Bidiagonal pairs, Tridiagonal pairs, Lie algebras, and Quantum Groups

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Definition

Let V denote a finite dimensional vector space. Let $A : V \to V$ and $A^* : V \to V$ denote linear transformations. We say A, A^* is a **bidiagonal pair** on V whenever (1)– (3) hold.

1. Each of A, A^* is diagonalizable.

2. There exists an ordering $V_0, V_1, \ldots, V_d$ (resp. $V_0^*, V_1^*, \ldots, V_d^*$) of the eigenspaces of A (resp. A^*) such that
$$(A - \theta_i I)V_i^* \subseteq V_{i+1}^* \qquad (0 \leq i \leq d)$$
$$(A^* - \theta_i^* I)V_i \subseteq V_{i+1} \qquad (0 \leq i \leq d)$$
where θ_i (resp. θ_i^*) is the eigenvalue of A (resp. A^*) associated with V_i (resp. V_i^*) and $V_{d+1} = 0, V_{d+1}^* = 0$.

3. For $0 \leq i \leq d/2$ the restrictions
$$(A - \theta_{d-i-1} I) \cdots (A - \theta_{i+1} I)(A - \theta_i I)|_{V_i^*} : V_i^* \to V_{d-i}^*$$
$$(A^* - \theta_{d-i-1}^* I) \cdots (A^* - \theta_{i+1}^* I)(A^* - \theta_i^* I)|_{V_i} : V_i \to V_{d-i}$$
are bijections.

Definition

Let V denote a finite dimensional vector space. Let $A : V \rightarrow V$ and $A^* : V \rightarrow V$ denote linear transformations. We say A, A^* is a **tridiagonal pair** on V whenever (1)–(4) hold.

1. Each of A, A^* is diagonalizable.

2. There exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d),$$

   where $V_{-1} = 0$, $V_{d+1} = 0$.

3. There exists an ordering $V^*_0, V^*_1, \ldots, V^*_d$ of the eigenspaces of A^* such that

$$A V^*_i \subseteq V^*_{i-1} + V^*_i + V^*_{i+1} \qquad (0 \leq i \leq d),$$

   where $V^*_{-1} = 0$, $V^*_{d+1} = 0$.

4. There does not exist a subspace W of V such that $AW \subseteq W, \quad A^*W \subseteq W, \quad W \neq 0, \quad W \neq V$.

The eigenvalues of bidiagonal and tridiagonal pairs

Let A, A^* denote a **bidiagonal** pair on V. For $0 \leq i \leq d$, let θ_i (resp. θ^*_i) denote the eigenvalue of A (resp. A^*) associated with V_i (resp. V^*_i).

Theorem (F.–N.): The expressions

$$\frac{\theta_{i-1} - \theta_i}{\theta_i - \theta_{i+1}}, \qquad \frac{\theta^*_{i-1} - \theta^*_i}{\theta^*_i - \theta^*_{i+1}}$$

are equal and independent of i for $1 \leq i \leq d-1$.

Let A, A^* be a **tridiagonal** pair on V. For $0 \leq i \leq d$, let θ_i (resp. θ^*_i) denote the eigenvalue of A (resp. A^*) associated with V_i (resp. V^*_i).

Theorem (Ito, Tanabe, Terwilliger): The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are equal and independent of i for $2 \leq i \leq d-1$.

Solving the bidiagonal eigenvalue recurrence

Solving the recurrence relation for the eigenvalues of a bidiagonal pair we find that the sequences $\theta_0, \theta_1, \ldots, \theta_d$ and $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ have one of the following types.

Type I: There exist scalars a_1, a_2, b_1, b_2 such that for $0 \leq i \leq d$

$$\theta_i = a_1 + a_2(2i - d),$$
$$\theta^*_i = b_1 + b_2(d - 2i).$$

Type II: There exist scalars q, a_1, a_2, b_1, b_2 such that for $0 \leq i \leq d$

$$\theta_i = a_1 + a_2\, q^{2i-d},$$
$$\theta^*_i = b_1 + b_2\, q^{d-2i}.$$

Solving the tridiagonal eigenvalue recurrence

Solving the recurrence relation for the eigenvalues of a tridiagonal pair we find that the sequences $\theta_0, \theta_1, \ldots, \theta_d$ and $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ have one of the following types.

Type I: There exist scalars $a_1, a_2, a_3, b_1, b_2, b_3$ such that for $0 \leq i \leq d$

$$\theta_i = a_1 + a_2\,(2i-d) + a_3\,(2i-d)^2,$$
$$\theta^*_i = b_1 + b_2\,(d-2i) + b_3\,(d-2i)^2.$$

Type II: There exist scalars $q, a_1, a_2, a_3, b_1, b_2, b_3$ such that for $0 \leq i \leq d$

$$\theta_i = a_1 + a_2\,q^{2i-d} + a_3\,q^{d-2i},$$
$$\theta^*_i = b_1 + b_2\,q^{d-2i} + b_3\,q^{2i-d}.$$

Type III: There exist scalars $a_1, a_2, a_3, b_1, b_2, b_3$ such that for $0 \leq i \leq d$

$$\theta_i = a_1 + a_2\,(-1)^i + a_3\,(2i-d)\,(-1)^i,$$
$$\theta^*_i = b_1 + b_2(-1)^i + b_3\,(d-2i)\,(-1)^i.$$

An alternative presentation of the Lie algebra $\mathfrak{sl}_2$

Definition: Let $\mathfrak{sl}_2$ denote the Lie algebra that has basis h, e, f and Lie bracket

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Theorem: $\mathfrak{sl}_2$ is isomorphic to the Lie algebra that has basis X, Y, Z and Lie bracket

$$[X, Y] = 2X + 2Y,$$
$$[Y, Z] = 2Y + 2Z,$$
$$[Z, X] = 2Z + 2X.$$

We call X, Y, Z the *equitable generators* of $\mathfrak{sl}_2$.

The quantum group $U_q(\mathfrak{sl}_2)$ and its alternate presentation

Definition: Let q denote a nonzero scalar which is not a root of unity. $U_q(\mathfrak{sl}_2)$ is the unital associative algebra generated by k, k^{-1}, e, f subject to the relations

$$\begin{aligned} kk^{-1} &= k^{-1}k = 1, \\ ke &= q^2 ek, \\ kf &= q^{-2} fk, \\ ef - fe &= \frac{k - k^{-1}}{q - q^{-1}}. \end{aligned}$$

Theorem: The algebra $U_q(\mathfrak{sl}_2)$ is isomorphic to the unital associative algebra generated by x, x^{-1}, y, z subject to the relations

$$\begin{aligned} xx^{-1} &= x^{-1}x = 1, \\ \frac{q\,xy - q^{-1}\,yx}{q - q^{-1}} &= 1, \\ \frac{q\,yz - q^{-1}\,zy}{q - q^{-1}} &= 1, \\ \frac{q\,zx - q^{-1}\,xz}{q - q^{-1}} &= 1. \end{aligned}$$

We call x, x^{-1}, y, z the *equitable generators* of $U_q(\mathfrak{sl}_2)$.

The quantum group $U_q(\widehat{\mathfrak{sl}}_2)$

Definition: The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is the unital associative algebra with generators $e_i^{\pm}$, $K_i^{\pm 1}$, $i \in \{0, 1\}$ which satisfy the following relations:

$$K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$K_0 K_1 = K_1 K_0,$$

$$K_i e_i^{\pm} K_i^{-1} = q^{\pm 2} e_i^{\pm},$$

$$K_i e_j^{\pm} K_i^{-1} = q^{\mp 2} e_j^{\pm}, \qquad i \neq j,$$

$$e_i^+ e_i^- - e_i^- e_i^+ = \tfrac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$e_0^{\pm} e_1^{\mp} = e_1^{\mp} e_0^{\pm},$$

$$(e_i^{\pm})^3 e_j^{\pm} - [3](e_i^{\pm})^2 e_j^{\pm} e_i^{\pm} + [3] e_i^{\pm} e_j^{\pm} (e_i^{\pm})^2 - e_j^{\pm} (e_i^{\pm})^3 = 0,$$
$$i \neq j,$$

where $[3] = (q^3 - q^{-3})/(q - q^{-1})$.

An alternative presentation for the quantum group $U_q(\widehat{\mathfrak{sl}}_2)$

Theorem: The algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is isomorphic to the unital associative algebra with generators x_i, y_i, z_i, $i \in \{0, 1\}$ and the following relations:

$$x_0 x_1 = x_1 x_0 \; = \; 1,$$

$$\frac{q x_i y_i - q^{-1} y_i x_i}{q - q^{-1}} \; = \; 1,$$

$$\frac{q y_i z_i - q^{-1} z_i y_i}{q - q^{-1}} \; = \; 1,$$

$$\frac{q z_i x_i - q^{-1} x_i z_i}{q - q^{-1}} \; = \; 1,$$

$$\frac{q z_i y_j - q^{-1} y_j z_i}{q - q^{-1}} \; = \; 1, \qquad i \neq j,$$

$$y_i^3 y_j - [3] y_i^2 y_j y_i + [3] y_i y_j y_i^2 - y_j y_i^3 = 0, \qquad i \neq j,$$

$$z_i^3 z_j - [3] z_i^2 z_j z_i + [3] z_i z_j z_i^2 - z_j z_i^3 = 0, \qquad i \neq j.$$

We call x_i, y_i, z_i the *equitable generators* of $U_q(\widehat{\mathfrak{sl}}_2)$.

From bidiagonal pairs to representations of $\mathfrak{sl}_2$ and $U_q(\mathfrak{sl}_2)$

Theorem (F.–N.): Let V denote a finite dimensional vector space. Let A, A^* denote a bidiagonal pair on V and assume the eigenvalues of A, A^* are of Type I. Then there exists a unique $\mathfrak{sl}_2$-module structure on V such that

$$(A - X)V = 0, \quad (A^* - Y)V = 0,$$

where X, Y are equitable generators of $\mathfrak{sl}_2$.

Theorem (F.–N.) Let V denote a finite dimensional vector space. Let A, A^* denote a bidiagonal pair on V and assume the eigenvalues of A, A^* are of Type II. Then there exists a unique $U_q(\mathfrak{sl}_2)$-module structure on V such that

$$(A - x)V = 0, \quad (A^* - y)V = 0,$$

where x, y are equitable generators of $U_q(\mathfrak{sl}_2)$.

Since the finite dimensional representations of $\mathfrak{sl}_2$ and $U_q(\mathfrak{sl}_2)$ are known these two theorems give a classification of all bidiagonal pairs.

From tridiagonal pairs to representations of $U_q(\widehat{\mathfrak{sl}}_2)$

Theorem (Ito, Terwilliger): Let V denote a finite dimensional vector space. Let A, A^* denote a tridiagonal pair on V and let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) be an ordering of the eigenvalues of A (resp. A^*). Assume there exist nonzero scalars q, a, b such that $\theta_i = aq^{2i-d}$ and $\theta_i^* = bq^{d-2i}$. Then there exists a unique irreducible $U_q(\widehat{\mathfrak{sl}}_2)$-module structure on V such that

$$(A - ay_0)V = 0, \quad (A^* - by_1)V = 0,$$

where y_0, y_1 are equitable generators of $U_q(\widehat{\mathfrak{sl}}_2)$. Moreover, there exists a unique irreducible $U_q(\widehat{\mathfrak{sl}}_2)$-module structure on V such that

$$(A - az_0)V = 0, \quad (A^* - bz_1)V = 0,$$

where z_0, z_1 are equitable generators of $U_q(\widehat{\mathfrak{sl}}_2)$.

The assumption on the eigenvalues in the above theorem says that the eigenvalues of A, A^* are (a special case) of Type II.

Since the finite dimensional irreducible $U_q(\widehat{\mathfrak{sl}}_2)$-modules are known the above theorem can be viewed as a first step toward the classification of tridiagonal pairs.