Transfer of gradings via derived equivalences and $\mathfrak{sl}_2$-categorification

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$\mathfrak{sl}_2$-categorification

Let $\mathcal{A}$ be an artinian, noetherian, k-linear abelian category such that the endomorphism ring of any simple object is k.

DEFINITION A weak $\mathfrak{sl}_2$-*categorification* on $\mathcal{A}$ is the data of an adjoint pair (E, F) of exact endo-functors on $\mathcal{A}$ such that:

- the action of $e = [E]$ and $f = [F]$ on $V = \mathbb{Q} \otimes K_0(\mathcal{A})$ gives a locally finite $\mathfrak{sl}_2$-representation
- the classes of simple objects of $\mathcal{A}$ are weight vectors
- F is isomorphic to a left adjoint of E

An $\mathfrak{sl}_2$-*categorification* on $\mathcal{A}$ is a weak $\mathfrak{sl}_2$-categorification with the extra data of $q \in k^\times$ and $a \in k$ with $a \neq 0$ if $q \neq 1$, and of $X \in \mathrm{End}(E)$ and $T \in \mathrm{End}(E^2)$ such that:

- $(1_E T) \circ (T 1_E) \circ (1_E T) = (T 1_E) \circ (1_E T) \circ (T 1_E)$ in $\mathrm{End}(E^3)$
- $(T + 1_{E^2}) \circ (T - q 1_{E^2}) = 0$ in $\mathrm{End}(E^2)$
- $T \circ (1_E X) \circ T = \begin{cases} qX1_E & \text{if } q \neq 1 \\ X1_E - T & \text{if } q = 1 \end{cases}$ in $\mathrm{End}(E^2)$
- $X - a$ is locally nilpotent

Applications: Symmetric groups, Hecke algebras, q-Schur algebras, General linear groups over a finite field, Category $\mathcal{O}$.

Categorification of a simple reflection

Let $\lambda \in \mathbb{Z}$. A complex of functors

$$\theta_\lambda : \mathrm{Comp}(\mathcal{A}_{-\lambda}) \longrightarrow \mathrm{Comp}(\mathcal{A}_\lambda),$$

where $(\theta_\lambda)^{-r} = E^{(\lambda+r)}F^{(r)}$ for $r, \lambda + r \geq 0$ and $(\theta_\lambda)^{-r} = 0$ otherwise, and differentials $d^{-r} : E^{(\lambda+r)}F^{(r)} \longrightarrow E^{(\lambda+r-1)}F^{(r-1)}$ are restriction of maps $1_{E^{(\lambda+r-1)}}\varepsilon 1_{F^{(r-1)}}$, is a categorification of a simple reflection s.

PROPOSITION The map $[\theta_\lambda] : V_{-\lambda} = K_0(\mathcal{A}_{-\lambda}) \longrightarrow V_\lambda = K_0(\mathcal{A}_\lambda)$ coincides with the action of s.

THEOREM The complex of functors $\theta = \bigoplus_\lambda \theta_\lambda$ induces a self-equivalence of $K^b(\mathcal{A})$ and of $D^b(\mathcal{A})$ and induces by restriction equivalences $K^b(\mathcal{A}_{-\lambda}) \cong K^b(\mathcal{A}_\lambda)$ and $D^b(\mathcal{A}_{-\lambda}) \cong D^b(\mathcal{A}_\lambda)$. Furthermore, $[\theta] = s$.

Blocks of symmetric groups and the Fock space

Let p be a prime number and $k = F_p$. Let $a \in k$. Given M a kS_n-module, we denote by $F_{a,n}(M)$ the generalized a-eigenspace of $X_n := \sum_{j=1}^{n-1}(j, n)$. This is a kS_{n-1}-module. We have a decomposition $\mathrm{Res}^{kS_n}_{kS_{n-1}} = \oplus_{a\in k} F_{a,n}$. There is a corresponding decomposition $\mathrm{Ind}^{kS_n}_{kS_{n-1}} = \oplus_{a\in k} E_{a,n}$, where $E_{a,n}$ is left and right adjoint to $F_{a,n}$. We set $E_a = \oplus_{n\geq 1} E_{a,n}$ and $F_a = \oplus_{n\geq 1} F_{a,n}$.

THEOREM The functors E_a and F_a for $a \in F_p$ give rise to an action of the affine Lie algebra $\widehat{\mathfrak{sl}}_p$ on $\oplus_{n\geq 0} K_0(kS_n-\mathrm{mod})$. The decomposition of $K_0(kS_n-\mathrm{mod})$ in blocks coincides with its decomposition in weight spaces. Two blocks of symmetric groups have the same weight if and only if they are in the same orbit under the adjoint action of the affine Weyl group. In particular for each $a \in F_p$ the functors E_a and F_a give a weak $\mathfrak{sl}_2$-categorification on $\mathcal{A} = \oplus_{n\geq 0} kS_n-\mathrm{mod}$.

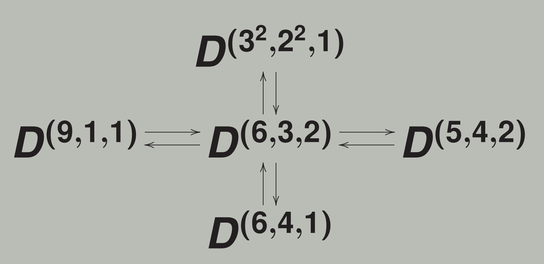
Weight 2 blocks of symmetric groups in characteristic 3

For any blocks B_1 and B_2 of weight 2 we have

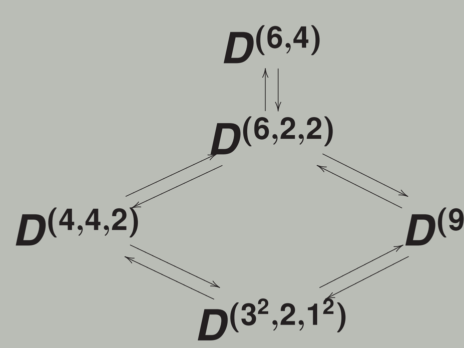
$$D^b(B_1) \cong D^b(B_2).$$

Up to Morita equivalence there are four blocks of weight 2 in characteristic 3.

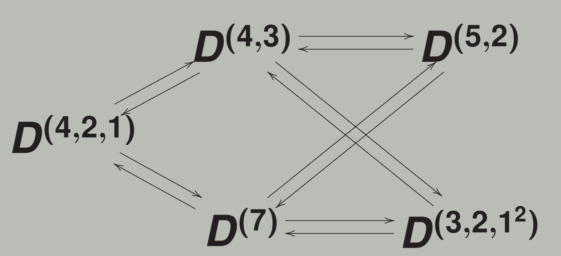
- $B(S_{11})$: the block of S_{11} with 3-core $(3, 1, 1)$ and the following quiver



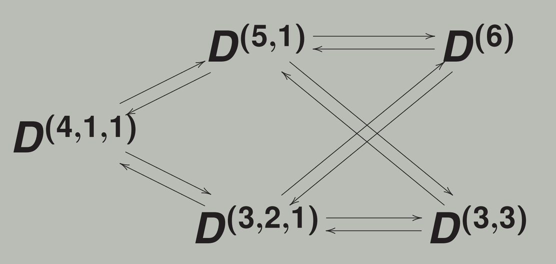
- $B(S_{10})$: the block of S_{10} with 3-core $(3, 1)$ and the following quiver



- $B_0(S_7)$: the principal block of S_7 with 3-core (1) and the following quiver



- $B_0(S_6)$: the principal block of S_6 with 3-core $\emptyset$ and the following quiver



Tilting complexes constructed using θ_λ

Let B_1 and B_2 be a $[2 : 1]$ pair in characteristic 3. To construct a tilting complex that tilts from B_1 to B_2 we apply θ_λ to projective indecomposable B_2-modules. If the 3-core of B_1 is larger than the 3-core of B_2, then we have that $\lambda = 1$ and that θ_λ is

$$0 \longrightarrow E^{(2)}F \longrightarrow E \longrightarrow 0,$$

with the differential being a map of maximal rank. We have that

$$T = \oplus_\mu \theta_\lambda(P^\mu),$$

where the sum runs over all 3-regular partitions μ that belong to B_2.

- A tilting complex that tilts from $B(S_{11})$ to $B(S_{10})$ is the sum of the following complexes:

$$\begin{array}{l} 0 \to P^{(3^2,2^2,1)} \to P^{(9,1,1)} \to 0 \\ 0 \to 0 \to P^{(6,4,1)} \to 0 \\ 0 \to P^{(3^2,2^2,1)} \to P^{(6,3,2)} \to 0 \\ 0 \to P^{(3^2,2^2,1)} \to P^{(5,4,2)} \to 0 \\ 0 \to P^{(3^2,2^2,1)} \to 0 \to 0 \end{array}$$

- A tilting complex that tilts from $B(S_{10}) \cong B_0(S_8)$ to $B_0(S_7)$ is the sum of the following complexes:

$$\begin{array}{l} 0 \to P^{(5,3)} \to P^{(8)} \to 0 \\ 0 \to P^{(5,3)} \to 0 \to 0 \\ 0 \to P^{(5,3)} \to P^{(4,3,1)} \to 0 \\ 0 \to P^{(5,3)} \to P^{(5,2,1)} \to 0 \\ 0 \to 0 \to P^{(3^2,1^2)} \to 0 \end{array}$$

- A tilting complex that tilts from $B_0(S_7)$ to $B_0(S_6)$ is the sum of the following complexes:

$$\begin{array}{l} 0 \to P^{(4,2,1)} \to P^{(7)} \to 0 \\ 0 \to P^{(4,2,1)} \to P^{(5,2)} \to 0 \\ 0 \to P^{(4,2,1)} \to 0 \to 0 \\ 0 \to P^{(4,2,1)} \to P^{(4,3)} \to 0 \\ 0 \to P^{(4,2,1)} \to P^{(3,2,1^2)} \to 0 \end{array}$$

Transfer of gradings via derived equivalences

THEOREM Let A and B be symmetric k-algebras. Assume $D^b(A) \cong D^b(B)$ and that A is graded. If T is a tilting complex of A-modules that induces derived equivalence between A and B, then there exists a grading on B and a structure of a graded complex T' on T, such that T' induces an equivalence $D^b(A-\mathrm{modgr}) \cong D^b(B-\mathrm{modgr})$.

Gradings and crystal decomposition matrices

PROPOSITION Let B be a weight 2 block in characteristic 3. If $C_q(B)$ is the graded Cartan matrix of B with respect to the tight grading and if $C^q(B)$ is the crystal Cartan matrix of B, then

$$C_q(B) = C^q(B).$$

CONJECTURE Let B be a weight 2 block. Then,

$$C_q(B) = C^q(B).$$

References

- J. Chuang and R. Rouquier, *Derived equivalences for symmetric groups and* $\mathfrak{sl}_2$*-categorification*, Annals of Mathematics, 167(2008), 245-298.
- R. Rouquier, *Automorphismes, Graduations et Catégories Triangulées*, preprint, 2000.