\xi_n} d\xi_n \right] \hat{g}(\xi')\psi(\xi')e^{ix'\cdot\xi'} d\xi'. \tag{109}$$

It is not difficult to see that, for $\xi' \neq 0$, the limits,

$$\lim_{x_n \to 0^\pm} r_{\pm j}(x', \xi') = \frac{1}{2\pi} \int\limits_{\|\xi'\|\Gamma_\pm} q_j(x', 0, \xi', \xi_n) d\xi_n, \tag{110}$$

exist, and define homogeneous symbols of degree $-j$. This shows that $\mathcal{P}_{\pm}$ are classical pseudodifferential operators of order 0, with symbols $r_{\pm}$ satisfying

$$r_{\pm} \sim \sum_{j=0}^{\infty} r_{\pm j} \sigma_1(P, \pm dr). \tag{111}$$

We now carry out the detailed computation of the principal symbol. For each ω' we let $M^{\pm}(x', \omega')$ denote the span of generalized nullspaces of

$$\{p_0(x', 0, \omega', \eta_l^{\pm}(\omega', x')) : \ l = 1, \dots, L_{\pm}\}. \tag{112}$$

The fiber $E_{(x',0)}$ is the direct sum $M_+(\omega', x') \oplus M_-(\omega', x')$. The subspaces $M_{\pm}(\omega', x')$ consists of directions υ such that the system of ODEs:

$$\begin{aligned} &p_0(x', 0, \omega', \partial_{x_n})\upsilon(x_n) = 0 \\ &\upsilon(0) = \upsilon, \end{aligned} \tag{113}$$

has a solution, which is exponentially decaying as $\pm x_n \to \infty$. The principal symbols of Q is $[p_0(x, \xi)]^{-1}$ and therefore, up a constant of modulus 1,

$$r_{\pm 0}(x', \omega') = \frac{1}{2\pi} \int\limits_{\Gamma_{\pm}} [p_0(x', 0, \omega', \xi_n)]^{-1} d\xi_n. \tag{114}$$

are easily seen to be projections, with $r_{\pm 0}(x', \omega')$ the projection onto $M_{\pm}(x', \omega')$, along $M_{\mp}(x', \omega')$.

A good treatment of the Calderon projector, in the general case, can be found in [9]; the case of Dirac operators can be found in [4].

6 Fredholm Boundary Value Problems for First Order Operators

We now examine boundary value problems for the elliptic first order operator P, considered in the previous section. The domain of the maximal extension of P as an unbounded operator on L^2, $\mathrm{Dom}_{\max}(P)$, consists of L^2-sections u of $E \to \Omega$, such that the distributional derivative Pu is in L^2 as well. It follows from Corollary 1 that if $u \in \mathrm{Dom}_{\max}(P)$, then u has distributional boundary values in $H^{-\frac{1}{2}}(b\Omega)$. Hence, if $\mathcal{R}$ is a pseudodifferential operator acting on sections of $E \restriction_{b\Omega}$, then we can define the domain of a closed, unbounded operator acting on $L^2(\Omega)$, by

$$\mathrm{Dom}(P, \mathcal{R}) = \{u \in \mathrm{Dom}_{\max}(P) : \ \mathcal{R}(u \restriction_{b\Omega}) = 0\}. \tag{115}$$

We use the notation $(P, \mathcal{R})$ to denote this unbounded operator acting on $L^2(\Omega)$.

In this section we consider boundary conditions defined by pseudodifferential projections. This is not a serious restriction, since the nullspace, $\mathcal{N}_{\mathcal{R}}$, of $\mathcal{R}$ is a closed subspace. Under fairly mild conditions, (for example: 0 is isolated in the spectrum of $\mathcal{R}$), the orthogonal projection, $\mathcal{R}_{\text{pr}}$, onto $\mathcal{N}_{\mathcal{R}}$ is a pseudodifferential operator. Evidently $(P, \mathcal{R})$ and $(P, \mathcal{R}_{\text{pr}})$ are the same operator on L^2. It is not necessary to assume that $\mathcal{R}$ is a classical pseudodifferential operator, but merely that it acts on $\mathcal{D}'(b\Omega)$. We give a condition on $\mathcal{R}$ that ensures that $(P, \mathcal{R})$ is a Fredholm operator, that is, has a finite dimensional nullspace and a closed range, in L^2, of finite codimension.

As in the example of $\bar{\partial}$ on D_1, our analysis centers on the comparison operator. We let $\mathcal{P}$ denote the Calderon projector for P on Ω. If $\mathcal{R}$ is a projector defining a boundary condition for P, then we consider the operator:

$$\mathcal{T} = \mathcal{R}\mathcal{P} + (\mathrm{Id} - \mathcal{R})(\mathrm{Id} - \mathcal{P}). \tag{116}$$

Assuming that $\mathcal{R} : H^s(b\Omega) \to H^s(b\Omega)$ for all $s \geq -\frac{1}{2}$, it follows from the fact that $\mathcal{P}$ is a classical pseudodifferential operator of order 0, that $\mathcal{T}$ preserves the same Sobolev spaces.

Definition 2. We say that $\mathcal{R}$ is μ-elliptic if $\mathcal{T}$ has parametrix $\mathcal{U}$, for which there exists a $\mu \in \mathbb{R}$, such that for every $s \geq -\frac{1}{2}$,

$$\mathcal{U} : H^s(b\Omega) \to H^{s-\mu}(b\Omega), \tag{117}$$

boundedly.

In this case we can select $\mathcal{U}$ so that

$$\mathcal{U}\mathcal{T} = \mathrm{Id} - K_1 \text{ and } \mathcal{T}\mathcal{U} = \mathrm{Id} - K_2, \tag{118}$$

where K_1, K_2 are finite rank, smoothing operators.

The classical elliptic case corresponds to $\mu = 0$. A small modification of the $\bar{\partial}$-Neumann condition on a strictly pseudoconvex, almost complex manifold gives an example where $\mu = \frac{1}{2}$, see [7, 6, 8].

Theorem 10. *Let Ω be a smooth manifold with boundary and $P : \mathcal{C}^\infty(\Omega; E) \to \mathcal{C}^\infty(\Omega; F)$ a first elliptic differential operator, with fundamental solution Q. Suppose that $\mathcal{R}$ is a pseudodifferential projection acting on sections of $E \restriction_{b\Omega}$. If $\mathcal{R}$ is μ-elliptic, with $\mu \leq 1$, then $(P, \mathcal{R})$ is a Fredholm operator; if $\mu < 1$, then the operator has a compact resolvent.*

Before proceeding with the proof of this theorem we observe that Lemma 1 has the following generalization:

Lemma 4. *If* $\mathcal{T} f \in \operatorname{Im} \mathcal{R}$ *then* $\mathcal{T}\mathcal{P} f = \mathcal{T} f$.

Proof. This follows immediately from the fact that $\mathcal{R}\mathcal{T} = \mathcal{T}\mathcal{P}$. $\square$

Proof of the Theorem. First we observe that $(P, \mathcal{R})$ has a finite dimensional nullspace. Suppose that $u \in \operatorname{Dom}(P, \mathcal{R})$ and $Pu = 0$. Corollary 1 implies that u has distributional boundary values in $H^{-\frac{1}{2}}(b\Omega)$, which therefore satisfy $\mathcal{R}(u \restriction_{b\Omega}) = 0$. Since $u \in \ker P$, it is clear that $\mathcal{P}(u \restriction_{b\Omega}) = u \restriction_{b\Omega}$. This implies that

$$\mathcal{R}(u \restriction_{b\Omega}) = \mathcal{T}(u \restriction_{b\Omega}) = 0. \tag{119}$$

On the other hand (118) then implies that

$$(\operatorname{Id} - K_1)u \restriction_{b\Omega} = 0. \tag{120}$$

As K_1 is a smoothing operator, the nullspace of $(\operatorname{Id} - K_1)$ is finite dimensional. The existence of the fundamental solution Q easily implies that elements of $\ker P$ are determined by their boundary values on $b\Omega$. This shows that the nullspace of $(P, \mathcal{R})$ is finite dimensional.

Now we turn to the proof that the range is of finite codimension, and closed. Let $f \in L^2(\Omega, F)$, and let $u_1 = Qf$, where, as usual, we extend f, by zero, to all of $\widetilde{\Omega}$, and

$$u_0 = -Q\sigma(P, dr)\left[\mathcal{U}\mathcal{R}(u_1 \restriction_{b\Omega}) \otimes \delta(r)\right]. \tag{121}$$

We need to show that $u = u_0 + u_1 \in \operatorname{Dom}(P, \mathcal{R})$. That $Pu = f$, in the sense of distributions, is clear. From Theorem 8 it follows that $u_1 \in H^1(\Omega)$, and therefore $\mathcal{U}\mathcal{R}(u_1 \restriction_{b\Omega}) \in H^{\frac{1}{2}-\mu}(b\Omega)$. Hence Theorem 9 and the embedding result $H_{(1,-\mu)}(\Omega) \subset H^{1-\mu}(\Omega)$, imply that $u_0 \in H^{1-\mu}(\Omega)$. If $\mu \leq 1$, then $u \in L^2(\Omega)$. To complete the argument, we need to show that $\mathcal{R}(u \restriction_{b\Omega}) = 0$. This is true, provided that f satisfies finitely many bounded linear conditions.

We note that

$$\mathcal{T}\mathcal{U}\mathcal{R}(u_1 \restriction_{b\Omega}) = (\operatorname{Id} - K_2)\mathcal{R}(u_1 \restriction_{b\Omega}). \tag{122}$$

Recall that K_2 is of finite rank, hence the requirement

$$K_2(\mathcal{R}(u_1 \restriction_{b\Omega})) = K_2\mathcal{R}(Qf \restriction_{b\Omega}) = 0 \tag{123}$$

is a finite set of linear conditions on f. As the map $f \mapsto Qf \restriction_{b\Omega}$ is bounded from $L^2(\Omega)$ to $H^{\frac{1}{2}}(b\Omega)$ is bounded, these are evidently defined by bounded linear functional. Let S denote the subset of $L^2(\Omega; F)$ where these conditions are satisfied.

If $f \in S$, then (122) and (123) imply that $\mathcal{T}(\mathcal{U}\mathcal{R}(u_1 \restriction_{b\Omega}) \in \operatorname{Im}\mathcal{R}$. Lemma 4 then implies that

$$\begin{aligned}\mathcal{R}(u_0 \restriction_{b\Omega}) &= -\mathcal{R}\mathcal{P}\mathcal{U}\mathcal{R}(u_1 \restriction_{b\Omega}) \\ &= -\mathcal{T}\mathcal{P}\mathcal{U}\mathcal{R}(u_1 \restriction_{b\Omega}) \\ &= -\mathcal{T}\mathcal{U}\mathcal{R}(u_1 \restriction_{b\Omega}) \\ &= -\mathcal{R}(u_1 \restriction_{b\Omega}).\end{aligned} \tag{124}$$

To pass to the final line we use (123). Thus, if $f \in S$, then $\mathcal{R}(u \restriction_{b\Omega}) = 0$, and therefore S is a subspace of the range of $(P, \mathcal{R})$. This is a closed subspace of finite codimension; hence the range of the operator is itself closed and of finite codimension. This completes the proof that $(P, \mathcal{R})$ is Fredholm operator provided $\mu \leq 1$.

Suppose that $u \in \operatorname{Dom}(P, \mathcal{R})$ and let $u_1 = QP(u) \in H^1(\Omega)$. The difference, $u - u_1$ is in the (formal) nullspace of P, hence

$$\mathcal{P}(u - u_1) \restriction_{b\Omega} = (u - u_1) \restriction_{b\Omega} \text{ and } \mathcal{R}(u - u_1) \restriction_{b\Omega} = -\mathcal{R}(u_1 \restriction_{b\Omega}). \tag{125}$$

A priori, $(u - u_1) \restriction_{b\Omega} \in H^{-\frac{1}{2}}(b\Omega)$. The identities in (125) imply that

$$\mathcal{T}(u - u_1) \restriction_{b\Omega} = -\mathcal{R}(u_1 \restriction_{b\Omega}) \in H^{\frac{1}{2}}(b\Omega). \tag{126}$$

Applying $\mathcal{U}$, we see that

$$(\operatorname{Id} - K_1)(u - u_1) \restriction_{b\Omega} = -\mathcal{U}\mathcal{R}(u_1 \restriction_{b\Omega}) \in H^{\frac{1}{2}-\mu}(b\Omega). \tag{127}$$

As K_1 is a smoothing operator, this shows that $(u - u_1) \restriction_{b\Omega} \in H^{\frac{1}{2}-\mu}(b\Omega)$. Theorem 9 implies that $u - u_1 \in H^{1-\mu}(\Omega)$ and therefore u is as well. Thus the domain of $(P, \mathcal{R})$ is contained in $H^{1-\mu}(\Omega; E)$, which, if $\mu < 1$, is compactly embedded into L^2, showing that the resolvent of $(P, \mathcal{R})$ is a compact operator. This completes the proof of the theorem. □

Using the same argument we can also prove higher norm estimates.

Theorem 11. *Under the hypotheses of Theorem 10, if $f \in H^s(\Omega; F)$ satisfies finitely many linear conditions, then there exists a solution u to*

$$Pu = f \text{ and } \mathcal{R}(u \restriction_{b\Omega}) = 0. \tag{128}$$

For each $s \geq 0$ there is a C_s such that

$$\|u\|_{H^{s+1-\mu}} \leq C_s \|f\|_{H^s}. \tag{129}$$

If $\mathcal{R}$ is a classical pseudodifferential operator, then we can easily give a symbolic condition for $\mathcal{R}$ to be 0-elliptic operator. The conditions are that for every $(x', \xi') \in T^*b\Omega \setminus \{0\}$, the retrictions

$$\sigma_0(\mathcal{R})(x', \xi') \restriction_{\operatorname{Im}\sigma_0(\mathcal{P})(x',\xi')} \text{ and } (\operatorname{Id} - \sigma_0(\mathcal{R})(x', \xi')) \restriction_{\operatorname{Im}(\operatorname{Id} - \sigma_0(\mathcal{P})(x',\xi'))}, \tag{130}$$

are injective. This of course implies that $\sigma_0(\mathcal{T})(x', \xi')$ is invertible away from the zero section. If the projections are orthogonal, then $\sigma_0(\mathcal{R})(x', \xi') \restriction_{\operatorname{Im}\sigma_0(\mathcal{P})(x',\xi')}$ gives an isomorphism on $\operatorname{Im}\sigma_0(\mathcal{R})(x', \xi')$ if and only if the complementary restriction $(\operatorname{Id} - \sigma_0(\mathcal{R})(x', \xi')) \restriction_{\operatorname{Im}(\operatorname{Id} - \sigma_0(\mathcal{P})(x',\xi'))}$ gives an isomorphism onto the orthogonal complement $\operatorname{Im}(\operatorname{Id} - \sigma_0(\mathcal{R})(x', \xi'))$.

As noted above, it is not necessary for $\mathcal{R}$ to be a classical pseudodifferential operator. In a series of papers, [7, 6, 8, 5], the case of a strictly pseudoconvex, $\text{Spin}_{\mathbb{C}}$-manifold is analyzed. In this context, a modification of the $\bar{\partial}$-Neumann condition can be defined that gives a $\frac{1}{2}$-elliptic operator.

References

[1] M. ATIYAH, V. PATODI, AND I. SINGER, *Spectral asymmetry and Riemannian geometry, I*, Math. Proc. Camb. Phil. Soc, 77 (1975), pp. 43–69.

[2] ——, *Spectral asymmetry and Riemannian geometry, II*, Math. Proc. Camb. Phil. Soc, 78 (1975), pp. 405–432.

[3] ——, *Spectral asymmetry and Riemannian geometry, III*, Math. Proc. Camb. Phil. Soc, 79 (1976), pp. 71–99.

[4] B. BOOSS-BAVNBEK AND K. P. WOJCIECHOWSI, *Elliptic Boundary Problems for the Dirac Operator*, Birkhäuser, Boston, 1996.

[5] C. L. EPSTEIN, *Subelliptic boundary conditions for Spin$_{\mathbb{C}}$ Dirac operators, gluing, relative indices, and tame fredholm pairs*, Proc. Nat Acad. of Sci., 103 (2006), pp. 15364–15369.

[6] ——, *Subelliptic Spin$_{\mathbb{C}}$ Dirac operators,I*, to appear Annals of Math., (2006), pp. 1–36.

[7] ——, *Subelliptic Spin$_{\mathbb{C}}$ Dirac operators,II*, to appear Annals of Math., (2006), pp. 1–54.

[8] ——, *Subelliptic Spin$_{\mathbb{C}}$ Dirac operators,III*, to appear Annals of Math., (2006), pp. 1–72.

[9] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators*, vol. 3, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.

[10] R. SEELEY, *Singular integrals and boundary value problems*, Amer. Jour. of Math., 88 (1966), pp. 781–809.

[11] M. E. TAYLOR, *Partial Differential Equations, Vol. 2*, vol. 116 of Applied Mathematical Sciences, Springer, New York, 1996.