Endoscopy and the geometry of the Hitchin fibration

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Orbital integrals

- Let F be a local field ($\mathbb{R}, \mathbb{C}$ or a finite extension of $\mathbb{Q}_p$). Let G be a connected reductive group over F.
- Amongst the most important invariant distributions on $G(F)$ are the orbital integrals associated to regular semisimple elements $\gamma \in G(F)$:

$$\mathcal{O}_\gamma^G(f) = \int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)\, d\dot{g}$$

where

- $f \in C_c^\infty(G(F))$ is a test function
- G_γ is the centralizer of γ
- $\mathcal{O}_\gamma^G$ depends on the choice of an invariant measure $d\dot{g}$ on the orbit $G_\gamma(F)\backslash G(F)$. We may assume that $\mathcal{O}_\gamma^G$ depends only the conjugacy class of γ.

Stable orbital integrals

- We can only expect a transfer of stable conjugacy classes between inner forms of the group G.
- Here stable means conjugacy classes of $G(\overline{F})$ where $\overline{F}$ is an algebraic closure of F.
- The stable orbital integral attached to a regular semisimple stable conjugacy class σ is

$$\mathcal{SO}_{\sigma}^{G}(f) = \sum_{\gamma} \mathcal{O}_{\gamma}^{G}(f)$$

  where the sum is over the finite set of conjugacy classes of γ inside σ.

The Arthur-Selberg trace formula

- In this slide the group G is over a number field F.
- Langlands functoriality predicts deep reciprocity laws between the automorphic spectra of G and its inner forms.
- The Arthur-Selberg trace formula is roughly the equality

$$\mathrm{trace}(f|\text{automorph. spectrum}) = \sum_{\gamma} a_{\gamma} \prod_{v} \mathcal{O}_{\gamma}^{v}(f)$$

  where
  - f is a test function.
  - The sum is over regular semi-simple conjugacy classes γ in $G(F)$.
  - $\prod_v \mathcal{O}_{\gamma}^{v}(f)$ is a product over completions F_v of F of local orbital integrals of $G(F_v)$.
  - a_{γ} is a global coefficient (a volume).
- A basic strategy to prove Langlands functoriality for inner forms is to compare the geometric sides of the trace formulas.

The endoscopy

- Main Problem : The trace formula is not stable: it is not a sum of products of local stable orbital integrals.
- The difference between the trace formula and its stable counterpart can be expressed as a sum of products of local distributions

$$\sum_{\gamma \in G(F)/\sim} \Delta_H(\sigma, \gamma) \mathcal{O}_\gamma^G(f)$$

  indexed by endoscopic groups H and regular semisimple stable conjugacy classes σ of $H(F)$. The function $\Delta_H(\sigma, \gamma)$ is the Langlands-Shelstad transfer factor: it vanishes unless the stable conjugacy class of γ matches σ.
- It is in fact possible to interpret the unstable part of the trace formula as a stable trace formula for endoscopic groups. But for this we need the following two statements in local harmonic analysis.

Two statements in local Harmonic Analysis

Theorem (Langlands-Shelstad transfer)

Let H be an endoscopic group of G. For any $f \in C_c^\infty(G(F))$, there exists $f^H \in C_c^\infty(H(F))$ s.t. for any stable conjugacy class σ of $H(F)$

$$\sum_{\gamma \in G(F)/\sim} \Delta_H(\sigma,\gamma)\mathcal{O}_\gamma^G(f) = \mathcal{S}\mathcal{O}_\sigma^H(f^H)$$

Theorem (Langlands-Shelstad fundamental lemma)

F is p-adic and G and H are unramified.
If f is the characteristic function of a hyperspecial maximal compact subgroup of $G(F)$, one may take for f^H the characteristic function of a hyperspecial maximal compact subgroup of $H(F)$.

3 reductions

1. Reduction to the units
   - Shelstad proved the transfer for archimedean fields.
   - The Fundamental Lemma (FL) $\Longrightarrow$ the p-adic transfer for the spherical Hecke algebra (Hales).
   - (FL) $\Longrightarrow$ the p-adic transfer (Waldspurger).
2. From the group to the Lie algebra
   - (FL) $\Longleftrightarrow$ a variant of (FL) for Lie algebras (Hales, Waldspurger)
3. Reduction to the case of local fields of equal characteristics

   For Lie algebras, we have
   - (FL) for p-adic field with residual field $\mathbb{F}_q$ is equivalent to (FL) for local fields $\mathbb{F}_q((\varepsilon))$. (Waldspurger / Cluckers-Hales-Loeser)

The fundamental lemma for the Lie algebra of $SL(2)$

- Let $F = \mathbb{F}_q((\varepsilon))$, $\mathcal{O}_F = \mathbb{F}_q[[\varepsilon]]$, $\mathbb{F}_q$ is finite of *char.* > 2.
- Let $G = SL(2)$ and $\mathfrak{g} = Lie(G)$.
- Let $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ s.t. $\alpha^2 \in \mathbb{F}_q$ and $E = F[\alpha] \supset \mathcal{O}_E$.
- The group $H(F) = \{x \in E \mid \mathrm{Norm}_{E/F}(x) = 1\}$ is an unramified endoscopic group of G.
- Any $a \in F^\times$ determines a regular characteristic polynomial

$$X^2 - (\alpha a)^2 \in F[X]$$

  and two distinct $G(F)$-conjugacy classes in $\mathfrak{g}(F)$ namely those of

$$\gamma_a = \begin{pmatrix} 0 & (\alpha a)^2 \\ 1 & 0 \end{pmatrix} \text{ and } \gamma'_a = \begin{pmatrix} 0 & \varepsilon^{-1}(\alpha a)^2 \\ \varepsilon & 0 \end{pmatrix}$$

- The (FL) is the equality

$$q^{-\mathrm{val}(a)} \mathcal{O}^G_{\gamma_a}(\mathbf{1}_{\mathfrak{g}(\mathcal{O}_F)}) - q^{-\mathrm{val}(a)} \mathcal{O}^G_{\gamma'_a}(\mathbf{1}_{\mathfrak{g}(\mathcal{O}_F)}) = \mathbf{1}_{\mathcal{O}_E}(a\alpha)$$

Cohomological interpretation

In the case of the Fundamental Lemma for Lie algebras over $\mathbb{F}_q((t))$, we have:

- The orbital integrals 'compute' the number of rational points of varieties over $\mathbb{F}_q$, some quotients of Affine Springer fibers.
- Thanks to the Grothendieck function-sheaf dictionary this gives a cohomological approach to the (FL).
- Ngô indeed proves the (FL) by a cohomological study of the elliptic part of the Hitchin fibration.

The example of $GL(n)$

Let $F = \mathbb{F}_q((\varepsilon)) \supset \mathcal{O} = \mathbb{F}_q[[\varepsilon]]$.
Let $G = GL(n)$ and $\mathfrak{g} = Lie(G)$ with $n > char(\mathbb{F}_q)$.

- Let $\gamma \in \mathfrak{g}(F)$ be regular semisimple.
- Let $\Lambda_\gamma \subset G_\gamma(F)$ be the image of the discrete group of F-rational cocharacters of G_γ by $\varepsilon \mapsto \varepsilon^\lambda$.
- Let $d\dot{g}$ be the quotient of Haar measures on $G(F)$ and $G_\gamma(F)$ normalized by

$$\mathrm{vol}(G(\mathcal{O}_F)) = 1 \text{ and } \mathrm{vol}(\Lambda_\gamma \backslash G_\gamma(F)) = 1$$

Proposition *We have*

$$\int_{G_\gamma(F)\backslash G(F)} \mathbf{1}_{\mathfrak{g}(\mathcal{O})}(g^{-1}\gamma g)\, d\dot{g} = |\Lambda_\gamma \backslash \mathfrak{X}_\gamma|$$

where $\mathfrak{X}_\gamma$ is the set of lattices $\mathcal{L} \subset F^n$ s.t. $\gamma\mathcal{L} \subset \mathcal{L}$.

The group Λ_γ acts on $\mathfrak{X}_\gamma$ through the action of $G(F)$ on the set of lattices.

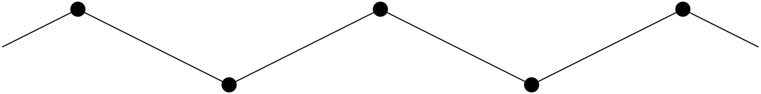
Affine Springer fiber ...

The set of lattices $\mathfrak{X}$ is an increasing union of projective varieties called the Affine Grassmaniann.
The Affine Springer fiber is the closed (ind-)subvariety $\mathfrak{X}_\gamma \subset \mathfrak{X}$.

Theorem (Kazhdan-Lusztig)

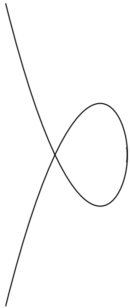
- *$\mathfrak{X}_\gamma$ is a variety locally of finite type and of finite dimension.*
- *The quotient $\Lambda_\gamma \backslash \mathfrak{X}_\gamma$ is a projective variety.*

Example $G = GL(2)$ and $\gamma = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}$.
Then $\mathfrak{X}_\gamma$ is $\mathbb{Z}\times$ an infinite chain of $\mathbf{P}^1$



... and its quotient

When one takes the quotient by $\Lambda_\gamma \simeq \mathbb{Z}^2$, one gets

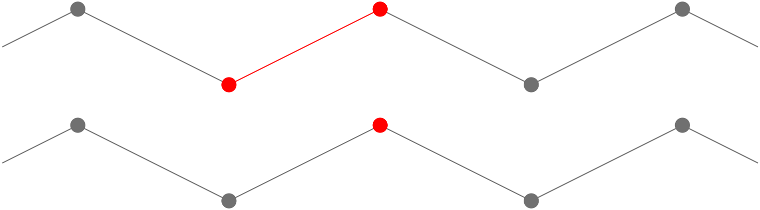


Back to the (FL) for $SL(2)$

Let $G = SL(2)$ and $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$

$$\gamma_\varepsilon = \begin{pmatrix} 0 & \alpha^2\varepsilon^2 \\ 1 & 0 \end{pmatrix} \text{ and } \gamma'_\varepsilon = \begin{pmatrix} 0 & \alpha^2\varepsilon \\ \varepsilon & 0 \end{pmatrix} \in \mathfrak{g}(F)$$

$\mathcal{O}_{\gamma_\varepsilon} = q + 1$ and $\mathcal{O}_{\gamma'_\varepsilon} = 1$ are the number of fixed points of two twisted Frobenius of a connected component of $\mathfrak{X}_\gamma$.



(FL) is given by the equality $q^{-1}(q+1) - q^{-1} \times 1 = 1$

Work of Goresky-Kottwitz-MacPherson

- For γ "equivalued" and unramified, they computed the cohomology of $\mathfrak{X}_\gamma$.
- $\mathcal{O}_\gamma = |(\Lambda_\gamma \backslash \mathfrak{X}_\gamma)(\mathbb{F}_q)| = \mathrm{trace}(Frob_q, H^\bullet(\Lambda_\gamma \backslash \mathfrak{X}_\gamma, \bar{\mathbb{Q}}_\ell))$.
- For such γ, they proved the Fundamental Lemma.

Remarks

- They need that γ is "equivalued" to prove that the cohomology of $\mathfrak{X}_\gamma$ is pure.
- It is conjectured that this cohomology is always pure.
- They need that γ is unramified since they first compute the equivariant cohomology of $\mathfrak{X}_\gamma$ for the action of a "big" torus.

Ngô's global approach

- Let C be a connected, smooth, projective curve over $k = \overline{\mathbb{F}_q}$
- Let $D = 2D'$ be an even and effective divisor on C of degree $> 2g$ with g the genus of C. Let $n > char(k)$.

A Higgs bundle is a pair $(\mathcal{E}, \theta)$ s.t.

- $\mathcal{E}$ is a vector bundle on C of rank n and degree 0
- $\theta : \mathcal{E} \to \mathcal{E}(D) = \mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{O}_C(D)$ is a twisted endomorphism.

For such a pair, we have

- trace$(\theta) : \mathcal{O}_C \overset{id}{\to} \mathcal{E}nd(\mathcal{E}) \overset{\theta}{\to} \mathcal{O}_C(D) \in H^0(C, \mathcal{O}_C(D))$
- $a_i(\theta) :=$ trace$(\wedge^i \theta) \in H^0(C, \mathcal{O}_C(iD))$

The characteristic polynomial of $(\mathcal{E}, \theta)$ is then defined by

$$\chi_\theta = X^n - a_1(\theta) X^{n-1} + \ldots + (-1)^n a_n(\theta) \in \bigoplus_i H^0(C, \mathcal{O}_C(iD))$$

Hitchin fibration

- Let $\mathbf{M}$ be the algebraic k-stack of Higgs bundles $(\mathcal{E}, \theta)$
- Let $\mathbf{A}$ be the affine space of characteristic polynomials

$$X^n - a_1 X^{n-1} + \ldots + (-1)^n a_n$$

with $a_i \in H^0(C, \mathcal{O}_C(iD))$. By Riemann-Roch theorem

$$\dim_k(\mathbf{A}) = \frac{n(n+1)}{2} \deg(D) + n(1-g)$$

- The Hitchin fibration is the morphism

$$f : \mathbf{M} \to \mathbf{A}$$

defined by

$$f(\mathcal{E}, \theta) = \chi_\theta$$

Adelic description of Hitchin fibers

- Let $F = k(C)$ the function field of C.
- Let $G = GL(n)$ and $\mathfrak{g} = Lie(GL(n))$.
- $\mathbb{A}$ ring of adèles of F and $\mathcal{O} = \prod_{c \in |C|} \hat{\mathcal{O}}_c \subset \mathbb{A}$
- Let $\varpi_D = (\varpi_c^{mult_c(D)})_{c \in |C|} \in \mathbb{A}^\times$
- Let $\chi \in \mathbf{A}(k)$ and $\mathcal{H}_\chi$ be the set of

$$(g, \gamma) \in G(\mathbb{A})/G(\mathcal{O}) \times \mathfrak{g}(F) \text{ s.t.}$$

  1. $\deg(\det(g)) = 0$
  2. $\chi_\gamma = \chi$
  3. $g^{-1}\gamma g \in \varpi_D^{-1}\mathfrak{g}(\mathcal{O})$

- The group $G(F)$ acts on $\mathcal{H}_\chi$ by $\delta \cdot (g, \gamma) = (\delta g, \delta\gamma\delta^{-1})$

Lemma

The Hitchin fibre $f^{-1}(\chi)(k)$ is the quotient groupoid $[G(F)\backslash\mathcal{H}_\chi]$.

Counting points of elliptic Hitchin fibers

Let $\mathbf{A}^{\text{ell}} \subset \mathbf{A}^{rss} \subset \mathbf{A}$ be the open subsets defined by

- $\mathbf{A}^{\text{ell}} = \{\chi \in \mathbf{A}^{\text{ell}} \mid \chi \text{ is irreducible in } F[X]\}$
- $\mathbf{A}^{rss} = \{\chi \in \mathbf{A}^{\text{ell}} \mid \chi \text{ is square-free in } F[X]\}$

Lemma (Ngô)

Let $\chi \in \mathbf{A}^{rss}$ and $\gamma \in \mathfrak{g}(F)$ s.t. $\chi_\gamma = \chi$. Let $(\gamma_c)_c = \varpi_D \gamma \in \mathfrak{g}(\mathbb{A})$. We have

$$f^{-1}(\chi)(k) \simeq [G(F) \backslash \mathcal{H}_\chi] \simeq [T(F) \backslash \prod_{c \in |C|} \mathfrak{X}_{\gamma_c}(k)]$$

where T is the centralizer of γ in G and $\mathfrak{X}_{\gamma_c}$ is an affine Springer fiber. Moreover if $k = \mathbb{F}_q$, we have

$$|f^{-1}(\chi)(\mathbb{F}_q)| = \text{vol}(T(F) \backslash T(\mathbb{A})^0) \prod_c \mathcal{O}_{\gamma_c}$$

where $\text{vol}(T(F) \backslash T(\mathbb{A})^0) < \infty$ iff $\chi \in \mathbf{A}^{\text{ell}}(\mathbb{F}_q)$.

A slight variant of the Hitchin fibration

Let $\infty \in C$ a closed point, $\infty \notin \mathrm{supp}(D)$.
Let $\mathbf{A}^\infty \subset \mathbf{A}^{rss}$ be the open subset of $\chi \in \mathbf{A}$ such that χ_∞ has only simple roots.
Let $\mathcal{A}$ be the étale Galois cover of $\mathbf{A}^\infty$ of group $\mathfrak{S}_n$ given by

$$\mathcal{A} = \{(\chi, \tau) \in \mathbf{A}^\infty \times k^n | \chi_\infty = \prod_{i=1}^{n} (X - \tau_i)\}$$

Let $(\mathcal{E}, \theta, \chi_\theta, \tau) \in \mathbf{M} \times_{\mathbf{A}} \mathcal{A}$. Then θ_∞ is a regular semi-simple endomorphism of $\mathcal{E}_\infty$. Let

$$\mathcal{M} \to \mathbf{M} \times_{\mathbf{A}} \mathcal{A}$$

be the $\mathbb{G}_m$-torsor we obtain by choosing an eigenvector e_1 in the line $\mathrm{Ker}(\theta_\infty - \tau_1 \, \mathrm{Id}_{\mathcal{E}_\infty})$.
Remark The additional datum e_1 "kills" the automorphisms coming from the center of G.

By base change, we have a Hitchin fibration still denoted f

$$\mathcal{M} \to \mathbf{M} \times_{\mathbf{A}} \mathcal{A} \to \mathcal{A}$$

So $\mathcal{M}$ classifies $(\mathcal{E}, \theta, \tau, e_1)$ s.t.

- $(\mathcal{E}, \theta)$ is Higgs bundle s.t. θ_∞ is regular semi-simple
- $\tau = (\tau_1, \ldots, \tau_n)$ is the ordered collection of eigenvalues of θ_∞
- $e_1 \in \mathcal{E}_\infty$ is an eigenvector of (θ_∞, τ_1).

By deformation theory, we have

Theorem (Biswas-Ramanan)

The algebraic stack $\mathcal{M}$ is smooth over k.

The spectral curve of Hitchin-Beauville-Narasimhan-Ramanan

Let $\Sigma_D = Spec\big(\bigoplus_{i=0}^{\infty} \mathcal{O}_C(-iD)X^i\big) \to C$ the whole space of the divisor D.
Let $a = (\chi, \tau) \in \mathcal{A}$.
The spectral curve Y_a is the closed curve in Σ_D defined by the equation

$$\chi(X) = X^n - a_1 X^{n-1} + \ldots + (-1)^n a_n = 0.$$

The canonical projection $\pi_a : Y_a \to C$ is a finite cover of degree n, which is étale over ∞. We have a natural identification

$$\pi_a^{-1}(\infty) = \{\infty_1, \ldots, \infty_n\} \cong \{\tau_1, \ldots, \tau_n\}.$$

Properties of the spectral curve Y_a

Recall $a = (\chi, \tau) \in \mathcal{A}$

- Y_a is reduced (since $\chi \in \mathbf{A}^{rss}$)
- Y_a is connected
- Y_a is not always irreducible: Y_a is irreducible $\Longleftrightarrow a \in \mathcal{A}^{\mathrm{ell}}$ (there are as many irreducible components of Y_a as irreducible factors of $\chi \in F[X]$)
- Its arithmetic genus defined by

$$q_{Y_a} = \dim(H^1(Y_a, \mathcal{O}_{Y_a})) = \dim(H^1(C, \pi_{a,*}\mathcal{O}_{Y_a}))$$

does not depend on a. In fact,

$$\pi_{a,*}\mathcal{O}_{Y_a} = \mathcal{O}_C \oplus \mathcal{O}_C(-D) \oplus \ldots \oplus \mathcal{O}((-n+1)D)$$

and $q_{Y_a} = \frac{n(n-1)}{2}\deg(D) + n(g-1) + 1$.

Hitchin-Beauville-Narasimhan-Ramanan correspondence

Theorem (H-BNR)

Let $a \in \mathcal{A}$. The Hitchin fiber $\mathcal{M}_a = f^{-1}(a)$ is isomorphic to the stack of torsion-free coherent $\mathcal{O}_{Y_a}$-modules $\mathcal{F}$ of degree 0 and rank 1 at generic points of Y_a, equipped with a trivialization of their stalk at ∞_1.

Construction: the multiplication by X gives a section

$$\mathcal{O}_{Y_a} \to \pi_a^* \mathcal{O}_C(D).$$

For such a $\mathcal{F}$, we get a morphism $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{Y_a}} \pi_a^* \mathcal{O}_C(D)$ and

$$\theta : \pi_{a,*}\mathcal{F} \to \pi_{a,*}(\mathcal{F} \otimes_{\mathcal{O}_{Y_a}} \pi_a^* \mathcal{O}_C(D)) = \pi_{a,*}(\mathcal{F})(D)$$

We associate to $\mathcal{F}$ the Higgs bundle $(\pi_{a,*}\mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}_C(\frac{n-1}{2}D), \theta)$.

Let $\mathcal{A}^{sm}$ the open set of a such that Y_a is smooth. One has $\mathcal{A}^{sm} \neq \emptyset$.

Corollary

For $a \in \mathcal{A}^{sm}$, the Hitchin fiber $\mathcal{M}_a$ is the Jacobian of Y_a. In particular, it is an abelian variety.

Let $a \in \mathcal{A}$.
Let $\mathrm{Pic}^0(Y_a)$ the smooth commutative group scheme of line bundles on Y_a of degree 0, equipped with a trivialization of their stalk at ∞_1.
By H-BNR correspondence, $\mathrm{Pic}^0(Y_a)$ acts on $\mathcal{M}_a$.
Let $\mathcal{M}_a^{\mathrm{reg}} \subset \mathcal{M}_a$ be the open sub-stack $(\mathcal{E}, \theta, \tau, e_1) \in \mathcal{M}_a$ such that θ_c is regular for any $c \in C$.

Lemma

$\mathcal{M}_a^{\mathrm{reg}}$ is a $\mathrm{Pic}^0(Y_a)$-torsor.

Dimension of Hitchin fibers $\mathcal{M}_a$

As a consequence of the work of Altmann-Iarrobino-Kleiman on compactified Jacobian, Ngô gets the following theorem

Theorem

- *$\mathcal{M}_a^{\text{reg}}$ is dense in $\mathcal{M}_a$.*
- $\dim(\mathcal{M}_a) = \dim(\mathcal{M}_a^{\text{reg}}) = \dim(\mathrm{Pic}^0(Y_a)) = q_{Y_a}$ *(=arithmétic genus of Y_a) does not depend on a.*
- *$Irr(\mathcal{M}_a)$ is a torsor under the abelian group $\pi_0(\mathrm{Pic}^0(Y_a)) \simeq \{(n_i) \in \mathbb{Z}^{Irr(Y_a)} \mid \sum_i n_i = 0\}$*

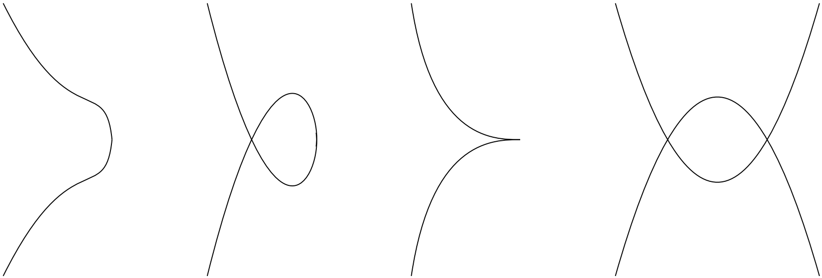
Corollary

- $\dim(\mathcal{M}) = n^2 \deg(D) + 1$.
- *$\mathcal{M}_a$ is irreducible if and only if $a \in \mathcal{A}^{\text{ell}}$.*

Some examples

Let $C = \mathbb{P}^1_k \supset \mathrm{Spec}(k[y]) \ni \infty$, $D = 2[0]$, $n = 2$.
Let $p(y) \in k[y]$ of degree 4 and $\tau \in k^\times$ s.t. $\tau^2 = p(0) \neq 0$.
Let $a = (X^2 - p(y), (\tau, -\tau)) \in \mathcal{A}$.
Y_a is of genus $q_{Y_a} = 1 = \dim(\mathcal{M}_a)$.

Examples of spectral curves Y_a



In the first 3 pictures, Y_a is irreducible and $\mathcal{M}_a \simeq Y_a$.

Support theorem on the elliptic locus

As a consequence of results of Altmann-Kleiman, the elliptic Hitchin morphism

$$f^{\mathrm{ell}} : \mathcal{M}^{\mathrm{ell}} = \mathcal{M} \times_{\mathcal{A}} \mathcal{A}^{\mathrm{ell}} \to \mathcal{A}^{\mathrm{ell}}$$

is proper and $\mathcal{M}^{\mathrm{ell}}$ is a smooth scheme over k.
By Deligne theorem, the complex of ℓ-adic sheaves $Rf^{\mathrm{ell}}_{*}\bar{\mathbb{Q}}_\ell$ is pure.
By Beilinson-Bernstein-Deligne-Gabber decomposition theorem, the direct sum of its perverse cohomology sheaves is semi-simple:

$$^{p}\mathcal{H}^{\bullet}(Rf^{\mathrm{ell}}_{*}\bar{\mathbb{Q}}_\ell) = \bigoplus_i {}^{p}\mathcal{H}^{i}(Rf^{\mathrm{ell}}_{*}\bar{\mathbb{Q}}_\ell)$$

Theorem (Ngô's support theorem)

The support of any irreducible constituent of $^{p}\mathcal{H}^{\bullet}(Rf^{\mathrm{ell}}_{G,*}\bar{\mathbb{Q}}_\ell)$ *is* $\mathcal{A}^{\mathrm{ell}}$.

Remarks

- The theorem is in fact only proved on a big subset of $\mathcal{A}$.
- Orbital integrals are "limits" of the simplest orbital integrals.

For other reductive groups G ?

- The support theorem is not true as stated.
- Let's consider the example $G = SL(2)$. The Hitchin space $\mathcal{M}_G$ classifies $(\mathcal{E}, \theta, \tau, e_1)$ as before with
  - $\mathcal{E}$ is a vector bundle of degree 2 and trivial determinant $\det(\mathcal{E}) = \mathcal{O}_C$.
  - $\theta : \mathcal{E} \to \mathcal{E}(D)$ is a traceless twisted endomorphism.
- The Hitchin base $\mathcal{A}_G$ classifies pairs $a = (X^2 - a_2, \tau)$ where $a_2 \in H^0(C, \mathcal{O}(2D))$ s.t. $a_2(\infty) = \tau^2 \neq 0$.
- We have a Hitchin morphism $f : \mathcal{M}_G \to \mathcal{A}_G$ defined by $f(\mathcal{E}, \theta, \tau, e_1) = (\det(\theta), \tau)$.
- A Hitchin fiber $\mathcal{M}_a$ is isomorphic to the stack of rank 1, torsionfree $\mathcal{O}_{Y_a}$-modules $\mathcal{F}$ which satisfy $\det(\pi_{a,*}\mathcal{F}(\frac{D}{2})) = \mathcal{O}_C$
- The group P_a acts on $\mathcal{M}_a$.

$$P_a := \mathrm{Ker}(\mathrm{Norm} : \mathrm{Pic}^0(Y_a) \to \mathrm{Pic}^0(C)).$$

The example of SL(2)

- Let $a \in \mathcal{A}^{\text{ell}}$ and $\rho_a : X_a \to C$ obtained from the normalization $X_a \to Y_a$ and $\pi_a : Y_a \to C$.
- Either the group P_a is connected or $\pi_0(P_a) = \mathbb{Z}/2\mathbb{Z}$.
- P_a is not connected iff $\rho_a : X_a \to C$ is étale.
  Let $\mathcal{L} \in Pic^0(C)[2]$ attached to X_a. Moreover there exists

$$b \in H^0(C, \mathcal{L}(D))$$

  s.t. $b^{\otimes 2} = a_2$.
- The groups P_a come in a family $P/\mathcal{A}^{\text{ell}}$ with a natural morphism

$$\mathbb{Z}/2\mathbb{Z} \to \pi_0(P/\mathcal{A}^{\text{ell}}).$$

- The group P acts on ${}^p\mathcal{H}^\bullet(Rf^{\text{ell}}_{G,*}\bar{\mathbb{Q}}_\ell)$ through $\pi_0(P/\mathcal{A}^{\text{ell}})$

$${}^p\mathcal{H}^\bullet(Rf^{\text{ell}}_{G,*}\bar{\mathbb{Q}}_\ell) = {}^p\mathcal{H}^\bullet(Rf^{\text{ell}}_{G,*}\bar{\mathbb{Q}}_\ell)_+ \oplus {}^p\mathcal{H}^\bullet(Rf^{\text{ell}}_{G,*}\bar{\mathbb{Q}}_\ell)_-$$

Support theorem for $SL(2)$

- For any non-trivial $\mathcal{L} \in Pic^0(C)[2]$,

$$\mathcal{A}_{\mathcal{L}} = \{b \in H^0(C, \mathcal{L}(D)) \mid b(\infty) \neq 0\}.$$

- The map $b \mapsto (b^{\otimes 2}, b(\infty))$ defines a closed immersion $\mathcal{A}_{\mathcal{L}} \hookrightarrow \mathcal{A}_G^{\text{ell}}$.
- The $\mathcal{A}_{\mathcal{L}}$ are disjoint.

Theorem (Ngô's support theorem)

1. *The support of any irreducible constituent of* ${}^p\mathcal{H}^\bullet(Rf_{G,*}^{\text{ell}}\bar{\mathbb{Q}}_\ell)_+$ *is* $\mathcal{A}_G^{\text{ell}}$*.*
2. *The supports of irreducible constituents of* ${}^p\mathcal{H}^\bullet(Rf_{G,*}^{\text{ell}}\bar{\mathbb{Q}}_\ell)_-$ *are the* $\mathcal{A}_{\mathcal{L}}$*.*

Cohomological fundamental lemma for $SL(2)$

- Any non-trivial $\mathcal{L} \in Pic^0(C)[2]$ defines an étale cover $X_{\mathcal{L}} \to C$ and an endoscopic group scheme on C

$$H_{\mathcal{L}} = (X_{\mathcal{L}} \times \mathbb{G}_m)/\{\pm 1\}$$

- For $H = H_{\mathcal{L}}$, we have a Hitchin morphism $f^H : \mathcal{M}_H \to \mathcal{A}_H$ with $\mathcal{A}_H = \mathcal{A}_{\mathcal{L}}$.

Theorem (Ngô)

Let $\iota_H : \mathcal{A}_H \to \mathcal{A}_G$. We have up to a shift and a twist

$$\iota_H^* \, {}^p\mathcal{H}^\bullet(Rf^{\mathrm{ell}}_{G,*}\bar{\mathbb{Q}}_\ell)_- \simeq {}^p\mathcal{H}^\bullet(Rf_{H,*}\bar{\mathbb{Q}}_\ell)$$

By the Grothendieck-Lefschetz trace formula, this gives a global version of the fundamental lemma for $G = SL(2)$.

$GL(n)$ case : outside the elliptic locus

- The properness of f^{ell} is crucial in Ngô's proof.
- Outside $\mathcal{A}^{\text{ell}}$, the Hitchin fibration is neither of finite type nor separated.
- To get Arthur's weighted fundamental lemma, we have to look outside $\mathcal{A}^{\text{ell}}$.
- For each $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, let's say that $m = (\mathcal{E}, \theta, \tau, e_1) \in \mathcal{M}$ is ξ-stable iff for any θ-invariant sub-bundle

  $$0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$$

  one has

  $$\deg(\mathcal{F}) + \sum_i \xi_i < 0$$

  where the sum is over i s.t. τ_i is an eigenvalue $of \theta_{|\mathcal{F}_\infty}$.

Remarks there is only a finite number of θ-invariant $\mathcal{F}$ and none if $(\mathcal{E}, \theta)$ is elliptic.

Properness of $\mathcal{M}^\xi$

Let $\mathcal{M}^\xi$ be the ξ-stable sub-stack of $\mathcal{M}$ for a generic ξ.

Theorem (Laumon-C.)

1. *$\mathcal{M}^\xi$ is an smooth open sub-stack of $\mathcal{M}$ which contains $\mathcal{M}^{\mathrm{ell}}$.*
2. *The ξ-stable Hitchin fibration is proper.*

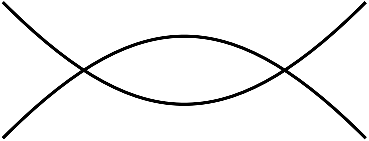
$$f^\xi : \mathcal{M}^\xi \to \mathcal{A}$$

3. *For $a \in \mathcal{A}(\mathbb{F}_q)$, $|\mathcal{M}^\xi_a(\mathbb{F}_q)|$ does not depend on ξ and is a global Arthur's weighted orbital integral.*
4. *Support theorem. The support of any irreducible constituent of ${}^p\mathcal{H}^\bullet(Rf^\xi_{G,*}\bar{\mathbb{Q}}_\ell)$ is $\mathcal{A}$.*

Here ξ generic means $\sum_{i\in I} \xi_i \notin \mathbb{Z}$ for any $\emptyset \neq I \subsetneq \{1, \ldots, n\}$

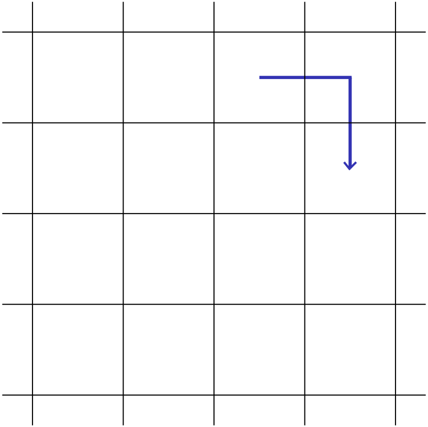
A spectral curve with 2 components

Let's go back to the example: $C = \mathbb{P}^1_k \supset \mathrm{Spec}(k[y]) \ni \infty$,
$D = 2[0]$, $n = 2$.
Let $a = (X^2 - (y^2 - 1)^2, (1, -1)) \in \mathcal{A}$.
In this case, Y_a has 2 irreducible components and looks like



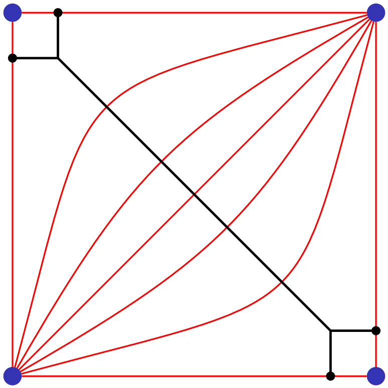
An non-elliptic fiber

$\mathcal{M}_a$ is the quotient of the product of 2 Affine Springer fibers by the diagonal action of $\mathbb{G}_m$ and the antidiagonal action of $\mathbb{Z}$



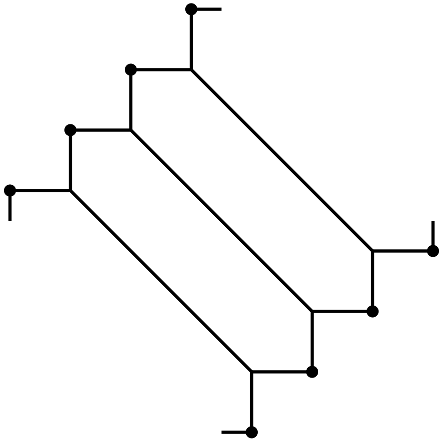
An non-elliptic fiber

The action of $\mathbb{G}_m$ stabilizes each square with 1-dim. orbits, fixed points and in black the quotient by $\mathbb{G}_m$

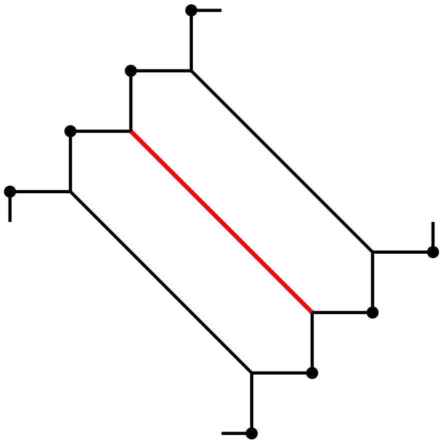


An non-elliptic fiber

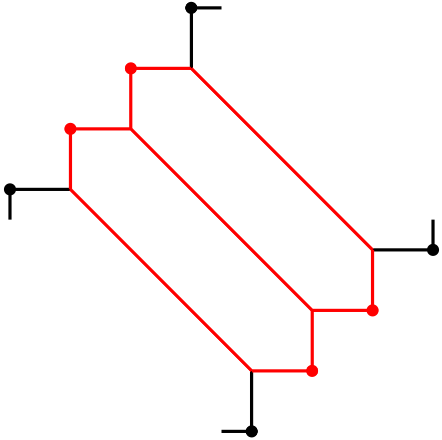
Up to some $B\mathbb{G}_m$, $\mathcal{M}_a$ looks like an infinite chain of non-separeted $\mathbf{P}^1$ with double 0 and double ∞.



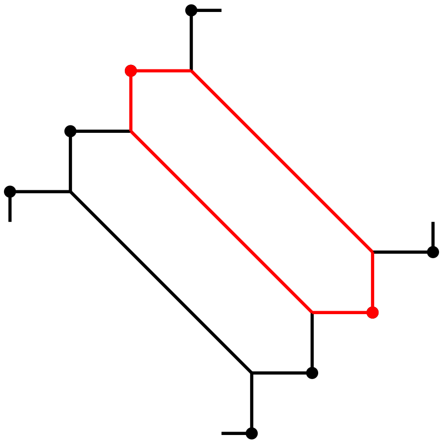
Stable part of $\mathcal{M}_a$



Semi-stable part of $\mathcal{M}_a$



ξ-stable part of $\mathcal{M}_a$, ξ generic



We get ...