CONTRIBUTED PROBLEMS
NUMBER THEORY CONFERENCE,
CARLETON UNIVERSITY, OTTAWA

Assembled and prepared by Shanta Laishram and Gary Walsh

Problem I: (Contributed by Shanta Laishram) Let n, d, k be positive integers with $\gcd(n, d) = 1$. Consider the arithmetic progression $n, n + d, \ldots, n + (k - 1)d$ and let P be the greatest prime divisor of $n(n + d) \cdots (n + (k - 1)d)$. Then for $n \geq kd$, we have

$$P > \frac{dk}{200}$$

except possibly for finitely many triplets (n, d, k).

This is confirmed for $d \leq 400$ by Shorey and Tijdeman ($d \leq 200$) and Laishram and Shorey($d \leq 400$). A proof of this statement will prove the general conjecture of Erdős that the equation

$$n(n + d) \cdots (n + (k - 1)d) = y^2$$

has no solution in positive integers n, d, k, y with $n \geq 1, k \geq 4, d > 1$ and $y > 1$. Erdős and Selfridge showed that when $d = 1$, there are no solutions. Also the proof of the above statement is likely to give results on the irreducibility of generalised Schur polynomials.

Problem II: (Contributed by Shanta Laishram) Let $B(n)$ denote the number of $1'$s in the binary expansion of n. We consider the equations $B(n) = B(n^2)$ when n is odd. For each $k \geq 1$, we observe that $B(n) = B(n^2) = k$ for $n = 2^k - 1$. Hare and Laishram showed that $B(n) = B(n^2) = k$ has finitely many solutions in odd integers n when $k \leq 8$. For $k \geq 16$ or $k \in \{12, 13\}$, we can show that there are infinite odd integers n such that $B(n) = B(n^2) = k$. The cases $k \in \{9, 10, 11, 14, 15\}$ remain a mystery even though its expected that for $k \in \{9, 10\}$, there are only finitely many odd integers n with $B(n) = B(n^2) = k$.

Problem III: (Contributed by Steve Gonek) Prove that if there is one counterexample to the Riemann Hypothesis, then there are infinitely many counterexamples.

Problem IV: (Contributed by Gary Walsh) Find all positive integer solutions to $\frac{x^2-1}{y^2+1} = (z^2 \pm 1)^2$.

Problem V: (Contributed by Kenneth S. Williams) Let N be a positive integer. Lagrange's Theorem asserts that N is a sum of four squares. Is there a proof of Lagrange's Theorem, which uses only the arithmetic condition for an integer to be a sum of two squares and which proceeds by showing the existence of an integer m such that both m and $N - m$ are sums of two squares.

Problem VI: (Contributed by Kenneth S. Williams) Let $q \in \mathbb{C}$ and N be a positive integer. Is there an identity

$$A_N(q) = B_N(q) \tag{1}$$

where $A_N(q)$ is a finite sum and $B_N(q)$ is a finite product, such that (1) with $q = e^{\frac{2\pi i}{N}}$ gives the Gauss sum evaluation

$$\sum_{s=1}^{N} e^{\frac{2\pi i s^2}{N}} = \frac{1}{2}\sqrt{N}(1+i)(1+i^{-N})?$$

Problem VII: (Contributed by Thomas Stoll) Prove or disprove that $\sum_{p \neq N}(-1)^{B(p)} < 0$ for all $N \geq 31$, where the summation is extended over primes p and $B(p)$ denotes the binary digits sum of p. (Conjecture posed by V. Shevelev, Generalized Newman Phenomena and Digit Conjectures on Primes, International Journal of Mathematics and Mathematical Sciences, Volume 2008 (2008), Article ID 908045)

Problem VIII: (Contributed by Thomas Stoll) Prove or disprove that $\sum_{1 \leq n \leq N}(-1)^{B(n^2)} < 0$ for all $N \geq 1$. (Problem first addressed in the Special Session "Analytic Number Theory" (Murty/Michel) of the Second Canada-France Congress 2008, Montreal)

Problem IX: (Contributed by George Andrews) New Generalizations of Alder's Conjecture: Let $q_d^*(n)$ denote the number of partitions of n with differences $\geq d$ between parts and $> d$ between multiples of d. Let $Q_d^*(n)$ denote the number of partitions of n into parts congruent to ± 1, $\pm(d+2)(\text{mod } 4d)$. Let $Q_d^{**}(n)$ denote the number of partitions of n into parts congruent to $\pm 1, \pm(d+2), \pm(d+6), \ldots, \pm(d+4j+2)(\text{mod } 4d)$ where $j = \left\lfloor \frac{d-2}{4} \right\rfloor$.

Weak new Alder conjecture: For $d > 1$,

$$q_d^*(n) - Q_d^*(n) \geq 0$$

Strong new Alder conjecture: For $d > 1$,

$$q_d^*(n) - Q_d^{**}(n) \geq 0.$$

Clearly both conjectures are identical for $d < 6$, and each expression is identically equal to 0 for $d = 2$ (Goellnitz-Gordon) and $d = 3$ (Schur). Obviously the strong conjecture implies the weak conjecture (This follows immediately from Theorem 3 in Pac. J. Math., 36(1971), 279-284). One can strengthen the conjectures to " $>$ " provided $d > 3$ and $n > d + 3$ (with the one exception $n = 12$, $d = 4$).