Character formulas for Feigin-Stoyanovsky's type subspaces of standard $A_2^{(1)}$-modules

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Abstract

Exact sequences of Feigin-Stoyanovsky's type subspaces for affine Lie algebra $\widetilde{\mathfrak{sl}}(l+1,\mathbb{C})$ lead to systems of recurrence relations for formal characters of those subspaces. By solving the corresponding system for $\widetilde{\mathfrak{sl}}(3,\mathbb{C})$, we obtain a new family of character formulas for all Feigin-Stoyanovsky's type subspaces at general level.

Affine Lie algebra and standard modules

$\mathfrak{g} = \mathfrak{sl}(\ell+1,\mathbb{C}) \rightsquigarrow$ simple Lie algebra
$\mathfrak{h} \rightsquigarrow$ a Cartan subalgebra of $\mathfrak{g}$
$R \rightsquigarrow$ root system
$\alpha_1,\dots,\alpha_\ell \rightsquigarrow$ fixed simple roots
x_α, $\alpha \in R \rightsquigarrow$ fixed root vectors
$\langle\cdot,\cdot\rangle \rightsquigarrow$ Killing form
$\omega_1,\dots,\omega_\ell \rightsquigarrow$ fundamental weights (with $\omega_0 = 0$)

$\tilde{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c\oplus\mathbb{C}d \rightsquigarrow$ affine Lie algebra associated to $\mathfrak{g}$
$c \rightsquigarrow$ canonical central element
$d \rightsquigarrow$ degree operator: $[d, x\otimes t^n] = nx\otimes t^n$
$[x\otimes t^m, y\otimes t^n] = [x,y]\otimes t^{m+n} + m\langle x,y\rangle\delta_{m+n,0}c \rightsquigarrow$ Lie product
$\mathfrak{h}^e = \mathfrak{h}\oplus\mathbb{C}c\oplus\mathbb{C}d$
$\{\alpha_0,\alpha_1,\dots,\alpha_\ell\}\subset(\mathfrak{h}^e)^* \rightsquigarrow$ simple roots
$\Lambda_0,\Lambda_1,\dots,\Lambda_\ell \rightsquigarrow$ fundamental weights

$\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_\ell\Lambda_\ell$, $\quad k_i\in\mathbb{Z}_+ \rightsquigarrow$ dominant integral weights
$L(\Lambda) \rightsquigarrow$ standard (integrable highest weight) $\tilde{\mathfrak{g}}$-module
$k = \Lambda(c) = k_0 + k_1 + \cdots + k_\ell \rightsquigarrow$ level of $L(\Lambda)$
$v_\Lambda \rightsquigarrow$ a highest weight vector

Feigin-Stoyanovsky's type subspaces

For fixed minuscule weight $\omega = \omega_\ell$ define

$$\Gamma = \{\alpha\in R \mid \langle\alpha,\omega\rangle = 1\} = \{\gamma_1,\gamma_2,\dots,\gamma_\ell \mid \gamma_i = \alpha_i+\cdots+\alpha_\ell\}.$$

This gives us a $\mathbb{Z}$-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1, \tag{1}$$

with $\mathfrak{g}_0 = \mathfrak{h} + \sum_{\langle\alpha,\omega\rangle=0}\mathfrak{g}_\alpha$, $\mathfrak{g}_{\pm1} = \sum_{\alpha\in\pm\Gamma}\mathfrak{g}_\alpha$, and correspondingly the $\mathbb{Z}$-grading

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} + \tilde{\mathfrak{g}}_0 + \tilde{\mathfrak{g}}_1,$$

having denoted $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c\oplus\mathbb{C}d$, $\tilde{\mathfrak{g}}_{\pm1} = \mathfrak{g}_{\pm1}\otimes\mathbb{C}[t,t^{-1}]$.

Definition 1 *For a standard $\tilde{\mathfrak{g}}$-module $L(\Lambda)$, Feigin-Stoyanovsky's type subspace of $L(\Lambda)$ is*

$$W(\Lambda) = U(\tilde{\mathfrak{g}}_1)\cdot v_\Lambda,$$

where $U(\tilde{\mathfrak{g}}_1)$ is the universal enveloping algebra of $\tilde{\mathfrak{g}}_1$.

Combinatorial bases

From Poincaré-Birkhoff-Witt theorem it follows that a Feigin-Stoyanovsky's type subspace $W(\Lambda)$ is spanned by set of monomial vectors

$$\{x(\pi)v_\Lambda | x(\pi) = \dots x_{\gamma_1}(-2)^{a_\ell}x_{\gamma_\ell}(-1)^{a_{\ell-1}}\cdots x_{\gamma_1}(-1)^{a_0}, a_i\in\mathbb{Z}_+, i\in\mathbb{Z}_+\}.$$

It is an important and interesting problem to reduce the above spanning set to monomial basis (basis consisting of monomial vectors) of $W(\Lambda)$.

Definition 2 *For level k standard $\tilde{\mathfrak{g}}$-module $L(\Lambda)$ with highest weight $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_\ell\Lambda_\ell$, we say that a monomial vector $x(\pi)v_\Lambda = \dots x_{\gamma_1}(-2)^{a_\ell}x_{\gamma_\ell}(-1)^{a_{\ell-1}}\cdots x_{\gamma_1}(-1)^{a_0}v_\Lambda \in W(\Lambda)$ is $(k,\ell+1)$-admissible for Λ if the following inequalities are met:*

initial conditions: $\quad a_0+\cdots+a_i \le k_0+\cdots+k_i, \quad i = 0,\dots,\ell-1$
difference conditions: $\quad a_i+\cdots+a_{i+\ell}\le k, \quad i\in\mathbb{Z}_+.$

Theorem 3 *The set of $(k,\ell+1)$-admissible monomial vectors for Λ is a basis for $W(\Lambda)$.*

Inspired by Capparelli, Lepowsky and Milas's use of intertwining operators in [2, 3], Primc in [8] obtained an elegant proof of Theorem 3.
Working also on $\mathfrak{g} = \mathfrak{sl}(\ell+1,\mathbb{C})$, but in more general setting of an arbitrary choice for ω (allowing it to be *any* of the fundamental weights $\omega_1,\dots,\omega_\ell$) - therefore covering all possible $\mathbb{Z}$-gradings (1), Trupčević in [9, 10] also uses intertwining operators to prove linear independence of combinatorial bases for Feigin-Stoyanovsky's type subspaces of all standard $\tilde{\mathfrak{g}}$-modules at arbitrary integer level.
Baranović in [1] gives a combinatorial description (in terms of difference and initial conditions) of bases for Feigin-Stoyanovsky's type subspaces for level 1 standard modules for affine Lie algebra of type $D_\ell^{(1)}$, and for a specific choice of (1). She then extends her method to obtain combinatorial bases in the case of level 2 standard modules of affine Lie algebra $D_4^{(1)}$.

Exact sequences

We now provide an exposition of new results obtained for $\mathfrak{g} = \mathfrak{sl}(\ell+1,\mathbb{C})$ in [6, 7]. For $\lambda_i := \omega_i - \omega_{i-1}$, $i = 1,\dots,\ell$, define

$$[i] := \mathrm{Res} z^{-1-\langle\lambda_i,\omega_{i-1}\rangle}c_i\mathcal{Y}(1\otimes e^{\lambda_i}, z),$$

where $\mathcal{Y}$ are Dong-Lepowsky's level 1 intertwining operators (cf. [4]). For suitably chosen constants c_i we have $L(\Lambda_{i-1}) \xrightarrow{[i]} L(\Lambda_i)$, with $[i]v_{\Lambda_{i-1}} = v_{\Lambda_i}$. Also, $[i]$ commute with the action of $x(\pi)$: $x(\pi)[i] = [i]x(\pi)$.
Furthermore, we use the so-called simple current operator, a linear bijection $[\omega]$ such that

$$L(\Lambda_0)\xrightarrow{[\omega]}L(\Lambda_\ell)\xrightarrow{[\omega]}L(\Lambda_{\ell-1})\xrightarrow{[\omega]}\dots\xrightarrow{[\omega]}L(\Lambda_1)\xrightarrow{[\omega]}L(\Lambda_0)$$
$$[\omega]v_{\Lambda_0} = v_{\Lambda_\ell},\quad [\omega]v_{\Lambda_i} = x_{\gamma_i}(-1)v_{\Lambda_{i-1}},\quad i = 1,\dots,\ell,$$

together with important property $x(\pi)[\omega] = [\omega]x(\pi^+)$, with $x(\pi^+)$ denoting monomial obtained from $x(\pi)$ by raising degrees of all factors by one.
Denote by $[i]_j = 1^{\otimes(k-j)}\otimes[i]\otimes 1^{\otimes(j-1)}$, $i = 1,\dots,\ell$ and $j = 0,\dots,\ell$, linear maps between higher level k standard $\tilde{\mathfrak{g}}$-modules that will keep the above mentioned properties of $[i]$. Similarly, we use $[\omega]^{\otimes k}$.
Fix $K = (k_0,\dots,k_\ell)$ such that $k_0+\cdots+k_\ell = k$, $k_i\in\mathbb{Z}_+$, $i = 0,\dots,\ell$.
Denote $W = W_{k_0,k_1,\dots,k_\ell} = W(\Lambda)$ for $\Lambda = k_0\Lambda_0+\cdots+k_\ell\Lambda_\ell$, and by v highest weight vector of $L(\Lambda)$.
Define also $m = \sharp\{i = 0,\dots,\ell-1 \mid k_i\neq 0\}$ and for $t = 0,\dots,m-1$ set

$$I_t = \{\{i_0,\dots,i_{t-1}\} | 0\le i_0\le\cdots\le i_{t-1}\le\ell-1, k_{i_j}\neq 0, j = 0,\dots,t-1\}.$$

Now, denote $W_{I_t} = W_{k_0,\dots,k_{i_0}-1,k_{i_0+1}+1,\dots,k_{i_{t-1}}-1,k_{i_{t-1}+1}+1,\dots,k_\ell}$, and by v_{I_t} the corresponding highest weight vector.
Introduce $U(\tilde{\mathfrak{g}}_1)-$homogeneous mappings $\varphi_t : \sum_{I_t}W_{I_t}\to\sum_{I_{t+1}}W_{I_{t+1}}$ by

$$\varphi_t|_{W_{I_t}} = \sum_{\substack{i,k_i\neq0\\ i\notin I_t}}(-1)^{\sharp\{j\in I_t|j<i\}}[i]_{k_0+\cdots+k_{i-1}}.$$

Theorem 4 *The following sequence is exact:*

$$0\to W_{k_\ell,k_0,k_1,\dots,k_{\ell-1}}\xrightarrow{[\omega]^{\otimes k}}W\xrightarrow{\varphi_0}\sum_{I_1}W_{I_1}\xrightarrow{\varphi_1}\dots\xrightarrow{\varphi_{m-1}}W_{I_m}\to 0.$$

Example 5 *For Feigin-Stoyanovsky's type subspaces of level 2 standard $\widetilde{\mathfrak{sl}}(3,\mathbb{C})$-modules we have the following family of exact sequences:*

$$0\to W_{0,2,0}\to W_{2,0,0}\to W_{1,1,0}\to 0$$
$$0\to W_{0,1,1}\to W_{1,1,0}\to W_{0,2,0}\oplus W_{1,0,1}\to W_{0,1,1}\to 0$$
$$0\to W_{1,1,0}\to W_{1,0,1}\to W_{0,1,1}\to 0$$
$$0\to W_{0,0,2}\to W_{0,2,0}\to W_{0,1,1}\to 0$$
$$0\to W_{1,0,1}\to W_{0,1,1}\to W_{0,0,2}\to 0$$
$$0\to W_{2,0,0}\to W_{0,0,2}\to 0$$

The proof of Theorem 4 relies on the interplay between initial conditions for various dominant integral weights at the fixed level k (note that difference conditions are the same for all Feigin-Stoyanovsky's type subspaces at the same integer level), and on use of properties for $[i]$ and $[\omega]$, cf. [6] for details.

Recurrences and characters

We proceed by defining formal character of $W = W(\Lambda)$.

Definition 6 *For $x(\pi) = \dots x_{\gamma_1}(-2)^{a_\ell}x_{\gamma_\ell}(-1)^{a_{\ell-1}}\cdots x_{\gamma_1}(-1)^{a_0}$ define degree $d(x(\pi)) = \sum_{j=0}^{\infty}\sum_{i=1}^{\ell}(j+1)a_{i+j\ell-1}$ and weight $w(x(\pi)) = \sum_{j=0}^{\infty}\sum_{i=1}^{\ell}\gamma_i a_{i+j\ell-1}$. Formal character of $W = W(\Lambda)$ is given by*

$$\chi(W)(z_1,\dots,z_\ell;q) = \sum\dim W^{m,n_1,\dots,n_\ell}q^m z_1^{n_1}\cdots z_\ell^{n_\ell},$$

with $W^{m,n_1,\dots,n_\ell}$ denoting the component of W spanned by basis monomial vectors $x(\pi)v$ of degree m and weight $n_1\gamma_1+\cdots+n_\ell\gamma_\ell$.

As a direct consequence of Theorem 4 we obtain systems of relations connecting characters of *all* Feigin-Stoyanovsky's type subspaces of arbitrary integer level k standard $\widetilde{\mathfrak{sl}}(\ell+1,\mathbb{C})$-modules:

$$\sum_{I\in D(K)}(-1)^{|I|}\chi(W_I)(z_1,\dots,z_\ell;q) = \tag{2}$$
$$= (z_1q)^{k_0}\dots(z_\ell q)^{k_{\ell-1}}\chi(W_{k_\ell,k_0,\dots,k_{\ell-1}})(z_1q,\dots,z_\ell q;q),$$

where $D(K)$ denotes the set of all I_{t+1} as defined in the previous section.

Example 7 *For Feigin-Stoyanovsky's type subspaces of level 2 standard $\widetilde{\mathfrak{sl}}(3,\mathbb{C})$-modules we have the following system of relations:*

$$\chi(W_{2,0,0})(z_1,z_2;q) = \chi(W_{1,1,0})(z_1,z_2;q) + (z_1q)^2\chi(W_{0,2,0})(z_1q,z_2q;q)$$
$$\chi(W_{1,1,0})(z_1,z_2;q) = \chi(W_{0,2,0})(z_1,z_2;q) + \chi(W_{1,0,1})(z_1,z_2;q) -$$
$$- \chi(W_{0,1,1})(z_1,z_2;q) + (z_1q)(z_2q)\chi(W_{0,1,1})(z_1q,z_2q;q)$$
$$\chi(W_{1,0,1})(z_1,z_2;q) = \chi(W_{0,1,1})(z_1,z_2;q) + z_1q\chi(W_{1,1,0})(z_1q,z_2q;q)$$
$$\chi(W_{0,2,0})(z_1,z_2;q) = \chi(W_{0,1,1})(z_1,z_2;q) + (z_2q)^2\chi(W_{0,0,2})(z_1q,z_2q;q)$$
$$\chi(W_{0,1,1})(z_1,z_2;q) = \chi(W_{0,0,2})(z_1,z_2;q) + z_2q\chi(W_{1,0,1})(z_1q,z_2q;q)$$
$$\chi(W_{0,0,2})(z_1,z_2;q) = \chi(W_{2,0,0})(z_1q,z_2q;q).$$

After introducing

$$\chi(W_{k_0,\dots,k_\ell})(z_1,\dots,z_\ell;q) = \sum_{n_1,\dots,n_\ell\ge0}A_{k_0,\dots,k_\ell}^{n_1,\dots,n_\ell}(q)z_1^{n_1}\cdots z_\ell^{n_\ell}$$

into (2) we get (for every choice of $n_1,\dots,n_\ell\ge0$)

$$\sum_{I\in D(K)}(-1)^{|I|}A_I^{n_1,\dots,n_\ell}(q) = q^{n_1+\cdots+n_\ell}A_{k_\ell,k_0,\dots,k_{\ell-1}}^{n_1-k_0,\dots,n_\ell-k_{\ell-1}}(q). \tag{3}$$

It is not hard to prove that the system (3) consists of relations that are *recursive* and that it has a *unique* solution (not obvious by itself), cf. Propositions 6.2 and 6.3 in [6].
We were also able to prove the following result:

Theorem 8 *Let $\Lambda = k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2$ be the highest weight of the level k standard $\widetilde{\mathfrak{sl}}(3,\mathbb{C})$-module $L(\Lambda)$. The following formula holds:*

$$\chi(W(k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2))(z_1,z_2;q) =$$
$$= \sum_{n_1,n_2\ge0}\ \sum_{\substack{\sum_{i=1}^k N_{1,i}=n_1\\ N_{1,1}\ge\cdots\ge N_{1,k}\ge0\\ \sum_{i=1}^k N_{2,i}=n_2\\ N_{2,k}\ge\cdots\ge N_{2,1}\ge0}}\frac{q^{\sum_{i=1}^k(N_{1,i}^2+N_{2,i}^2+N_{1,i}N_{2,i})}\cdot L_{k_0,k_1,k_2}^{N_1,N_2}(q)}{(q)_{N_{1,1}-N_{1,2}}\cdots(q)_{N_{1,k}}(q)_{N_{2,k}-N_{2,k-1}}\cdots(q)_{N_{2,1}}}z_1^{n_1}z_2^{n_2}$$

$$L_{k_0,k_1,k_2}^{N_1,N_2}(q) = \sum_{\substack{p_1\in P_{k_1+k_2}^1\\ p_2\sim p_1}}q^{\sum_{i=1}^k p_{1,i}N_{1,i}+p_{2,i}N_{2,i}}\prod_{i=1}^k(1-\delta_{p_{1,i}-p_{1,i+1},-1}q^{N_{1,i}-N_{1,i+1}}),$$

where $P_{k_1+k_2}^1 = \{(p_{1,1},\dots,p_{1,k})\in\{0,1\}^k | \sum_{i=1}^k p_{1,i} = k_1+k_2\}$ and $p_2 = (p_{2,1},\dots,p_{2,k})\sim p_1$ means $p_{2,i} = 1$ if $p_{1,i} = 1$ and $\sharp\{p_{1,j}|p_{1,j} = 1, j\le i\}\le k_2$, and $p_{2,i} = 0$ otherwise. Also, $N_1 = (N_{1,1},\dots,N_{1,k})$, $N_2 = (N_{2,1},\dots,N_{2,k})$, and $N_{1,k+1} = p_{1,k+1} = 0$.

Example 9 *Let us present "linear" terms in the nominators of character formulas for Feigin-Stoyanovsky's type subspaces of level 2 standard $\widetilde{\mathfrak{sl}}(3,\mathbb{C})$-modules:*

$$L_{2,0,0}^{N_{1,1},N_{1,2},N_{2,1},N_{2,2}}(q) = 1$$
$$L_{1,1,0}^{N_{1,1},N_{1,2},N_{2,1},N_{2,2}}(q) = q^{N_{1,2}}$$
$$L_{1,0,1}^{N_{1,1},N_{1,2},N_{2,1},N_{2,2}}(q) = q^{N_{1,1}+N_{2,1}} + q^{N_{1,2}+N_{2,2}}(1-q^{N_{1,1}-N_{1,2}})$$
$$L_{0,2,0}^{N_{1,1},N_{1,2},N_{2,1},N_{2,2}}(q) = q^{N_{1,1}+N_{1,2}}$$
$$L_{0,1,1}^{N_{1,1},N_{1,2},N_{2,1},N_{2,2}}(q) = q^{N_{1,1}+N_{1,2}+N_{2,1}}$$
$$L_{0,0,2}^{N_{1,1},N_{1,2},N_{2,1},N_{2,2}}(q) = q^{N_{1,1}+N_{1,2}+N_{2,1}+N_{2,2}}.$$

One should mention that formulas in Theorem 8 represent a complete set of full characters for Feigin-Stoyanovsky's type subspaces of all standard $\widetilde{\mathfrak{sl}}(3,\mathbb{C})$-modules, which specially reinstalls but also strengthens the result obtained in [5].
Theorem is proved by checking that character formulas for $W(k_0\Lambda_0+k_1\Lambda_1+k_2\Lambda_2)$ presented in Theorem 8 satisfy the corresponding system (2); more precisely, by checking that matching $A_{k_0,k_1,k_2}^{n_1,n_2}(q)$ satisfy (3), cf. [7].

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