Dynamical formation of correlations in the the Bose-Einstein condensate

László Erdős
University of Munich

Toronto, 2009 Dec

MANY BODY QUANTUM DYNAMICS

$\mathbf{x} = (x_1, x_2, \ldots, x_N)$ position of the particles, $x_j \in \mathbb{R}^d$

Wave function: $\Psi_N(x_1, \ldots, x_N) \in L^2_{symm}$ (bosons)

The time evolution

$$i\partial_t \Psi_{N,t} = H_N \Psi_{N,t}$$

is governed by the Hamiltonian (energy) operator

$$H_N = \sum_{j=1}^{N} \Big[-\Delta_{x_j} + U(x_j) \Big] + \sum_{i<j} V(x_i - x_j)$$

U one-body potential (typically trapping, i.e. $\lim_{x\to\infty} U(x) = \infty$)
V is the interaction, typically repulsive, $V \geq 0$.

In density matrix formalism, $\gamma_{N,t} = |\Psi_{N,t}\rangle\langle\Psi_{N,t}|$ (projection)

$$i\partial_t \Psi_{N,t} = H_N \Psi_{N,t} \quad \Longleftrightarrow \quad i\partial_t \gamma_{N,t} = \Big[H_N, \gamma_{N,t} \Big]$$

SOFT MEAN FIELD POTENTIAL $\Longrightarrow$ *HARTREE EQUATION*

$$H_N = \sum_{j=1}^{N} \Big[-\Delta_{x_j} + U(x_j) \Big] + \frac{1}{N} \sum_{i<j} V(x_i - x_j)$$

THEOREM: If $\Psi_0 = \prod_j \varphi_0(x_j)$, then $\Psi_t \approx \prod_j \varphi_t(x_j)$ as $N \to \infty$

where $$i\partial_t \varphi_t = (-\Delta + U)\varphi_t + \Big(V \star |\varphi_t|^2 \Big) \varphi_t$$

Each particle: subject to the same mean-field potential

$$\frac{1}{N} \sum_{j=1}^{N} V(x - x_j) |\varphi(x_j)|^2 \approx (V \star |\varphi|^2)(x)$$

(Law of large numbers if the state is indeed a product)

Hepp, Spohn, Ginibre–Velo, Bardos–Golse–Mauser, E-Yau

GOAL: $V =$ Dirac delta interaction $\Longrightarrow$ GP eq. (cubic NLS)

BOSE-EINSTEIN CONDENSATION (BEC)

Free (non-interacting) bosons in a trap $U_L(x) = U(x/L)$

$$H_0 = \sum_{j=1}^{N} (-\Delta_{x_j} + U_L(x_j))$$

is the direct sum of the one-body operator $-\Delta + U_L$.

Prob. to find an eigenstate with energy E is $\sim e^{-\beta E}$
($\beta = 1/T$ inverse temperature)

BEC ($d = 3$): At low temperature, the prob to find the ground state of $-\Delta + U_L$ is strictly positive uniformly for all L.
(Remark: no BEC in $d = 2$ for positive temperature)

1) Does the same hold with interaction? (OPEN)

2) Experiment: Trap BEC and observe the evolution of the condensate as the trap removed $\Longrightarrow$ Gross-Pitaevskii (GP) equation (this talk)

MATHEMATICAL DEFINITION OF BEC

One-particle marginal density of a general N-body state Ψ_N

$$\gamma_N^{(1)}(x;x') := \int \Psi_N(x,Y)\overline{\Psi}_N(x',Y)\mathrm{d}Y, \qquad Y = (x_2, \dots x_N)$$

Operator on the one-particle space, $0 \le \gamma_N^{(1)} \le 1$, $\mathrm{Tr}\, \gamma_N^{(1)} = 1$.

$\gamma_N^{(k)}(x_1, \dots x_k; x_1', \dots x_k')$, is defined similarly ($k$-particle marg. dens.)

Spectral decomposition: $\gamma_N^{(1)} = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$.

DEFINITION: Ψ_N is a (sequence of) condensate states if

$$\liminf_{N\to\infty} \max_j \lambda_j > 0$$

Example: $\Psi_N = \varphi^{\otimes N} = \prod_j \varphi(x_j)$, then $\gamma_N^{(1)} = |\phi\rangle\langle\phi|$ (projection)

TYPICAL SCALES

$$H_N = \sum_{j=1}^{N} \Big[-\Delta_{x_j} + U_L(x_j) \Big] + \sum_{i<j} V(x_i - x_j)$$

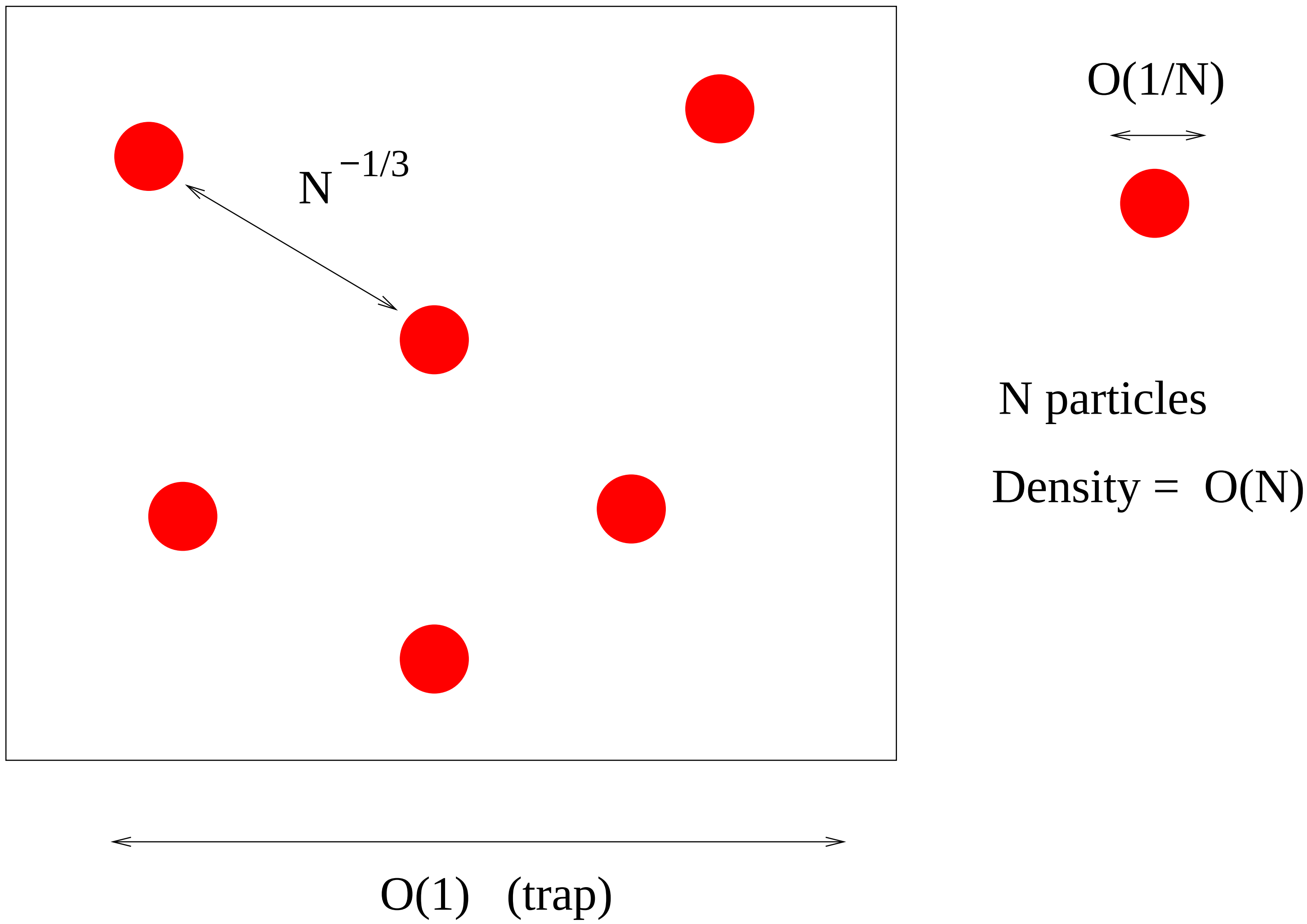
U_L is a one-body "trapping" potential with lengthscale L
$V \geq 0$ is the (repulsive) interaction potential with lengthscale a.

Parameters of a typical experiment (rubidium atom at Cornell)

$a \sim 10^{-3} \mu m$, $L \sim 1\ \mu m$, $N = 10^3$, density $\varrho = N/L^3 = 10^3 \mu m^{-3}$

Note that $a/L \sim O(1/N)$

Key parameter: $\varrho a^3 \ll 1$
Effectively low density system with a strong local interaction.



In units where the trap $L = O(1)$, the system is at high density on the scale of the trap but it is in the dilute regime viewed on the scale of the interaction $a \sim O(1/N)$.

SCATTERING LENGTH

Characterizes the effective lengthscale of the interaction.

Let supp V be compact. Consider the zero energy scattering eq.

$$\Big[-\Delta + \frac{1}{2}V\Big]f = 0, \qquad f(x) \to 1,\ \ |x| \to \infty$$

Then $f = 1 - w$ with $w(x) = \frac{a_0}{|x|}$, for some a_0.

a_0 is called the scattering length of V

Alternatively:

$$\int |\nabla w|^2 + V(1-w)^2 = \int V(1-w) = 8\pi a_0$$

Rescaling: $N^2 V(Nx)$ has scattering length $a = a_0/N$.

In a dilute gas of neutral bosons, the scattering length is the only characteristic lengthscale of the interaction.

HAMILTONIAN

$$H_N = \sum_{j=1}^{N} \left[-\Delta_{x_j} + U(x_j) \right] + \sum_{i<j} N^2 V(N(x_i - x_j))$$

Interaction potential has scattering length $1/N$.

GP-theory: Many-body interactions and correlations $\rightarrow$ nonlinear, on-site self-interaction with coupling = scattering length

Lieb, Seiringer, Yngvason (1999) proved that the GP functional is asymptotically exact for the ground state energy, i.e.

$$\lim_{N\to\infty} \inf \text{Spec} \frac{H_N}{N} = \inf_u \int_{\mathbb{R}^3} \left[|\nabla u|^2 + U|u|^2 + 4\pi a_0 |u|^4 \right]$$

Lieb and Seiringer (2001) showed that the one particle density matrix of the ground state of H_N converges to the minimizer u.

Dynamics: the ground state of trapped BEC is a highly excited state for the system without the trap. The (time dependent) GP theory describes also excited states and their evolutions!

DERIVATION OF TIME DEPENDENT GP EQUATION

THEOREM [E-Schlein-Yau] Assume $V \geq 0$, smooth, spherical, and external potential is removed $U = 0$.

$$\text{Initial state} \qquad \Psi_N(\mathbf{x}) = \prod_{j=1}^{N} \varphi(x_j), \qquad \varphi \in H^1(\mathbb{R}^3)$$

Then, for fixed k, t, for the k-particle density matrix of $\Psi_{N,t}$

$$\gamma_{N,t}^{(k)} \to |\varphi_t\rangle\langle\varphi_t|^{\otimes k} \qquad N \to \infty \quad \text{(in trace norm)}$$

where φ_t is the solution of the GP equation

$$i\partial_t \varphi_t = \Big[-\Delta + 8\pi a_0 |\varphi_t|^2 \Big] \varphi_t, \qquad \varphi_{t=0} = \varphi$$

The theorem also holds if the initial state Ψ_N has finite energy per particle, $\langle \Psi_N, H_N \Psi_N \rangle \leq CN$, and it exhibits BEC, in particular, it holds for the trapped ground state (experiment)

[Alternative proof announced by Pickl with different conditions]

HARTREE EQUATION WITH DIRAC DELTA ??

$$H = \sum_j (-\Delta_j) + \frac{1}{N}\sum_{i<j} V(x_i - x_j) \quad \Longrightarrow \quad i\partial_t \varphi_t = -\Delta\varphi_t + (V * |\varphi_t|^2)\varphi_t$$

The interaction can be written as

$$V_N(x) = N^2 V(Nx) = \frac{1}{N}\, N^3 V(Nx) \approx \frac{b_0}{N}\delta(x), \quad \text{with} \quad b_0 := \int V,$$

and $(\delta * |\varphi|^2)\varphi = |\varphi|^2\varphi$ but $8\pi a_0 < b_0$ (strictly!)

It is not just a Dirac-delta version of the mean field model.

Explanation: The wave function has a specific stationary short scale correlation structure:

$$\Psi_{N,t} \sim \prod_{j=1}^N \varphi_t(x_j) \prod_{i<j} (1 - w_N(x_i - x_j))$$

where $1 - w_N(x) = 1 - w(Nx)$ is the zero energy scattering mode.

The interaction energy for such a state is

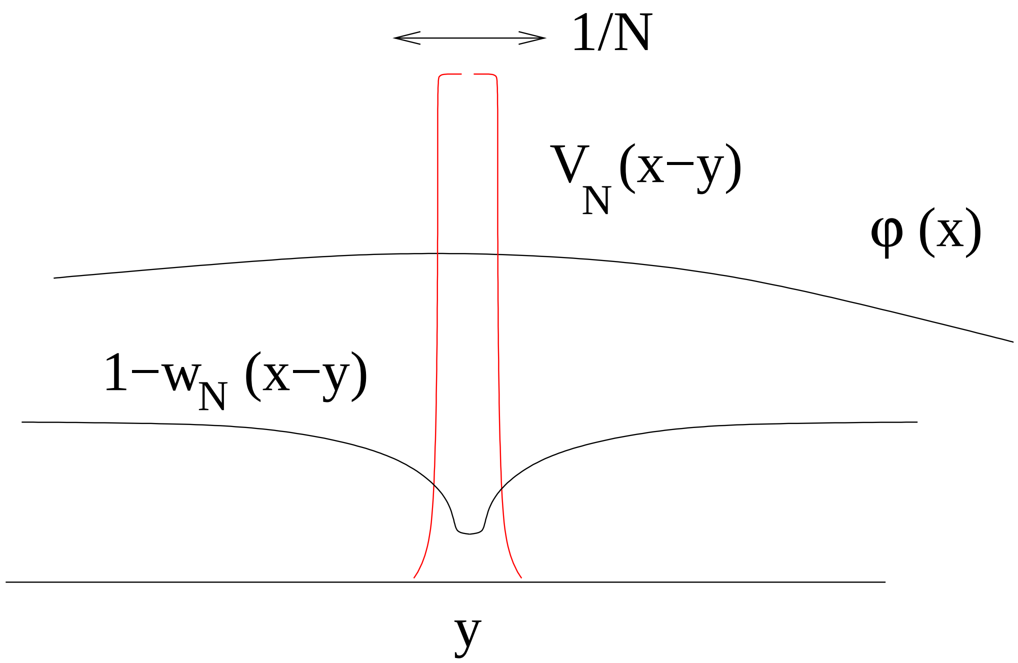
$$\left\langle \Psi, \sum_{k<j} V_N(x_j - x_k)\Psi \right\rangle = \frac{N^2}{2}\int V_N(x-y)\Big[1 - w_N(x-y)\Big]^2 |\varphi(x)|^2|\varphi(y)|^2 \mathrm{d}x\mathrm{d}y$$

$$\approx \frac{N}{2}\int V(1-w)^2 \int |\varphi|^4$$

if φ is "smooth", i.e. essentially constant on the range of V_N.

But V_N, $(1-w_N)^2$ live on the same scale and $\int V(1-w)^2 < \int V$.

The kinetic energy also picks up contribution from ∇w.



BBGKY HIERARCHY

$$H_N = -\sum_{j=1}^{N} \Delta_j + \frac{1}{N} \sum_{j<k} V(x_j - x_k)$$

$V = V_N$ may depend on N so that $\int V_N = O(1)$.

Recall the Schrödinger equation in commutator form

$$i\partial_t \gamma_{N,t} = [H_N, \gamma_{N,t}]$$

Take the partial trace wrt. $2, 3, \dots N$ particles

$$i\partial_t \gamma_{N,t}^{(1)} = \Big[-\Delta_1, \gamma_{N,t}^{(1)} \Big] + \frac{N-1}{N} \mathrm{Tr}_{x_2} \Big[V(x_1 - x_2), \gamma_{N,t}^{(2)} \Big]$$

Similar equations for each k-marginals form a system of N coupled coupled equation – BBGKY hierarchy.

$$i\partial_t \gamma^{(1)}_{N,t}(x_1;x_1') = (-\Delta_{x_1} + \Delta_{x_1'})\gamma^{(1)}_{N,t}(x_1;x_1') + \int \mathrm{d}x_2 \left(V(x_1-x_2) - V(x_1'-x_2)\right)\gamma^{(2)}_{N,t}(x_1,x_2;x_1',x_2) + o(1)\,.$$

To get a closed equation for $\gamma^{(1)}_{N,t}$, we need some relation between $\gamma^{(1)}_{N,t}$ and $\gamma^{(2)}_{N,t}$. Most natural: independence

Propagation of chaos: No production of correlations

If initially $\gamma^{(2)}_{N,0} = \gamma^{(1)}_{N,0} \otimes \gamma^{(1)}_{N,0}$, then hopefully $\gamma^{(2)}_{N,t} \approx \gamma^{(1)}_{N,t} \otimes \gamma^{(1)}_{N,t}$

No exact factorization for finite N, but maybe it holds for $N \to \infty$.

Suppose $\gamma^{(k)}_{\infty,t}$ is a (weak) limit point of $\gamma^{(k)}_{N,t}$ with

$$\gamma^{(2)}_{\infty,t}(x_1,x_2;x_1',x_2') = \gamma^{(1)}_{\infty,t}(x_1,x_1')\gamma^{(1)}_{\infty,t}(x_2;x_2').$$

$$i\partial_t \gamma^{(1)}_{N,t}(x_1;x_1') = (-\Delta_{x_1} + \Delta_{x_1'})\gamma^{(1)}_{N,t}(x_1;x_1')$$
$$+ \int \mathrm{d}x_2 \Big(V(x_1 - x_2) - V(x_1' - x_2) \Big) \underbrace{\gamma^{(2)}_{N,t}(x_1, x_2; x_1', x_2)}_{\to \gamma^{(1)}_{\infty,t}(x_1,x_1')\gamma^{(1)}_{\infty,t}(x_2;x_2)} + o(1)$$

With the notation $\varrho_t(x) := \gamma^{(1)}_{\infty,t}(x;x)$, it converges, to

$$i\partial_t \gamma^{(1)}_{\infty,t}(x_1;x_1') = (-\Delta_{x_1} + \Delta_{x_1'})\gamma^{(1)}_{\infty,t}(x_1;x_1')$$
$$+ \Big(V * \varrho_t(x_1) - V * \varrho_t(x_1') \Big)\, \gamma^{(1)}_{\infty,t}(x_1;x_1')$$

$i\partial_t \gamma^{(1)}_{\infty,t} = \Big[-\Delta + V * \varrho_t\, ,\ \gamma^{(1)}_{\infty,t} \Big]$ Hartree eq for density matrix

If $V = V_N = N^2 V(Nx)$, then the short scale structure is relevant.

For $\gamma^{(2)}_{N,t}(x_1, x_2; x_1', x_2) = (1 - w_N(x_1 - x_2))\gamma^{(1)}_{N,t}(x_1, x_1')\gamma^{(1)}_{N,t}(x_2, x_2)$,

$$\Longrightarrow \quad \int V_N(x_1 - x_2)(1 - w_N(x_1 - x_2))\Big[\text{Smooth}\Big] = 8\pi a_0$$

GENERAL SCHEME OF THE PROOF

$$i\partial_t \gamma_{N,t}^{(k)} = \sum_{j=1}^{k} \Big[-\Delta_j, \gamma_{N,t}^{(k)}\Big] + \frac{1}{N}\sum_{i<j}^{k} \Big[V(x_i - x_j), \gamma_{N,t}^{(k)}\Big] + \frac{N-k}{N}\sum_{j=1}^{k} \mathrm{Tr}_{k+1}\Big[V(x_j - x_{k+1}), \gamma_{N,t}^{(k+1)}\Big]$$

formally converges to the ∞ Hartree hierarchy: $(k = 1, 2, \ldots)$

$$i\partial_t \gamma_{\infty,t}^{(k)} = \sum_{j=1}^{k} \Big[-\Delta_j, \gamma_{\infty,t}^{(k)}\Big] + \sum_{j=1}^{k} \mathrm{Tr}_{x_{k+1}}\Big[V(x_j - x_{k+1}), \gamma_{\infty,t}^{(k+1)}\Big] \qquad (*)$$

$\Big\{\gamma_t^{(k)} = \otimes_1^k \gamma_t^{(1)}\Big\}_{k=1,2\ldots}$ solves $(*)$ $\iff$ $i\partial_t \gamma_t^{(1)} = \Big[-\Delta + V * \varrho_t \, , \gamma_t^{(1)}\Big]$

If we knew that $\begin{cases} (*) \text{ had a unique solution, and} \\ \lim_N \gamma_{N,t}^{(k)} \text{ exists and satisfies } (*), \end{cases}$

then the limit must be the factorized one

$\Longrightarrow$ Propagation of chaos + convergence to Hartree eq.

Step 1: Prove apriori bound on $\gamma_{N,t}^{(k)}$ uniformly in N.
Need a good norm and space $\mathcal{H}$! (Sobolev)

Step 2: Choose a convergent subsequence: $\gamma_{N,t}^{(k)} \to \gamma_{\infty,t}^{(k)}$ in $\mathcal{H}$

Step 3: $\gamma_{\infty,t}^{(k)}$ satisfies the infinite hierarchy (need regularity)

Step 4: Let $\gamma_t^{(1)}$ solve NLHE/NLS. Then $\gamma_t^{(k)} = \otimes \gamma_t^{(1)}$ solves
the ∞-hierarchy in $\mathcal{H}$.

Step 5: Show that the ∞-hierarchy has a unique solution in $\mathcal{H}$.

Key mathematical steps: Apriori bound and uniqueness

Apriori bound: conservation of $H^k \Longrightarrow$ mixed Sob. bound

Uniqueness: Many-body version of Strichartz via Feynman graphs

PERSISTENCE OF THE LOCAL STRUCTURE

One of the main ingredients of the previous proof is

$$\int \left| \nabla_i \nabla_j \frac{\Psi_N(\mathbf{x})}{1 - w_N(x_i - x_j)} \right|^2 \mathrm{d}\mathbf{x} \leq C \left\langle \Psi_N, \frac{H_N^2}{N^2} \Psi_N \right\rangle$$

Note that

$$\int \left| \nabla_i \nabla_j \frac{1}{1 - w_N(x_i - x_j)} \right|^2 \mathrm{d}\mathbf{x} \approx CN$$

thus boundedness of H_N^2 / N^2 detects the short scale structure.

Since H_N^2 is conserved, for initial data $\Psi_{N,0}$ with

$$\langle \Psi_{N,0}, H_N^2 \Psi_{N,0} \rangle \leq CN^2,$$

the same holds for $\Psi_{N,t}$, and thus the short scale structure present in the initial state $\Psi_{N,0}$ is preserved for later times.

Does it emerge dynamically?

A SURPRISE FOR THE PRODUCT INITIAL STATE

NL energy predicts the wrong NL evolution for $\Psi_N = \varphi^{\otimes N}$

$$\frac{1}{N}\langle \Psi_N, H_N \Psi_N \rangle \to \int \left[|\nabla \varphi|^2 + \frac{b_0}{2}|\varphi|^4 \right]$$

since, recalling $V_N(x) = \frac{1}{N} N^3 V(Nx)$,

$$\int \frac{1}{N} \sum_{i<j} V_N(x_i - x_j) \prod_{j=1}^{N} |\varphi(x_j)|^2 \to \frac{b_0}{2}|\varphi|^4$$

but the marginals of $\Psi_{N,t}$ factorize, $\gamma_{N,t}^{(1)} \to |\varphi_t\rangle\langle\varphi_t|$, with

$$i\partial \varphi_t = -\Delta \varphi_t + 8\pi a_0 |\varphi_t|^2 \varphi_t \qquad (b_0 > 8\pi a_0!!)$$

Energy lost?

No. In E-S-Y theorem, the limit holds in L^2 (trace norm) but not in H^1.

EXPLANATION

The product state instantenously builds up a short scale correlation to minimize its local energy. This short scale correlation then drives the orbitals according to a_0. The excess energy is diffused into incoherent modes on scales $1/N \ll \ell \ll 1$ and does not influence the evolution of the condensate.

Our main result: The dynamical emergence of the short scale structure, characterized by the correlation factor $1 - w_N(x_i - x_j)$.

The short scale structure must hold for scales

$$\frac{1}{N} \leq |x_i - x_j| \leq \ell$$

with some $\ell \leq N^{-1/3}$ (typical nearest neighbor distance).

For larger distances, three particle correlations may occur, but to maintain the GP dynamics, it is sufficient if $1 - w_N$ is present on the scale $|x_i - x_j| \approx N^{-1}$ (range of V_N).

DYNAMICAL EMERGENCE OF THE CORRELATION

Theorem: [E-Michelangeli-Schlein, 2008]

Let $V_N(x) = N^2 V(Nx)$, $\Psi_N = \varphi^{\otimes N}$, φ smooth, decaying. Define

$$\mathcal{F}_N(t) := \int \theta_\ell(x_1 - x_2) \Big| \frac{\Psi_{N,t}(\mathbf{x})}{1 - w_N(x_1 - x_2)} - \prod_{j=1}^N \varphi_t(x_j) \Big|^2 \mathrm{d}\mathbf{x}$$

with a smooth cutoff on scale $\ell \geq N^{-1}$. Then for $t \leq N^{-1}$

$$\mathcal{F}_N(t) \leq C \mathcal{F}_N(0) \Big(\frac{1}{N^{1/5}} \frac{(N^2 t)^2}{N\ell} + \frac{(N\ell)^4}{\langle N^2 t \rangle} \Big)$$

(modulo logs). Concretely, with $\ell = \frac{1}{N}$, we have

$$\mathcal{F}_N(t) \ll \mathcal{F}_N(0), \quad \text{for} \quad \frac{1}{N^2} \ll t \ll \frac{1}{N^{2-\frac{1}{10}}}$$

Remark: Natural lengthscale $\approx \frac{1}{N}$, natural timescale $\approx \frac{1}{N^2}$

$$\mathcal{F}_N(t) = \int \theta_\ell(x_1 - x_2)\left|\frac{\Psi_{N,t}(\mathbf{x})}{1 - w_N(x_1 - x_2)} - \prod_{j=1}^{N} \varphi_t(x_j)\right|^2 \mathrm{d}\mathbf{x}$$

$$\mathcal{F}_N(t) \le C\mathcal{F}_N(0)\left(\frac{1}{N^{1/5}}\frac{(N^2 t)^2}{N\ell} + \frac{(N\ell)^4}{\langle N^2 t\rangle}\right)$$

After an initial time layer of order $t \ge \frac{1}{N^2}$, it is expected that $\mathcal{F}_N(t) \ll \mathcal{F}_N(0)$ for all times, but we cannot control many body effects for larger times (first term).

The formation of the $1 - w_N$ structure is a two-body scattering event on time scale $t \sim \frac{1}{N^2}$ (second term). The effective scattering time increases as ℓ (window size) increases.

Strategy of proof: (i) reduce to the two-body problem locally; (ii) analyse the two body scattering with a constant initial data.

REDUCTION TO THE TWO-BODY ANALYSIS

Decouple the particles 1 and 2 from the rest:

$$\widetilde{H}_N = -\Delta_1 - \Delta_2 + V_N(x_1 - x_2) + \sum_{j=3}^{N}(-\Delta_j) + \sum_{j=1}^{2}\sum_{k=3}^{N} V_N(x_j - x_k)$$

and let $\widetilde{\Psi}_{N,t}$ be the time evolution of $\widetilde{H}_N$. Note that

$$\widetilde{\Psi}_{N,t}(\mathbf{x}) = \psi_t(x_1, x_2)\Phi_t(x_3, \ldots, x_N)$$

Then (essentially)

$$\begin{aligned}\mathcal{F}_N(t) \leq & C\int \theta_\ell(x_1 - x_2)\Big|\Psi_{N,t}(\mathbf{x}) - \widetilde{\Psi}_{N,t}(\mathbf{x})\Big|^2 \mathrm{d}\mathbf{x} \\ & + C\ell^2 \int \theta_\ell(x_1 - x_2)\left|\frac{\psi_t(x_1, x_2)}{1 - w_N(x_1 - x_2)} - \varphi(x_1)\varphi(x_2)\right|^2 \mathrm{d}x_1 \mathrm{d}x_2 + Error \\ \equiv & \, C\,\Big(\mathcal{G}_N(t) + \mathcal{K}_N(t)\Big) + Error\end{aligned}$$

$\swarrow$ deteriorates in time $\qquad$ $\searrow$ a 2-body problem

TWO-BODY SCATTERING

By Poincaré inequality and by a change of variables,

$$x = x_2 - x_1, \qquad \eta = \frac{x_1 + x_2}{2}$$

$$\mathcal{K}_N(t) = C\ell^2 \int \theta_\ell(x_1 - x_2) \left| \frac{\psi_t(x_1, x_2)}{1 - w_N(x_1 - x_2)} - \varphi(x_1)\varphi(x_2) \right|^2 \mathrm{d}x_1 \mathrm{d}x_2$$

$$\leq C\ell^2 \int \mathrm{d}x \mathrm{d}\eta \theta_{2\ell}(x) \left| \nabla_x \frac{e^{-it\mathfrak{h}_N} \psi_\eta(x)}{1 - w_N(x)} \right|^2 + Error$$

where

$$\mathfrak{h}_N = -2\Delta_x + V_N(x)$$

is the two-body Hamiltonian in relative coordinates and

$$\psi_\eta(x) = \varphi(\eta + x/2)\varphi(\eta - x/2)$$

Practically, think of $\psi_\eta(x) = 1$ on the relevant short scale, forget η and smuggle in $1 = \omega_N + (1 - \omega_N)$

Thus, modulo negligible errors,

$$\mathcal{K}_N(t) \le C\ell^2 \int \mathrm{d}x \theta_\ell(x) \left[\left| \nabla_x \frac{e^{-it\mathfrak{h}_N} w_N(x)}{1 - w_N(x)} \right|^2 + \left| \nabla_x \frac{e^{-it\mathfrak{h}_N}(1 - w_N)(x)}{1 - w_N(x)} \right|^2 \right]$$

The second term is (essentially) zero, since (modulo domains)

$$\mathfrak{h}_N(1 - w_N) = 0 \quad \Longrightarrow \quad e^{-it\mathfrak{h}_N}(1 - w_N) = 1 - w_N$$

For the first term, using the wave operator

$$\Omega = \lim_{t\to\infty} e^{it(-\Delta + \frac{1}{2}V)} e^{it\Delta}$$

after rescaling $x \to x/N$, $t = T/N^2$, we need to control

$$\int \theta_{N\ell} \left| \nabla \frac{\Omega e^{2iT\Delta} \Omega^* w}{1 - w} \right|^2 \le \|\nabla w\|_2^2 \|\Omega e^{2iT\Delta} \Omega^* w\|_\infty^2 + (N\ell)^3 \|\nabla \Omega e^{2iT\Delta} \Omega^* w\|_\infty^2$$

Recall that $w(x) \sim \frac{1}{|x|}$ at large distances and recall that

$$\Omega, \Omega^* : W^{k,p} \to W^{k,p} \quad \text{bounded} \quad 1 \le p \le \infty \quad \text{[Yajima]}$$

(assuming V is nice), we need only a dispersive estimate for slowly decaying initial data (like $\Omega^* w \in L^{3+\varepsilon}$, but $\notin L^p$, $p \le 3$).

NEW DISPERSIVE ESTIMATE

Theorem: In three dimensions,

$$\left\|e^{it\Delta}f\right\|_q \le \frac{C}{t^{\frac{3}{2}\left(\frac{1}{s}-\frac{1}{q}\right)}}\left(\|f\|_s + \|\nabla f\|_{\frac{3s}{s+3}}\right) + \frac{C}{t^{\frac{3}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-1}}\|\nabla^2 f\|_r$$

with

$$\frac{3}{2} \le s \le \infty, \qquad \max\{3, s\} \le q \le \infty, \qquad 1 \le r \le \frac{3q}{3+2q}$$

In particular, when $q = \infty$ and $3 \le s \le \infty$ and $r = \frac{3s}{3+2s}$, then

$$\left\|e^{it\Delta}f\right\|_\infty \le \frac{C}{t^{\frac{3}{2s}}}\left(\|f\|_s + \|\nabla f\|_{\frac{3s}{s+3}} + \|\nabla^2 f\|_{\frac{3s}{3+2s}}\right)$$

Standard dispersion estimate requires $f \in L^p$, $p < 2$. Here: some additional regularity can be used to perform integration by parts.

Putting all these together, we control the two body scattering term $\mathcal{K}_N(t)$. The

CONCLUSIONS

- We derived the GP equation from many-body Ham. with interaction on scale $1/N$. Coupling const. $=$ scattering length.

- Conservation of H^k can imply bounds in Sobolev space and Stricharz can be strengthened with Feynman diagrams in many body problems

- A specific short scale correlation structure is preserved and even emerges along the dynamics. In the $N \to \infty$ limit, this structure is negligible in L^2 sense but not in energy sense, thus it influences the dynamics via the emergence of the scatt. length.

- We proved a new dispersive estimate for slowly decaying but regular initial data.

- Open question: persistence of the short scale structure for all times

To estimate $\mathcal{G}_N$, enlarge the window to size $\widetilde{\ell} \gg \ell$ and control

$$\widetilde{\mathcal{G}}_N(t) := \int \theta_{\widetilde{\ell}}(x_1 - x_2)\Big|\Psi_{N,t}(\mathbf{x}) - \widetilde{\Psi}_{N,t}(\mathbf{x})\Big|^2 \mathrm{d}\mathbf{x} =: \int \theta_{12}|\delta\Psi|^2$$

and estimate its derivative

$$\begin{aligned} \left|\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{G}}_N(t)\right| &\lesssim \langle \nabla\sqrt{\theta}\cdot\nabla(\delta\Psi), \sqrt{\theta}(\delta\Psi)\rangle + \langle \delta\Psi, \theta_{12}\sum_{k\geq 3} V_{1k}\widetilde{\Psi}\rangle \\ &\lesssim C\Big(\widetilde{\ell}^{-1} + N\widetilde{\ell}^{3/2}\Big)\mathcal{G}_{N,t}^{1/2} \end{aligned} \tag{1}$$

by using energy conservation and the fact that

$$\|\psi_t\|_\infty \leq C \log N$$

for the two body solution

$$\psi_t = e^{-it\mathfrak{h}_N}\varphi^{\otimes 2}, \qquad \mathfrak{h}_N = -\Delta_1 - \Delta_2 + V_N(x_1 - x_2)$$

Optimizing in (1) gives $\widetilde{\ell} \sim N^{-2/5}$ and by Gronwall

$$|\mathcal{G}_N'(t)| \leq CN^{2/5}\mathcal{G}_N(t)^{1/2} \quad \Longrightarrow \quad \mathcal{G}_N(t) \leq CN^{-1/5}(Nt^2)$$

METHODS OF THE PROOF

Two main issues to handle:

1) Proving propagation of chaos, i.e. that the higher order density matrices (correlation functions) remain asymptotically factorized,

$$\gamma_{N,t}^{(k)} \approx \left[\gamma_{N,t}^{(1)}\right]^{\otimes k}$$

at least on larger scales or in the limit.

2) Justifying the short scale correlation structure which eventually vanishes in the L^2 limit, but does not vanish in H^1 sense and is thus influences the dynamics (via the scattering length).

1) is done via the limiting BBGKY hierarchy.

2) is done via conservation of H_N^2 along the time evolution.

FUNDAMENTAL DIFFICULTY OF N-BODY ANALYSIS

There is no good norm. The conserved L^2-norm is too strong. $\Psi(x_1, \dots x_N)$ carries info of all particles (too detailed).

Keep only information about the k-particle correlations:

$$\gamma_\Psi^{(k)}(X_k, X_k') := \int \Psi(X_k, Y_{N-k})\overline{\Psi}(X_k', Y_{N-k})dY_{N-k}$$

where $X_k = (x_1, \dots x_k)$. It is a partial trace

$$\gamma_\Psi^{(k)} = \mathrm{Tr}_k |\Psi\rangle\langle\Psi|$$

It monitors only k particles.

Good news: Most physical observables involve only $k = 1, 2$-particle marginals. Enough to control them.

Bad news: there is no closed equation for them.

BASIC TOOL: BBGKY HIERARCHY

$$H = -\sum_{j=1}^{N} \Delta_j + \frac{1}{N}\sum_{j<k} V(x_j - x_k)$$

$V = V_N$ may depend on N so that $\int V_N = O(1)$

Take the k-th partial trace of the Schrödinger eq.

$$i\partial_t \gamma_{N,t} = [H, \gamma_{N,t}] \quad \Longrightarrow$$

$$i\partial_t\gamma_{N,t}^{(k)} = \sum_{j=1}^{k}\Big[-\Delta_j, \gamma_{N,t}^{(k)}\Big] + \frac{1}{N}\sum_{i<j}^{k}\Big[V(x_i - x_j), \gamma_{N,t}^{(k)}\Big] + \frac{N-k}{N}\sum_{j=1}^{k}\mathrm{Tr}_{k+1}\Big[V(x_j - x_{k+1}), \gamma_{N,t}^{(k+1)}\Big]$$

A system of N coupled equations. $(k = 1, 2, \ldots, N)$

Last eq. is just the original N-body Schr. eq. Tautological?

Closure? Wish: Propag. of chaos: $\gamma_N^{(2)} \approx \gamma_N^{(1)} \otimes \gamma_N^{(1)}$ $(N \to \infty)$

GENERAL SCHEME OF THE PROOF

$$i\partial_t\gamma_{N,t}^{(k)} = \sum_{j=1}^{k}\left[-\Delta_j, \gamma_{N,t}^{(k)}\right] + \frac{1}{N}\sum_{i<j}^{k}\left[V(x_i - x_j), \gamma_{N,t}^{(k)}\right] + \frac{N-k}{N}\sum_{j=1}^{k}\text{Tr}_{k+1}\left[V(x_j - x_{k+1}), \gamma_{N,t}^{(k+1)}\right]$$

formally converges to the ∞ Hartree hierarchy: $(k = 1, 2, \ldots)$

$$i\partial_t\gamma_{\infty,t}^{(k)} = \sum_{j=1}^{k}\left[-\Delta_j, \gamma_{\infty,t}^{(k)}\right] + \sum_{j=1}^{k}\text{Tr}_{x_{k+1}}\left[V(x_j - x_{k+1}), \gamma_{\infty,t}^{(k+1)}\right] \qquad (*)$$

$$\left\{\gamma_t^{(k)} = \otimes_1^k \gamma_t^{(1)}\right\}_{k=1,2\ldots} \text{ solves } (*) \quad \Longleftrightarrow \quad i\partial_t\gamma_t^{(1)} = \left[-\Delta + V * \varrho_t\,, \gamma_t^{(1)}\right]$$

If we knew that
- (*) had a unique solution, and
- $\lim_N \gamma_{N,t}^{(k)}$ exists and satisfies (*),

then the limit must be the factorized one

$\Longrightarrow$ Propagation of chaos + convergence to Hartree eq.

Step 1: Prove apriori bound on $\gamma_{N,t}^{(k)}$ uniformly in N.
Need a good norm and space $\mathcal{H}$! (Sobolev)

Step 2: Choose a convergent subsequence: $\gamma_{N,t}^{(k)} \to \gamma_{\infty,t}^{(k)}$ in $\mathcal{H}$

Step 3: $\gamma_{\infty,t}^{(k)}$ satisfies the infinite hierarchy (need regularity)

Step 4: Let $\gamma_t^{(1)}$ solve NLHE/NLS. Then $\gamma_t^{(k)} = \otimes \gamma_t^{(1)}$ solves the ∞-hierarchy in $\mathcal{H}$.

Step 5: Show that the ∞-hierarchy has a unique solution in $\mathcal{H}$.

Key mathematical steps: Apriori bound and uniqueness

Apriori bound: conservation of $H^k \implies$ mixed Sob. bound

Uniqueness: Many-body version of Strichartz via Feynman graphs

APRIORI BOUNDS

Mixed Sobolev norm [E-Yau,01]

$$\|\gamma^{(k)}\|_{\mathcal{H}^k} := \text{Tr}\, \nabla_1 \dots \nabla_k \gamma^{(k)} \nabla_k \dots \nabla_1$$

can be used for potentials with weaker singularity (e.g. Coulomb).

$\langle \Psi_t, H^k \Psi_t \rangle$ is conserved, we turn it into Sobolev-type norms

$$(*) \qquad \langle \Psi, H^k \Psi \rangle \geq (CN)^k \int |\nabla_1 \dots \nabla_k \Psi|^2 \qquad = (CN)^k \|\gamma^{(k)}\|_{\mathcal{H}^k}$$

so mixed Sob. norms stay under control as time evolves.

(*) is incorrect for the GP, w_N is too singular; $w_N(x) \sim \frac{a}{|x|}$

$$\int \Big|\nabla_1 \nabla_2 (1 - w_N(x_1 - x_2))\Big|^2 \geq \int \frac{a^2}{(|x| + a)^6} \mathrm{d}x = O(a^{-1}) = O(N)$$

After removing the singular part:

Proposition: Define

$$\Phi_{12}(\mathbf{x}) := \frac{\Psi(\mathbf{x})}{1 - w_N(x_1 - x_2)}$$

Then

$$\langle \Psi, H^2 \Psi \rangle \geq (CN)^2 \int |\nabla_1 \nabla_2 \Phi_{12}|^2$$

Weak limit of Ψ and Φ_{12} are equal, but Φ_{12} can be controlled in Sobolev space. Use compactness for Φ_{12}! Similarly for $k > 2$.

Key observation: For singular potentials, the upper bound

$$\langle \Psi, H_N^2 \Psi \rangle \leq CN^2$$

implies that Ψ has a short scale structure in any $x_i - x_j$ variable.

It is essentially a two-body phenomenon, but one needs to control that no third particle gets close.

UNIQUENESS OF THE ∞-HIERARCHY IN SOBOLEV SPACE

$$i\partial_t \gamma_t^{(k)} = \sum_{j=1}^{k} \Big[-\Delta_j, \gamma_t^{(k)} \Big] \underbrace{-i\sigma \sum_{j=1}^{k} \mathrm{Tr}_{x_{k+1}} \Big[\delta(x_j - x_{k+1}), \gamma_t^{(k+1)} \Big]}_{B^{(k)}\gamma^{(k+1)}}$$

Iterate it in integral form:

$$\gamma_t^{(k)} = \mathcal{U}(t)\gamma_0^{(k)} + \int_0^t \mathrm{d}s\, \mathcal{U}(t-s) B^{(k)} \mathcal{U}(s) \gamma_0^{k+1} + \dots$$
$$+ \int_{\sum_k s_k = t} \mathrm{d}s_1 \dots \mathrm{d}s_n\, \mathcal{U}(s_1) B^{(k)} \mathcal{U}(s_2) B^{(k+1)} \dots B^{(k+n-1)} \gamma_{s_n}^{k+n}$$

$$\mathcal{U}(t)\gamma^{(k)} := e^{it\sum_{j=1}^k \Delta_j} \gamma^{(k)} e^{-it\sum_{j=1}^k \Delta_j}$$

Problem 1. $\|B^{(k)}\gamma^{(k+1)}\|_{\mathcal{H}^k} \le C\|\gamma^{(k+1)}\|_{\mathcal{H}^{k+1}}$ is wrong because $\delta(x) \not\le (1-\Delta)$. Need smoothing from $\mathcal{U}$!!

$$\gamma_t^{(k)} = \mathcal{U}(t)\gamma_0^{(k)} + \int_0^t \mathrm{d}s\, \mathcal{U}(t-s)B^{(k)}\mathcal{U}(s)\gamma_0^{k+1} + \dots$$
$$+ \int_{\sum_k s_k = t} \mathrm{d}s_1 \dots \mathrm{d}s_n\, \mathcal{U}(s_1)B^{(k)}\mathcal{U}(s_2)B^{(k+1)} \dots B^{(k+n-1)}\gamma_{s_n}^{k+n}$$

Stricharz inequality? Space-time smoothing of $e^{it\Delta}$.

$$\left\| e^{it\Delta}\psi \right\|_{L^p(L^q(\mathrm{d}x)\mathrm{d}t)} = \left[\int \mathrm{d}t \Big(\int \mathrm{d}x |e^{it\Delta}\psi|^q \Big)^{p/q} \right]^{1/p} \le C\|\psi\|_2$$

$(2 \le p \le \infty,\ 2/p + 3/q = 3/2)$

Problem 2. $B^{(k)}B^{(k+1)} \dots B^{(k+n-1)} \approx n!$, because $B^{(k)} = \sum_1^k \big[\dots \big]$.
This can destroy convergence. Gain back from time integral

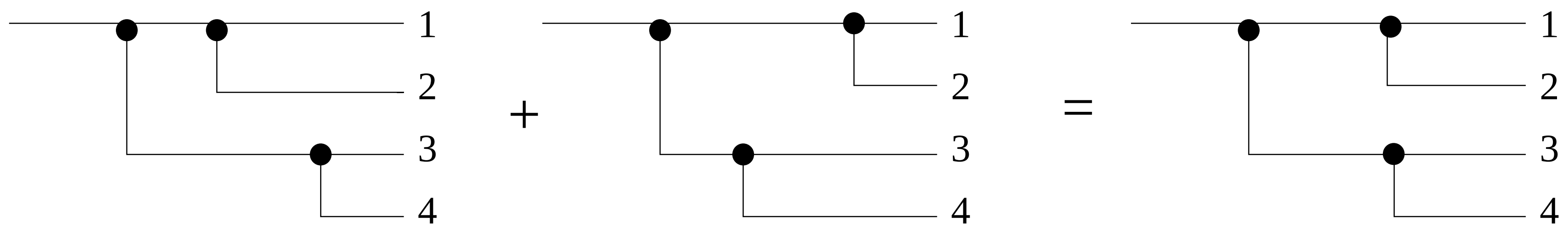
$$\int_{\sum_k s_k = t} \mathrm{d}s_1 \dots \mathrm{d}s_n \le \frac{1}{n!}$$

Here $L^1(\mathrm{d}s)$ was critically used, Stricharz destroys convergence.

We expand it into Feynman graphs, use combinatorial identities and do multiple integrals carefully.

An example for combinatorics:

The Duhamel expansion keeps track of the full time ordering and it counts the following two graphs separately:



Number of graphs on m vertices with time ordering: $m!$

Number of graphs on m vertices without time ordering $= C^m$

The resummation reduced $m!$ to C^m. The factorial was fake!

CONCLUSIONS

- We derived the GP equation from many-body Ham. with interaction on scale $1/N$. Coupling const. = scattering length.

- A specific short scale correlation structure is preserved or even emerges along the dynamics. In the $N \to \infty$ limit, this structure is negligible in L^2 sense (ensuring a closed eq. for the orbitals) but not in energy sense, thus it influences the dynamics via the emergence of the scatt. length.

- Conservation of H^k can imply bounds in Sobolev space

- Stricharz can be strengthened with Feynman diagrams in many body problems

IDEA OF THE H^2-APRIORI BOUND

Work in one particle setting, i.e. in $\mathbb{R}^3$.

$$H = -\Delta + V, \qquad V(x) = \frac{1}{N} N^3 V_0(Nx)$$

Let $f = 1 - w$ be the scattering solution

$$(-\Delta + V)f = 0$$

By scaling,

$$f(x) = f_0(Nx) \sim \begin{cases} 1 - \frac{a_0}{Nx} & x \geq N^{-1} \\ O(1) & x \leq N^{-1} \end{cases}$$

(here a_0 is the scattering length of V_0).

Let Ψ be any wavefunction, factorize f out $(0 < f \leq 1)$

$$\Psi = f\Phi$$

LEMMA: If V_0 is sufficiently small (e.g. a_0 is small), then

$$(\Psi, H^2\Psi) \geq c \int |\Delta\Phi|^2$$

$$(\Psi, H^2\Psi) \geq c \int |\Delta\Phi|^2 \qquad\qquad \Psi = f\Phi$$

$$H\Psi = (-\Delta + V)\Psi = fL\Big[\Psi/f\Big]$$

with

$$L := -\Delta + 2(\nabla \log f)\nabla$$

FACT: L is self-adjoint with respect to $f^2(x)dx$:

$$\int \bar{\Phi} L\Omega\, f^2 = \int L\bar{\Omega}\, \Phi\, f^2 = \int \nabla\bar{\Phi}\nabla\Omega\, f^2$$

$$\begin{aligned}
(\Psi, H^2\Psi) &= \int |H\Psi|^2 = \int |L\Phi|^2\, f^2 = \int \nabla\bar{\Phi}\nabla(L\Phi)\, f^2 \\
&= \int \nabla\bar{\Phi} L(\nabla\Phi)\, f^2 + \int \nabla\bar{\Phi}[\nabla, L]\Phi\, f^2 \\
&= \int |\nabla^2\Phi|^2\, f^2 + \int \nabla\bar{\Phi}\Big[\frac{\nabla^2 f}{f} + \frac{(\nabla f)^2}{f^2}\Big]\nabla\Phi\, f^2
\end{aligned}$$

$$(\Psi, H^2\Psi) = \int |\nabla^2\Phi|^2 f^2 + \int \nabla\bar{\Phi}\Big[\frac{\nabla^2 f}{f} + \frac{(\nabla f)^2}{f^2}\Big]\nabla\Phi\, f^2$$

From the scaling of f:

$$\nabla^2 f \sim \frac{a_0}{N|x|^3} \le \frac{a_0}{|x|^2}, \qquad (\nabla f)^2 \sim \Big(\frac{a_0}{N|x|^2}\Big)^2 \le \frac{a_0}{|x|^2},$$

thus

$$\Big|\int \nabla\bar{\Phi}\Big[\cdots\Big]\nabla\Phi\, f^2\Big| \le Ca_0 \int \frac{1}{|x|^2}|\nabla\Phi|^2 \le Ca_0 \int |\nabla^2\Phi|^2$$

thus, after estimating $f^2 \ge C > 0$, we have

$$(\Psi, H^2\Psi) \ge C\int |\nabla^2\Phi|^2 - Ca_0 \int |\nabla^2\Phi|^2 \ge C\int |\nabla^2\Phi|^2$$

if a_0 is small enough.

Special case: $k = 1$:

$$i\partial_t \gamma^{(1)}_{N,t}(x_1; x_1') = (-\Delta_{x_1} + \Delta_{x_1'})\gamma^{(1)}_{N,t}(x_1; x_1')$$
$$+ \int \mathrm{d}x_2 \Big(V(x_1 - x_2) - V(x_1' - x_2) \Big) \gamma^{(2)}_{N,t}(x_1, x_2; x_1', x_2) + o(1)\,.$$

To get a closed equation for $\gamma^{(1)}_{N,t}$, we need some relation between $\gamma^{(1)}_{N,t}$ and $\gamma^{(2)}_{N,t}$. Most natural: independence

Propagation of chaos: No production of correlations

If initially $\gamma^{(2)}_{N,0} = \gamma^{(1)}_{N,0} \otimes \gamma^{(1)}_{N,0}$, then hopefully $\gamma^{(2)}_{N,t} \approx \gamma^{(1)}_{N,t} \otimes \gamma^{(1)}_{N,t}$

No exact factorization for finite N, but maybe it holds for $N \to \infty$.

Suppose $\gamma^{(k)}_{\infty,t}$ is a (weak) limit point of $\gamma^{(k)}_{N,t}$ with

$$\gamma^{(2)}_{\infty,t}(x_1, x_2; x_1', x_2') = \gamma^{(1)}_{\infty,t}(x_1, x_1')\gamma^{(1)}_{\infty,t}(x_2; x_2').$$

$$i\partial_t\gamma^{(1)}_{N,t}(x_1;x_1') = (-\Delta_{x_1}+\Delta_{x_1'})\gamma^{(1)}_{N,t}(x_1;x_1') + \int \mathrm{d}x_2 \Big(V(x_1-x_2)-V(x_1'-x_2)\Big) \underbrace{\gamma^{(2)}_{N,t}(x_1,x_2;x_1',x_2)}_{\to \gamma^{(1)}_{\infty,t}(x_1,x_1')\gamma^{(1)}_{\infty,t}(x_2;x_2)} + o(1)$$

With the notation $\varrho_t(x) := \gamma^{(1)}_{\infty,t}(x;x)$, it converges, to

$$i\partial_t\gamma^{(1)}_{\infty,t}(x_1;x_1') = (-\Delta_{x_1}+\Delta_{x_1'})\gamma^{(1)}_{\infty,t}(x_1;x_1') + \Big(V*\varrho_t(x_1)-V*\varrho_t(x_1')\Big)\,\gamma^{(1)}_{\infty,t}(x_1;x_1')$$

$i\partial_t\gamma^{(1)}_{\infty,t} = \Big[-\Delta+V*\varrho_t\,,\ \gamma^{(1)}_{\infty,t}\Big]$ Hartree eq for density matrix

If $V = V_N$ scaled, then the short scale structure can be relevant.

For $\gamma^{(2)}_{N,t}(x_1,x_2;x_1',x_2) = (1-w_N(x_1-x_2))\gamma^{(1)}_{N,t}(x_1,x_1')\gamma^{(1)}_{N,t}(x_2,x_2)$,

$$\Longrightarrow \quad \int V_N(x_1-x_2)(1-w_N(x_1-x_2))\Big[\text{Smooth}\Big] = \begin{cases} 8\pi a_0 & \text{if} \quad \beta = 1 \\ b_0 & \text{if} \quad \beta < 1 \end{cases}$$

Feynman graphs

Iteration the ∞-hierarchy: $\gamma_{\infty,t} = \mathcal{U}_t\gamma_0 + \int_0^t \mathrm{d}s\, \mathcal{U}_{t-s}B\gamma_{\infty,s}$

$$\gamma_{\infty,t}^{(k)} = \sum_{m=0}^{n} \Xi_m^{(k)}(t) + \Omega_n^{(k)}(t)$$

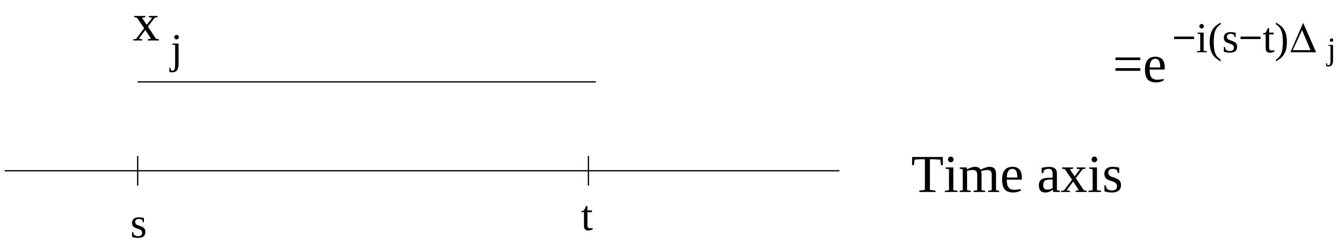
$$\Omega_n^{(k)} = \int \ldots \int \mathrm{d}s_1 \mathrm{d}s_2 \ldots \mathrm{d}s_n\, \mathcal{U}_{t-s_1} B \mathcal{U}_{s_1-s_2} B \ldots \mathcal{U}_{s_{n-1}-s_n} B \gamma_{\infty,s_n}^{(k+n)}$$

$\Xi_m^{(k)}$ are similar but with the initial condition γ_0 at the end.

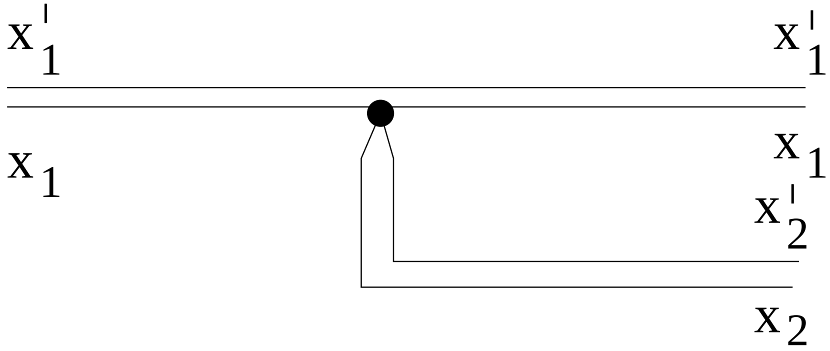
Feynman graphs: convenient representation of Ξ and Ω.

Lines represent free propagators.

E.g. the propagator line of the j-th particle between times s and t represent $\exp[-i(s-t)\Delta_j]$:

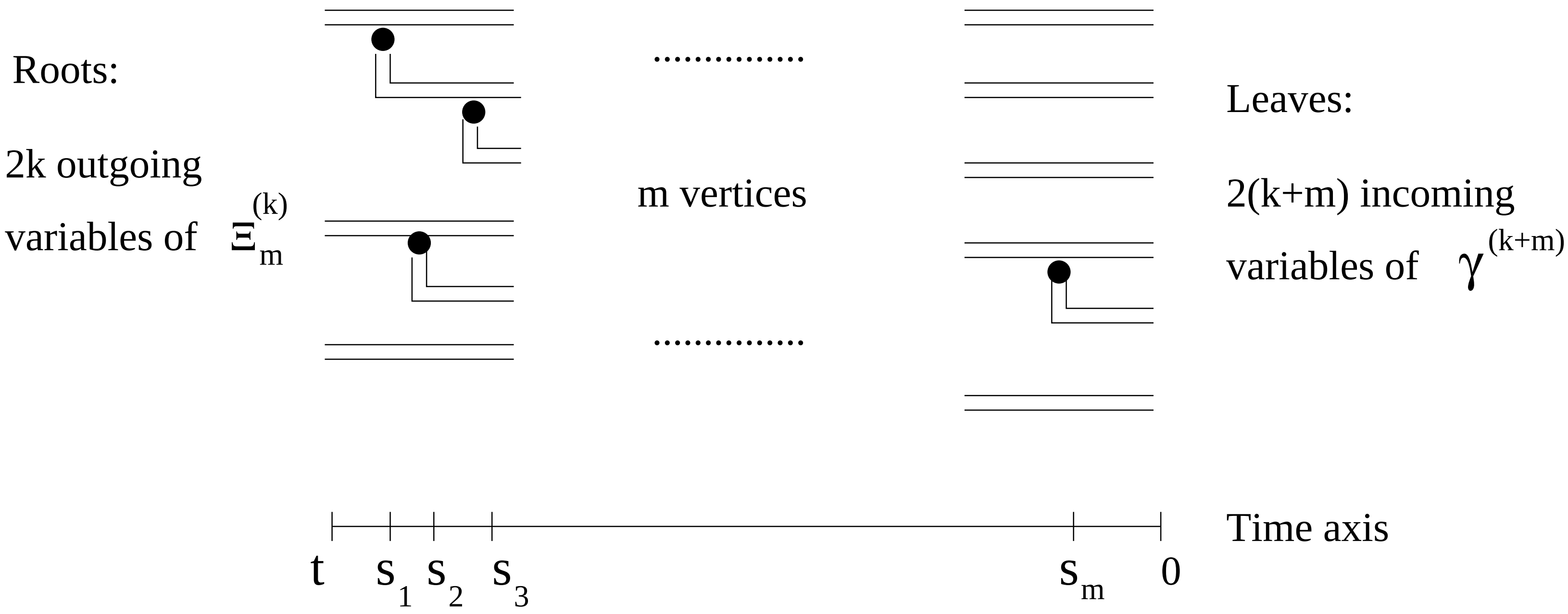


Vertices represent B, e.g. $V(x_1 - x_2)\gamma(x_1, x_2; x_1', x_2')\delta(x_2 - x_2')$



$$\Xi_m^{(k)} = \int \ldots \int \mathrm{d}s_1 \mathrm{d}s_2 \ldots \mathrm{d}s_m \, \mathcal{U}_{t-s_1} B \mathcal{U}_{s_1-s_2} B \ldots \mathcal{U}_{s_{m-1}-s_m} B \mathcal{U}_{s_m} \gamma_{\infty,0}^{(k+m)}$$

corresponds to summation over all graphs Γ of the form:



$$\mathrm{Tr}\, \mathcal{O}\, \Xi_m^{(k)} = \sum_{\Gamma} \mathrm{Val}(\Gamma)$$

Value of a graph Γ in momentum space

$$\mathrm{Val}(\Gamma) = \int\int \prod_{e\in E} \mathrm{d}\alpha_e \mathrm{d}p_e \prod_e \frac{1}{\alpha_e - p_e^2 + i\eta_e} \prod_{v\in V} \delta\Big(\sum_{e\in v} \alpha_e\Big)\delta\Big(\sum_{e\in v} p_e\Big)$$
$$\times\, e^{-it\sum_{e\in Root}(\alpha_e - i\eta_e)} \mathcal{O}(p_e\ :\ e \in \text{Root})\, \gamma_0(p_e\ :\ e\in \text{Leaves})$$

$p_e \in \mathbb{R}^3$ is the momentum on edge e
$\alpha_e \in \mathbb{R}$ variable dual to time running on the edge e.
$\eta_e = O(1)$ regularizations satisfying certain compatibitility cond.

Two main issues to look at

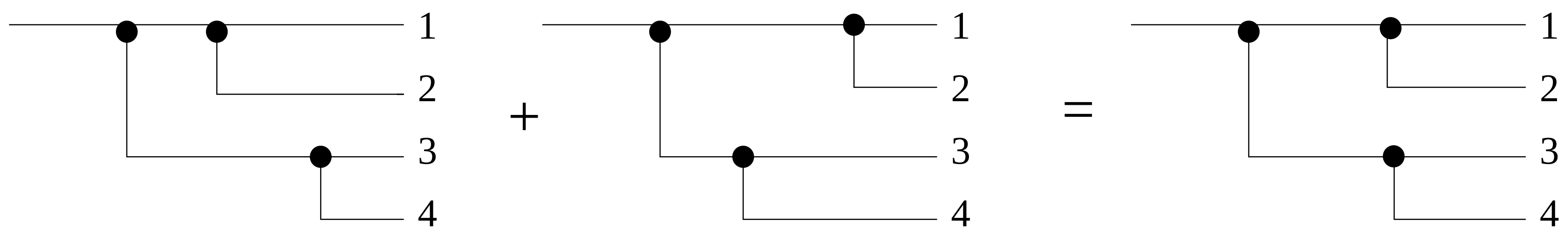
• What happens to the $m!$ problem (combinatorial complexity of the BBGKY hiearchy)?

• What happens to the singular interaction = large p problem
In other words: why is Val(Γ) UV-finite?

Combinatorics reduced by resummation: $m!$ is artificial

Let $k = 1$ for simplicity, i.e. we have a tree (not a forest).

The Duhamel expansion keeps track of the full time ordering and it counts the following two graphs separately:



Number of graphs on m vertices with time ordering: $m!$
(the j-th new vertex can join each of the $(j-1)$ earlier ones)

Number of graphs on m vertices without time ordering = Number of binary trees = Catalan numbers $\frac{1}{m+1}\binom{2m}{m} \le C^m$

UV regime: Finiteness of Val(Γ)

$$|\mathrm{Val}(\Gamma)| \leq \int\int \prod_{e\in E} \mathrm{d}\alpha_e \mathrm{d}p_e \prod_e \frac{1}{\langle \alpha_e - p_e^2 \rangle} \prod_{v\in V} \delta\Big(\sum_{e\in v} \alpha_e\Big)\delta\Big(\sum_{e\in v} p_e\Big)$$
$$\times\, \mathcal{O}(p_e : e \in \text{Root})\, \gamma_0(p_e : e \in \text{Leaves})$$

$\|\gamma_0\|_{\mathcal{H}^{(m+1)}}$ guarantees a $\langle p_e\rangle^{-5/2}$ decay on each leaf.

Power counting ($k = 1$, one root case).

\# of edges $= 3m + 2$, no. of leaves $= 2m + 2$
\# of effective p_e (and α_e) variables: $(3m + 2) - m = 2m + 2$
$2m + 2$ propagators are used for the convergence of α_e integrals
Remaining m propagators give $\langle p^2\rangle$ decay each.

Total p-decay: $\frac{5}{2}(2m + 2) + 2m = 7m + 5$ in $3(2m + 2)$ dim.
There is some room, but each variable must be checked. We follow the momentum decay on legs as we successively integrate out each vertex. There are 7 types of edges, 12 types of vertex integrations that form a closed system. □