QHWR of the Lie subalgebra of type orthogonal of matrix differential operators on the circle.

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Resumen

In this paper classify the irreducible quasifinite highest weight modules of the orthogonal Lie subalgebra of the Lie algebra of matrix differential operators on the circle and construct them in terms of representations theory of the complex Lie algebra $g\ell_\infty^{[m]}$ of infinite matrices with finite number of non-zero diagonals over the algebra $R_m = \mathbb{C}[u]/(u^{m+1})$ and its subalgebras of type B and D.

The algebra $\widehat{\mathfrak{D}^N}$

Let N be a positive integer. Denote by $\mathfrak{D}^N_{as}$ the associative algebra of all regular matrix differential operators on $\mathbb{C}^\times$, of the form

$$E = e_k(t)\partial_t^k + e_{k-1}(t)\partial_t^{k-1} + \ldots + e_0(t)$$

where

$$e_i(t) \in Mat_N\mathbb{C}[t, t^{-1}]$$

and denote by $\mathfrak{D}^N$ the corresponding Lie algebra.

Let

$$\widehat{\mathfrak{D}^N} = \mathfrak{D}^N + \mathbb{C}C$$

denote the central extension of $\mathfrak{D}^N$ by a one-dimensional center $\mathbb{C}C$.

The subalgebra $\widehat{\mathfrak{D}_{on}^N}$

For any $A \in \mathrm{Mat}_N(\mathbb{C})$ we set $(A)^\dagger_{i,j} = A_{N+1-j,N+1-i}$ and let the anti-involution on $\mathcal{D} = \mathcal{D}^1$ given by

$$\dot{\sigma}_{+,-1}(t^k f(D)) = (-t)^k f(-D-k-1).$$

We extend to a map on $\mathrm{Mat}_N(\mathcal{D}) = \mathcal{D} \otimes \mathrm{Mat}_N(\mathbb{C})$ by taking $[\dot{\sigma}_{+,-1}(A)]_{i,j} = \dot{\sigma}_{+,-1}(A_{i,j})$.
Consider the anti-involution τ in $\mathfrak{D}^N$ defined by

$$\tau(t^k f(D)A) = \dot{\sigma}_{+,-1}(t^k f(D)A^\dagger). \tag{0.1}$$

We denote by $\mathfrak{D}_o^N$ the Lie subalgebra of $\mathfrak{D}^N$ given by $-\tau$-fixed points in $\mathfrak{D}^N$.
Denote by $\widehat{\mathfrak{D}_o^N}$ the central extension of $\mathfrak{D}_o^N$ by the one dimensional center $\mathbb{C}C$. Letting $\mathrm{wt}(z^k f(D)E_{i,j}) = kN + i - j$, $\mathrm{wt}(C) = 0$ gives the *principal gradation* de $\widehat{\mathfrak{D}^N}$, which is inherithed by $\mathfrak{D}_o^N$.

Definition

A *parabolic subalgebra* $\mathcal{P}$ of $\widehat{\mathcal{D}_o^N}$ as a subalgebra of the form $\mathcal{P} = \bigoplus_{k,\, r\in\mathbb{Z}}(\mathcal{P})_{kN+r}$ where $(\mathcal{P})_{kN+r} = (\widehat{\mathcal{D}_o^N})_{kN+r}$ if $Nk+r \geq 0$, and $(\mathcal{P})_{Nk+r} \neq 0$ for some $Nk+r<0$.

Given $a \in (\widehat{D_o^N})_{-1}$, with $a \neq 0$, we define $\mathcal{P}^a = \bigoplus_{-N+1\leq r\leq N-1,\ k\in\mathbb{Z}}(\mathcal{P}^a)_{kN+r}$ where $(\mathcal{P}^a)_{kN+r} = (\widehat{\mathcal{D}_o^N})_{kN+r}$ if $Nk+r \geq 0$ and

$$\mathcal{P}^a{}_{-1} = \sum [\dots[[a,(\widehat{D_o^N})_0],(\widehat{D_o^N})_0],\dots] \quad \mathcal{P}^a{}_{-k-1} = [\mathcal{P}^a{}_{-1},\mathcal{P}^a{}_{-k}].$$

We call a parabolic subalgebra $\mathcal{P}$ *non-degenerate* if $\mathcal{P}_{-j}$ has finite codimension in $(\widehat{D_o^N})_{-j}$, for all $j > 0$, and an element $a \in (\widehat{D_o^N})_{-1}$ *non-degenerate* if $\mathcal{P}^a$ is non-degenerate.

Definition

Let $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ any $\mathbb{Z}$-graded Lie algebra over $\mathbb{C}$, and let $\mathfrak{g}_+ = \oplus_{j>0}\mathfrak{g}_j$. A $\mathfrak{g}$-module V is called *$\mathbb{Z}$-graded* if $V = \oplus_{j\in\mathbb{Z}}V_j$ and $\mathfrak{g}_i V_j \subset V_{i+j}$. A $\mathbb{Z}$-graded $\mathfrak{g}$-module V is called *quasifinite* if $\dim V_j < \infty$ for all j. Given $\lambda \in \mathfrak{g}_0^*$, and $\mathfrak{g}$-module $V(\mathfrak{g}, \lambda)$ a *highest weight module* with highest weight vector $v_\lambda \in V(\mathfrak{g}, \lambda)_0$. A non-zero vector $v \in V(\mathfrak{g}, \lambda)$ is called *singular* if $\mathfrak{g}_+ v = 0$.

Theorem

The following conditions on $\lambda \in (\widehat{D_o^N})^*$ *are equivalent:*

- *The Verma module* $M(\widehat{D_o^N};\, \lambda)$ *contains a singular vector* $av_\lambda \in M(\widehat{D_o^N}, \lambda)_{-1}$, *where a is non-degenerate ;*
- *There existe a non-degenerate element* $a \in (\widehat{D_o^N})_{-1}$, *such that* $\lambda([(\widehat{D_o^N})_1, a]) = 0$;
- *The irreducible module* $L(\widehat{D_o^N}; \lambda)$ *is quasifinite;*
- *There exist a non-degenerate element* $a \in (\widehat{D_o^N})_{-1}$, *such that* $L(\widehat{D_o^N}; \lambda)$ *is the irreducible quotient of a generalized Verma module* $M(\widehat{D_o^N};\, \mathcal{P}^a,\, \lambda)$,

Given $\lambda \in (\widehat{D_o^N})_0^*$ define the *labels*

$$\Delta_{i,l} = -\lambda((D_0)^l E_{i,i} - (-D_0)^l E_{N+1-i,N+1-i})$$

with $l \in \mathbb{Z}$, $i = 1 \dots N$ and the *central charge* $c = \lambda(C)$, and consider the generating series

$$\Delta_i(x) = \sum_{l \geq 0} \frac{x^l}{l!} \Delta_{i,l}, \qquad i = 1 \dots N.$$

Theorem

A $\widehat{\mathfrak{D}_o^N}$-module $L(\lambda)$ is quasifinite if and only if

$$\Delta_N(x) = \frac{\phi_N(x)}{e^{\frac{x}{2}}}$$

$$\Delta_i(x) - \Delta_N(x) = \sum_{i \leq k \leq N-1} \phi_k(x)$$

where $\phi_k(x)$ are all quasipolynomials.

Given $s \in \mathbb{C}$, we obtain a family of homomorphism of Lie algebras $\varphi_s : \mathcal{D}_o^N \to g\ell_\infty^{[m]}$ (resp. $\varphi_s : (\mathcal{D}_o^N)^{\mathcal{O}} \to g\ell_\infty^{[m]}$).

$$\begin{aligned}
\varphi_s^{[m]} \left(t^k \left(f(D_k)\, E_{i,j} - f(D_{-k})\, E_{N+1-j,N+1-i} \right) \right) = \\
= \sum_{r=0}^{m} \sum_{l \in \mathbb{Z}} \Bigg[& \frac{f^{(r)}\left(-l + \frac{k+1}{2} + s\right)}{r!} u^r E_{(l-k)N-i+1,lN-j+1} \\
& -(-1)^r \frac{f^{(r)}\left(l - \left(\frac{k+1}{2}\right) - s\right)}{r!} u^r \, E_{(l-k-1)N+j,(l-1)N+i} \Bigg]
\end{aligned} \tag{0.2}$$

with $1 \leq i < j \leq N$ and $f^{(r)}$ denote the rth derivative of f.

$$\varphi_s^{[m]}\left(t^k\left(f(D_k)E_{i,N+1-j}\right)\right) = \\ = \sum_{r=0}^{m}\sum_{l\in\mathbb{Z}} \frac{f^{(r)}\left(-l+\frac{k+1}{2}+s\right)}{r!} u^r E_{(l-k)N-i+1,j} \tag{0.3}$$

where again, $1 \le i \le N$ and $f^{(r)}$ denote the rth derivative of f. Fix $\vec{s} = (s_1, \cdots, s_M) \in \mathbb{C}^M$, such that $s_i - s_j \notin \mathbb{Z}$ if $i \neq j$ and $s_i + s_j \notin \mathbb{Z}$ for all i, j. Also fix $\vec{m} = (m_1, \cdots, m_M) \in \mathbb{Z}_+^M$. Let $g\ell_\infty^{[\vec{m}]} = \oplus_{i=1}^M g\ell_\infty^{[m_i]}$ and consider the homomorphism

$$\varphi_{\vec{s}}^{[\vec{m}]} = \oplus_{i=1}^M \varphi_{s_i}^{[m_i]} \ : \ \left(\mathcal{D}_n^N\right)^O \longrightarrow g\ell_\infty^{[\vec{m}]}. \tag{0.4}$$

Where $\mathcal{O}$ denote the algebra of all holomorphic functions on $\mathbb{C}$ with the topology of uniform convergence on compact sets and $(\mathcal{D}_o^N)^{\mathcal{O}}$ is a completion of $\mathcal{D}_o^N$

consisting of all differential operators in $\mathcal{D}_o^N$ with $f \in \mathcal{O}$.

Proposition

The homomorphism $\varphi_s^{[m]}$ lifts to a Lie algebra homomorphism $\widehat{\varphi}_s^{[m]}$ of the corresponding central extensions.

Given $\vec{m} = (m_1, \cdots m_M) \in \mathbb{Z}_+^N$ and $\vec{s} = (s_1, \cdots, s_M)$ such that, $s_i \in \mathbb{Z}$ implies $s_i = 0$; $s_i \in \mathbb{Z} + 1/2$ implies $s_i = 1/2$ and $s_i \neq \pm s_j \mod \mathbb{Z}$ for $i \neq j$

$$\widehat{\varphi}_{\vec{s}}^{[\vec{m}]} = \bigoplus_{i=1}^{M} \varphi_{s_i}^{[m_i]} : \widehat{\mathcal{D}_o^N} \longrightarrow \mathfrak{g}^{[\vec{m}]} := \oplus_{i=1}^{M} \mathfrak{g}^{[m_i]}, \tag{0.5}$$

where

$$\mathfrak{g}^{[m]} = \begin{cases} \widehat{g\ell}_{\infty}^{[m]} & \text{if } s \notin \mathbb{Z}/2 \\ b_{\infty}^{[m]} & \text{if } s = 1/2 \quad \text{and} \quad N \textit{ odd} \\ d_{\infty}^{[m]} & \text{if } s = 0 \text{ or } s = 1/2 \quad \text{and } N \textit{ even} \end{cases} \tag{0.6}$$

Proposition

The homomorphism $\widehat{\varphi}_{\vec{s}}^{[\vec{m}]}$ extends to a surjective homomorphism of Lie algebras which is denoted again by $\widehat{\varphi}_{\vec{s}}^{[\vec{m}]}$:

$$\widehat{\varphi}_{\vec{s}}^{[\vec{m}]} = \bigoplus_{i=1}^{M} \widehat{\varphi}_{s_i}^{[m_i]} : (\widehat{\mathcal{D}_o^N})^{\mathcal{O}} \longrightarrow \mathfrak{g}^{[\vec{m}]}.$$

Proposition

The $\mathfrak{g}^{[m]}$-module $L\left(\mathfrak{g}^{[m]},\ \lambda\right)$ is quasifinite if and only if all but finitely many of the ${}^*h_k^{(i)}$ are zero, where $*$ represents a, b or d depending on whether $\mathfrak{g}^{[m]}$ is $\widehat{g\ell}_\infty^{[m]}$, $b_\infty^{[m]}$ or $d_\infty^{[m]}$.

Given $\vec{m} = (m_1, \cdots m_M) \in \mathbb{Z}_+^M$, take a quasifinite $\lambda_i \in \left(\mathfrak{g}^{[m_i]}\right)_0^*$ for each $i = 1, \cdots, M$ and let $L\left(\mathfrak{g}^{[m_i]},\ \lambda_i\right)$ be the corresponding irreducible $\mathfrak{g}^{[m_i]}$-module. Let $\vec{\lambda} = (\lambda_1, \cdots, \lambda_M)$. Then the tensor product

$$L\left(\mathfrak{g}^{[\vec{m}]}, \vec{\lambda}\right) = \otimes_{i=1}^{M} L\left(\mathfrak{g}^{[m_i]},\ \lambda_i\right)$$

is an irreducible $\mathfrak{g}^{[\vec{m}]}$-module, with $\mathfrak{g}^{[\vec{m}]} = \oplus_{i=1}^{M} \mathfrak{g}^{[m_i]}$. The module $L(\mathfrak{g}^{[\vec{m}]}, \vec{\lambda})$ can be regarded as a $\widehat{\mathcal{D}_o^N}$- module via the homomorphism $\varphi_{\vec{s}}^{[\vec{m}]}$, and will be denoted by $L_{\vec{s}}^{[\vec{m}]}(\vec{\lambda})$.

Theorem

Let V be a quasifinite $\mathfrak{g}^{[\vec{m}]}$-module, which is regarded as a $\widehat{\mathcal{D}_o^N}$-module via the homomorphism $\varphi_{\vec{s}}^{[\vec{m}]}$. Then any $\widehat{\mathcal{D}_o^N}$-submodule of V is also a $\mathfrak{g}^{[\vec{m}]}$-submodule. In particular, the $\widehat{\mathcal{D}_o^N}$-module $L_{\vec{s}}^{[\vec{m}]}(\vec{\lambda})$ are irreducible if $\vec{s} = (s_1, \cdots, s_M)$ is such that, $s_i \in \mathbb{Z}$ implies $s_i = 0$; $s_i \in \mathbb{Z} + 1/2$ implies $s_i = 1/2$ and $s_i \neq \pm s_j \mod \mathbb{Z}$ for $i \neq j$.

Proposition

Consider the embedding $\widehat{\varphi}_s^{[m]} : \widehat{\mathcal{D}_o^N} \longrightarrow \widehat{g\ell_\infty}^{[m]}$ with $s \notin \mathbb{Z}/2$. The $\widehat{g\ell_\infty}^{[m]}$-module $L(\widehat{g\ell_\infty}^{[m]}, \lambda)$ regarded as a $\widehat{\mathcal{D}_o^N}$-module is isomorphic to $L(\widehat{\mathcal{D}_o^N}, e^+, e^-)$ where e^+ and e^- consist of exponents $-l + s + \frac{1}{2}$ with $l \in \mathbb{Z}$ with multiplicities

$$- \sum_{0 \leq r \leq m} \lambda_{lN-i+1}^{(r)} \frac{x^r}{r!}, \quad \text{and} \sum_{0 \leq r \leq m} \lambda_{(l-1)N+i}^{(r)} (-1)^r \frac{x^r}{r!}$$

respectively.

We obtain similar results for $s = \frac{1}{2}$, N odd and even and $s = 0$.