On the regularity of the infinity manifolds: the case of Sitnikov problem and some global aspects of the dynamics

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Preliminaries

One of the outstanding problems in Celestial Mechanics is the **detection and computation** of **capture and escape boundaries**.

They are related to the **existence of some invariant objects at infinity** which have **invariant manifolds**.

But these invariant objects **are not hyperbolic**. They are only **parabolic** in the sense of Dynamical Systems.

It is well known that this fact was **partially analysed by Moser and McGehee**. The manifolds **exists** and they are **analytic except, perhaps, at infinity**. Related results are due to **C.Robinson**.

Standing question: Which is the **regularity class** of these manifolds? How can we **compute them** with rigorous error control, so that they can be used to obtain **capture and escape boundaries**?

I shall use the same problem analysed in the past. The well known **Sitnikov problem**.

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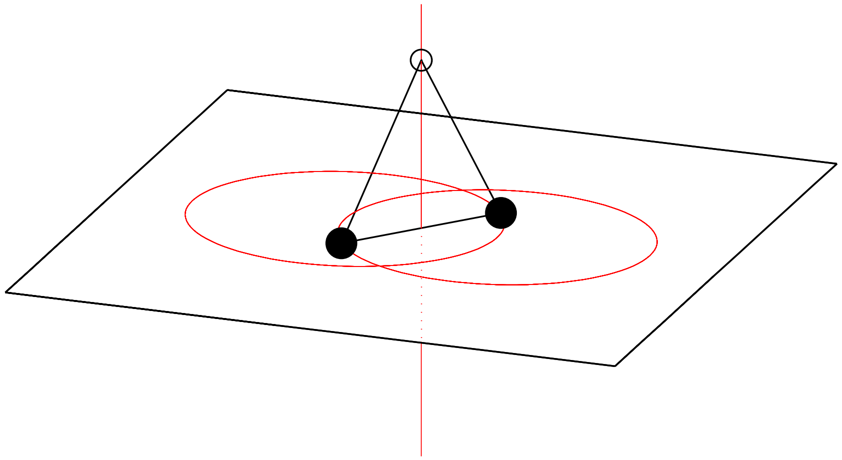
The problem

To decide about **escape/capture** on a given problem of Celestial Mechanics.

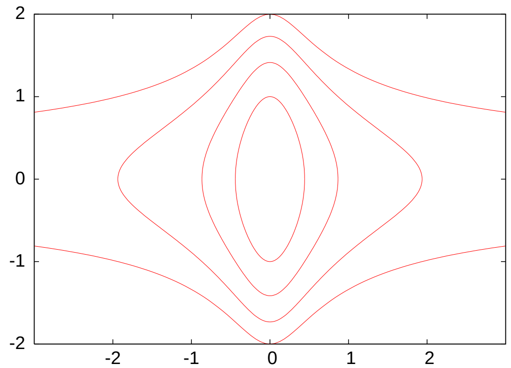
Related questions: **transversality of invariant manifolds, measure of the splitting, creation of chaotic zones, symbolic dynamics, non-integrability**.

We consider one of the simplest models: **Sitnikov problem**

$$\ddot{z} = -\frac{z}{(z^2 + r(t)^2/4)^{3/2}}, \quad r(t) = 1 - e\cos(E), \quad t = E - e\sin(E).$$



The problem



$(z, \dot{z})$ for $e = 0$,
$H = -1.5, -1.0, -0.5$ and 0.

The problem has **1 d.o.f. for $e=0$** and, hence, it is **integrable**
Historical notes: Chazy, Sitnikov, Alekseev, Moser, McGehee.

As a first order system

$$\dot{z} = v, \quad \dot{v} = z(z^2 + r(t)^2/4)^{-3/2}.$$

A **new time, E, $' = d/dE$** and **Hamiltonian formulation**:

$$H(z,\theta,v,J) = (1-e\cos(\theta))\left[\frac{1}{2}v^2 - (z^2+(1-e\cos(\theta))^2/4)^{-1/2}\right] - J.$$

A suitable **Poincaré section $\mathcal{S}$**: polar coordinates $(|v|,t)$ when $z=0$.
Better (APM) use $(|v|(1-e\cos(E))^{1/2}, E)$ **instead of** $(|v|,t)$.

Poincaré map: $(|v|_k, E_k) \to (|v|_{k+1}, E_{k+1})$.

Symmetries: $S_1 : (z,v,t) \leftrightarrow (z,-v,-t)$, $\quad S_2 : (z,v,t) \leftrightarrow (-z,v,-t)$,
$S_3 : (z,v,t) \leftrightarrow (-z,-v,t)$.

If the infinitesimal mass **escapes to infinity**, the massive bodies move in $\mathbb{S}^1$ (eventually, after regularisation of binary collisions using Levi-Civita variables). One talks of a **periodic orbit at infinity**.

Invariant manifolds at infinity

Theorem (Moser): The problem has periodic orbits at both z plus and minus infinity, with invariant manifolds (orbits going to or coming from infinity parabolically). For e small enough the manifolds intersect $\mathcal{S}$ in curves diffeomorphic to circles. These curves have transversal intersection, implying the existence of heteroclinic orbits from $+\infty$ to $-\infty$ and vice versa.

Consequences: **Non-integrability**, **embedding of the shift with infinitely many symbols, existence of** **oscillatory solutions**, **escape/capture domains**, etc.

The p.o. at ∞ is **parabolic** or, topologically, **weakly hyperbolic**. The linearised map around the p.o. is **the identity**.

McGehee variables: $z = 2/q^2, \dot{z} = -p$ and eccentric anomaly

$$q' = \Psi q^3 p, \quad p' = \Psi q^4 \left(1 + \Psi^2 q^4\right)^{-3/2}$$

where $\quad \Psi = \frac{1 - e\cos(E)}{4} \quad$ and $\quad ' = d/dE.$

If $e = 0$ the invariant manifolds are given as $p = \pm q(1 + q^4/16)^{-1/4}$.

Let us denote as $W_{\pm}^{u,s}$ the intersections of unstable/stable manifolds of $\pm\infty$ with $\mathcal{S}$. Due to S_3, $W_{\pm}^{u}$ coincide and also $W_{\pm}^{s}$ coincide, but W_{+}^{s}, W_{-}^{u} have $v > 0$, while W_{-}^{s}, W_{+}^{u} have $v < 0$.

Due to S_1, W_{+}^{u} **and** W_{-}^{s} **are symmetric with respect to** $E = 0$.

We look for a **parametric representation** of the manifolds of the p.o. as

$$p(E, e, q) = \sum_{k \geq 1} b_k(e, E) q^k = \sum_{k \geq 1} \sum_{j \geq 0} \sum_{i \geq 0} c_{i,j,k} e^i \,\mathrm{sc}(jE)\, q^k,$$

where $b_k(e, E)$ are trigonometric polynomials in E with polynomial coefficients in e, $c_{i,j,k}$ are rational coefficients, sc denotes sin or cos functions

McGehee proved: **The invariant manifolds are analytic except, perhaps, at $q = 0$.**

It can be reduced to obtain invariant manifolds of **fixed parabolic points** of discrete maps (thing about the intersection of the manifolds with $E = 0$).

In this context Baldomà-Haro proved: **Generically, 1D invariant manifolds of fixed parabolic points are of some Gevrey class**

Recall: a **formal power series $\sum_{n\geq 0} a_n \xi^n$ is of Gevrey class s if $\sum_{n\geq 0} a_n (n!)^{-s} \xi^n$ is analytic around the origin**.

Problem: To decide about the **regularity class** of $p(E, e, q)$ and to obtain **an explicit representation**.

Solution: First we look for a representation, asking for **invariance**:

$$\Psi q^4 \left(1 + \Psi^2 q^4\right)^{-3/2} = \sum_{k\geq 1} \frac{db_k}{dE}(e, E) q^k +$$

$$\sum_{k\geq 1} b_k(e, E) \Psi k q^{k+2} \sum_{m\geq 1} b_m(e, E) q^m.$$

Recurrence:

$$\binom{-3/2}{m}\left(\frac{1-e\cos(E)}{4}\right)^{2m+1} = b'{}_n(e,E) + \frac{1-e\cos E}{4}\sum_{k=1}^{n-3} k b_k(e,E) b_{n-2-k}(e,E),$$

where $m = n/4 - 1$, defined only for n multiple of 4.

Solving the recurrence: For W_+^u be have $b_1 = 1$. Then we start to compute iteratively. But $b'_n(e,E) =$ **known function** allows to compute **only the periodic part** $\tilde{b}_n$ of $b_n = \tilde{b}_n + \bar{b}_n$. The **constant part** $\bar{b}_n$ is computed previous to the solution of $b'_{n+3}(e,E) =$ known function, to have a zero average function when we integrate.

An easy induction gives the following result on the format of the solution:

Lemma The coefficients $b_k(e,E)$ satisfy

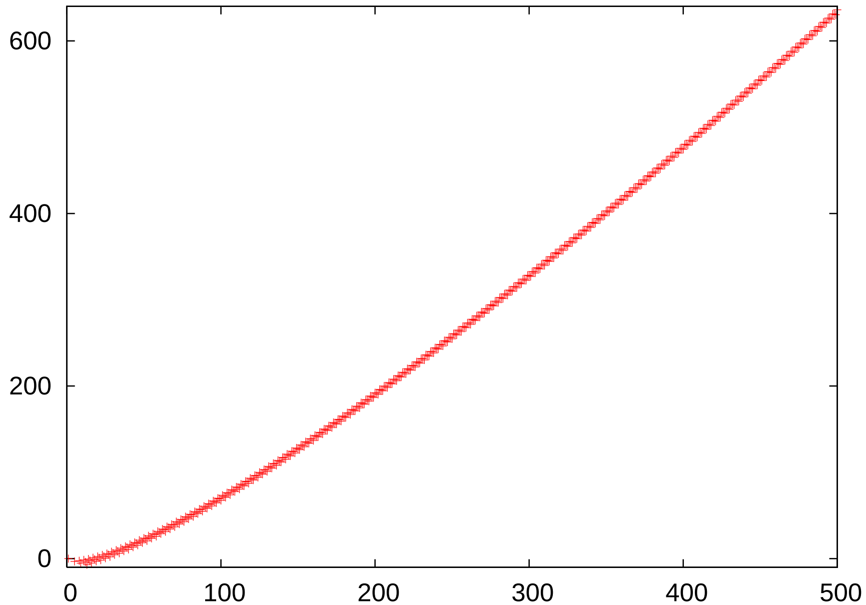
- For k odd (resp. even) b_k is even (resp. odd) in E. It contains harmonics from 0 to $(k-1)/2$, all of them cosinus (resp. from 1 to $(k-2)/2$, all of them sinus).
- The coefficients of the $j-$th harmonic in b_k contain, at most, powers of e with exponents from j to $(k-1)/2$ for k odd (resp. from j to $(k-2)/2$ for k even). The step in the exponents of e is 2.

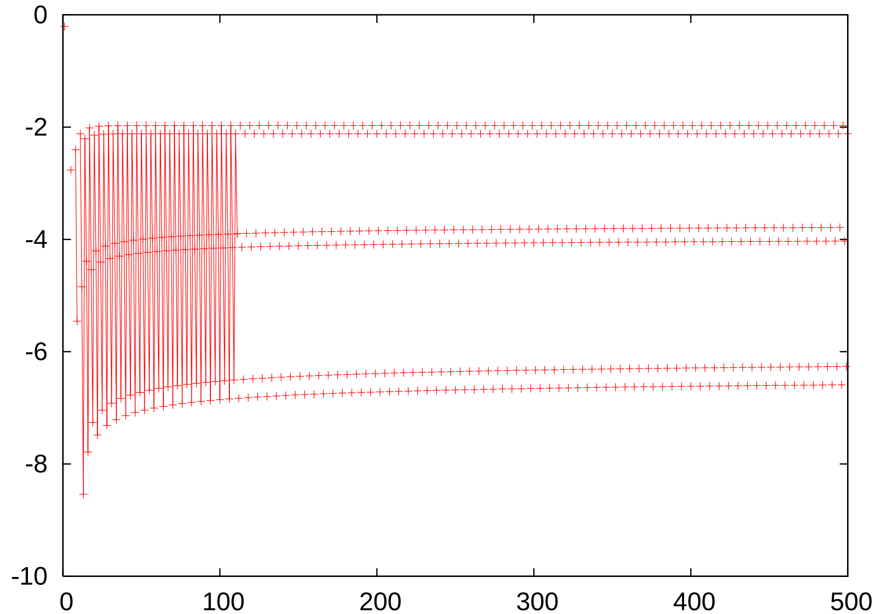
k	j	i	a	b
1	0	0	1	1
5	0	0	-1	2^{6}
5	0	2	-3	2^{7}
8	1	1	3	2^{6}
8	2	2	-9	2^{9}
8	1	3	-9	2^{9}
8	3	3	1	2^{9}
9	0	0	5	2^{13}
9	0	2	27	2^{13}
9	0	4	9	2^{13}
11	1	1	27	2^{8}
11	0	2	27	2^{9}
11	2	2	-189	2^{12}
11	1	3	-81	2^{12}
11	3	3	33	2^{12}
11	0	4	-81	2^{13}
11	2	4	9	2^{10}
11	4	4	-9	2^{14}
12	1	1	-15	2^{11}
12	2	2	75	2^{14}
12	1	3	-75	2^{14}
12	3	3	-25	2^{14}
12	2	4	75	2^{15}
12	4	4	75	2^{18}
12	1	5	75	2^{15}
12	3	5	-25	2^{17}
12	5	5	-3	2^{17}
13	0	0	-15	2^{19}
13	0	2	-351	2^{20}
13	0	4	-213	2^{19}
13	0	6	-121	2^{21}
14	1	1	-81	2^{8}
14	2	2	1215	2^{13}
14	1	3	1215	2^{13}
14	3	3	-255	2^{13}
14	2	4	-45	2^{11}
14	4	4	225	2^{16}
14	1	5	-135	2^{13}
14	3	5	135	2^{15}
14	5	5	-27	$2^{15} * 5$
15	1	1	-429	2^{14}
15	0	2	-447	2^{15}
15	2	2	3783	2^{18}
15	1	3	-6669	2^{18}
15	3	3	-1131	2^{18}
15	0	4	-13203	2^{20}
15	2	4	5265	2^{19}
15	4	4	3627	2^{22}
15	1	5	2301	2^{19}
15	3	5	-11791	$2^{21} * 3$
15	5	5	-1131	$2^{21} * 5$
15	0	6	2291	2^{20}
15	2	6	-4199	2^{21}
15	4	6	221	2^{21}
15	6	6	13	2^{21}
16	1	1	105	2^{17}
16	2	2	-735	2^{20}
16	1	3	2205	2^{20}
16	3	3	1225	$2^{20} * 3$
16	2	4	-1225	2^{20}
16	4	4	-1225	2^{23}
16	3	5	1225	2^{22}
16	5	5	147	2^{22}
16	2	6	-3675	2^{24}
16	4	6	-735	2^{24}
16	6	6	-245	$2^{24} * 3$
16	1	7	-3675	2^{24}
16	3	7	245	2^{24}
16	5	7	49	2^{24}
16	7	7	5	2^{24}

First $c_{i,j,k} = a/b$. Columns: k, j, i, a, b $\quad (q^k \operatorname{sc}(jE) e^i)$.

To have some insight, we look to the behaviour of the coefficients obtained numerically for **larger n**.

Let us introduce a **norm:** $a_k = \sum_{i,j} |c_{i,j,k}|$ **for** b_k.





Left: $\log(a_k)$ vs k. Right: $\log(a_k) - \log(\Gamma((k+1)/3)) + 0.095894k$ vs k.

One observes that the behaviour of a_k depends on the value of k **mod 6**. On the right plot, from top to bottom, the values of k mod 6 are 5,2,3,0,1,4, respectively. A suitable fit helps to display results in a nice way.

It is essential to remark: $b_2 = b_3 = b_4 = 0$. Also $b_6 = b_7 = b_{10} = 0$, but this is not so relevant.

Main result: the Gevrey character of the manifolds

Guided by the behaviour **suggested** by the numerical results, we can **scale** properly the recurrence and introduce $B_n(e,E)$

$$b_n(e,E) = \Gamma((n+1)/3)\rho^n B_n(e,E)$$

where $\rho = (3/4)^{1/3}$. This allows to obtain

Theorem: The manifolds $W_{\pm}^{u,s}$ are **exactly Gevrey-1/3** in q uniformly for $E \in \mathbb{S}^1$, $e \in (0,1]$. Concretely, let a_n denote the norm of b_n. Then there exist constants c_1, c_2, $0 < c_1 < c_2$ such that, for $n \geq 5$ except for $n = 6, 7, 10$ one has

$$\mathbf{c_1}\rho^{\mathbf{n}} < \mathbf{a_n}/\mathbf{\Gamma}((\mathbf{n}+\mathbf{1})/\mathbf{3}) < \mathbf{c_2}\rho^{\mathbf{n}}.$$

Furthermore the coefficient $B_{1,1,n}$ of $e\sin(E)$ in B_n satisfies $0 < C_1 < |B_{1,1,n}| < C_2$ for some constants C_1, C_2 and $n \geq 8$, except for $n = 9, 10, 13$.

Sketch of the proof

Steps:

- Rewrite the recurrence in terms of B_n,
- Note that with this scaling, and dividing by a suitable Γ the **term on the left becomes negligible**,
- Note that with this scaling, the **terms coming from** $b_1 b_{n-3}$ **are** $\mathcal{O}(1)$ and the effect of **all the other terms in the sum is** $\mathcal{O}(n^{-4/3})$,
- The **essential part reduces** to

$$\mathbf{B'_n} = -(\mathbf{1} - \mathbf{e}\cos(\mathbf{E}))\mathbf{B_{n-3}} + \mathcal{O}(\mathbf{n}^{-4/3}),$$

  where the $\mathcal{O}(n^{-4/3})$ is bounded by some $An^{-4/3}$, $A>0$, indep. of n.
- This gives the **purely periodic part** $\tilde{B}_n$ of B_n. The **average part** $\tilde{B}_n$ is obtained by requiring that $(1 - e\cos(E))B_n$ has zero average.
- The **operator** T $B_{n-3} \to B_n$, neglecting the $\mathcal{O}(n^{-4/3})$ and going from 2π-periodic to 2π-periodic satisfies: T^4 has 2 eigenvectors with **eigenvalue 1**. All other eigenvalues have $|\mu| < 1$.

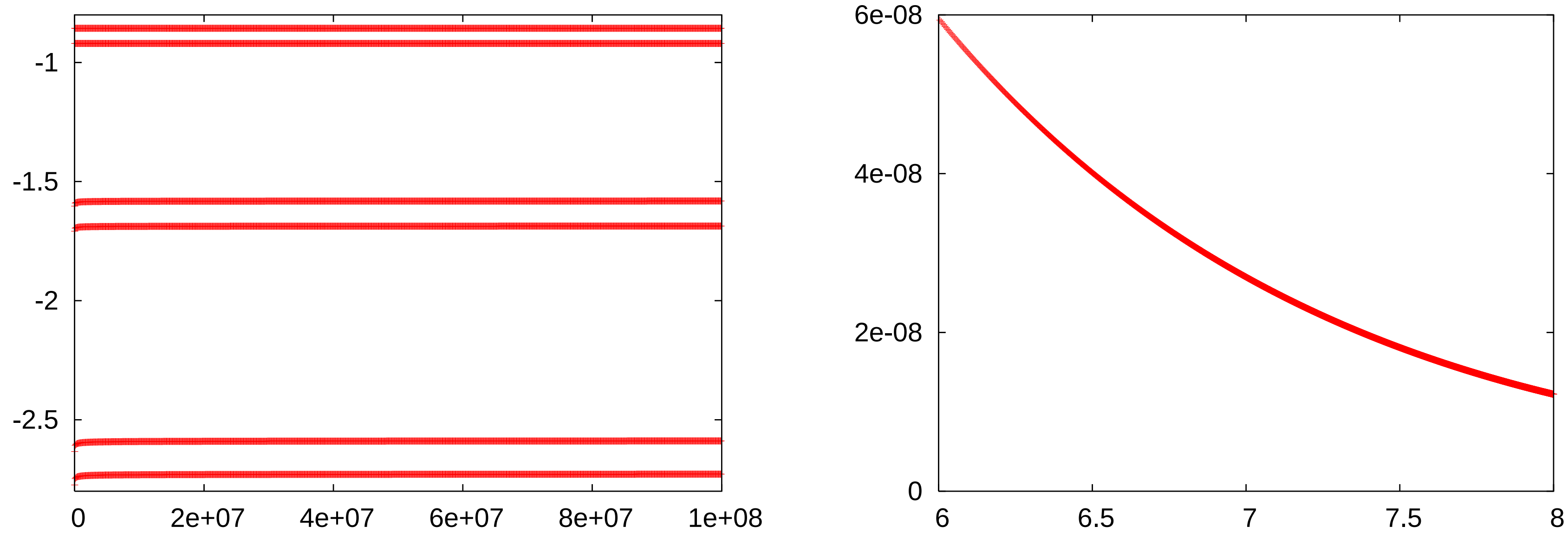
Effective expansions to high order and additional checks

Furthermore let $A_n = \sum_{i,j} |C_{i,j,n}|$ be the norm of $B_n = \sum_{i,j} C_{i,j,n} e^{i}$sc $\cos(jE)$. Then

$$\mathbf{\lim_{n=6m+k, m\to\infty} A_n = L_k, \quad k = 0 \ldots, 5}$$

and

$$\mathbf{A_{n=6m+k} = L_k + \delta_{k,1} n^{-1/3} + \delta_{k,2} n^{-2/3} + \ldots}$$



Left: $\log(A_n)$ as a function of n. Right: $A_n - L_5$ for $n = 6m + 5$, $L_5 \approx 0.139278497$

Asymptotic character of the formal series

The **formal series** introduced is **not convergent**, but can be **useful** to compute p for given q, e, E if we know its **asymptotic character**. The main result in this direction is the following

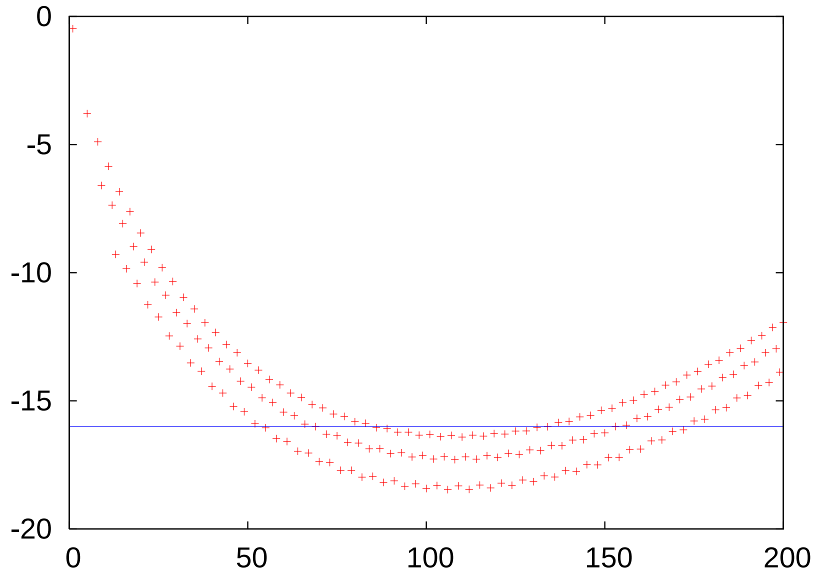
Theorem: The formal expansion gives an **asymptotic representation** of the invariant manifolds of p.o.$_\infty$. Concretely, the **truncation of the series at order** n has an error which is bounded by the sum of the norms of next three terms

$$C(a_{n+1}q^{n+1} + a_{n+2}q^{n+2} + a_{n+3}q^{n+3}),$$

where C is a constant which can be taken close to 1.

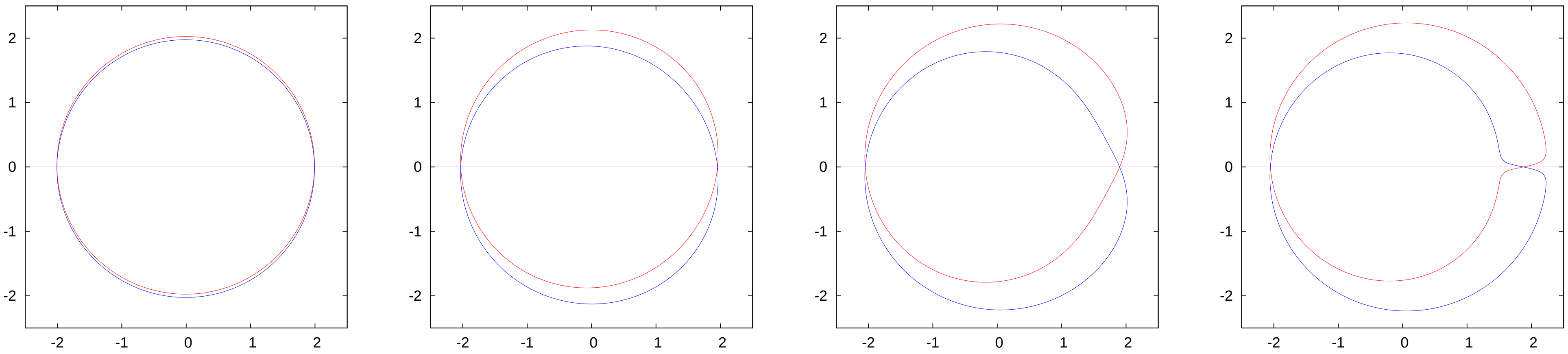
Optimal estimates

Given q the **optimal order is** $n_{\mathbf{opt}} \approx 4/q^3$. Using optimal order **the error bound is** $< N\exp(-4/(3q^3)), N < 1$. **Large** q **allows to start numerical integration at small** z: $z = 2/q^2 = 18$ if $q = 1/3$.

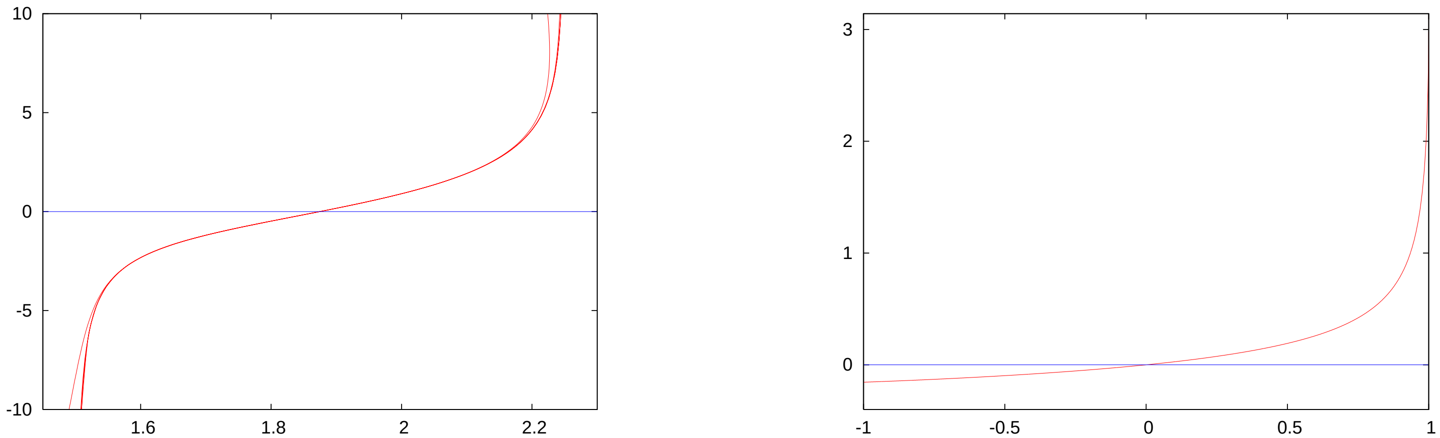


$\log_{10} a_n q^n$ as a function of n for $q = 1/3$.

Intersections with $z = 0$ and splitting



Manifolds of $z = 0$ for $e = 0.1, 0.5, 0.9, 0.999$.



Manifolds with $e = 1 - 10^{-k}, k = 3, \ldots, 8$ with vertical variable divided by $\delta = \sqrt{1-e}$. Right: Right (left) angle of splitting as a function of e $(-e)$. For $e \to 1$ the right splitting behaves as $2 \arctan(c/\delta)$.

Tending to the limit case $e = 1$

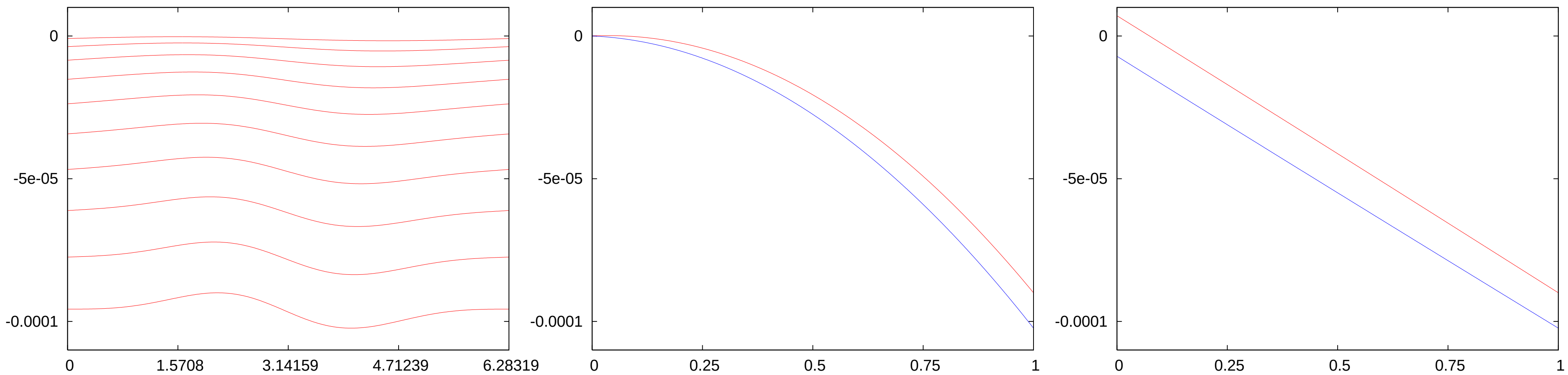
When $e \longrightarrow 1$ we observe two interesting facts:

- Close to $E = 0$ and scaling the vertical variable by $\delta = \sqrt{1-e}$ the manifolds are essentially independent of e. They have **the same shape**.
- In the limit the manifolds **have a radial jump** from $E \longrightarrow 0_-$ **to** $E \longrightarrow 0_+$. This has a **simple mechanical explanation** and is related to the **approach to triple collision**. Can also be explained using the limit case of mass ratio tending to zero in the **isosceles problem**, by analysing the invariant manifolds of the central configurations in the **triple collision manifold**.

The first part is analysed by introducing $E = \delta s, z = \delta^2 u, v = w/\delta$, writing the equations in the new variables, using a (large) compact set for u, s. **When $\delta \longrightarrow 0$ there exists a limit equation.**

The second part is analysed using the RTBP with the **primaries in collinear parabolic orbits**. The relevant parameter is the **time t_0 of passage through $z = 0$ assuming the primaries collide at t=0**. If $t_0 > 0$ or $t_0 < 0$, suitable scalings give **two different limit problems**.

Escape/capture boundaries



Left: Plot of the **corrections for p with respect to the case $e = 0$ for e=0.1,0.2,. . . ,1.0** for $q = 1/3$ as a function of E. This is useful for **early detection of escape/capture**.

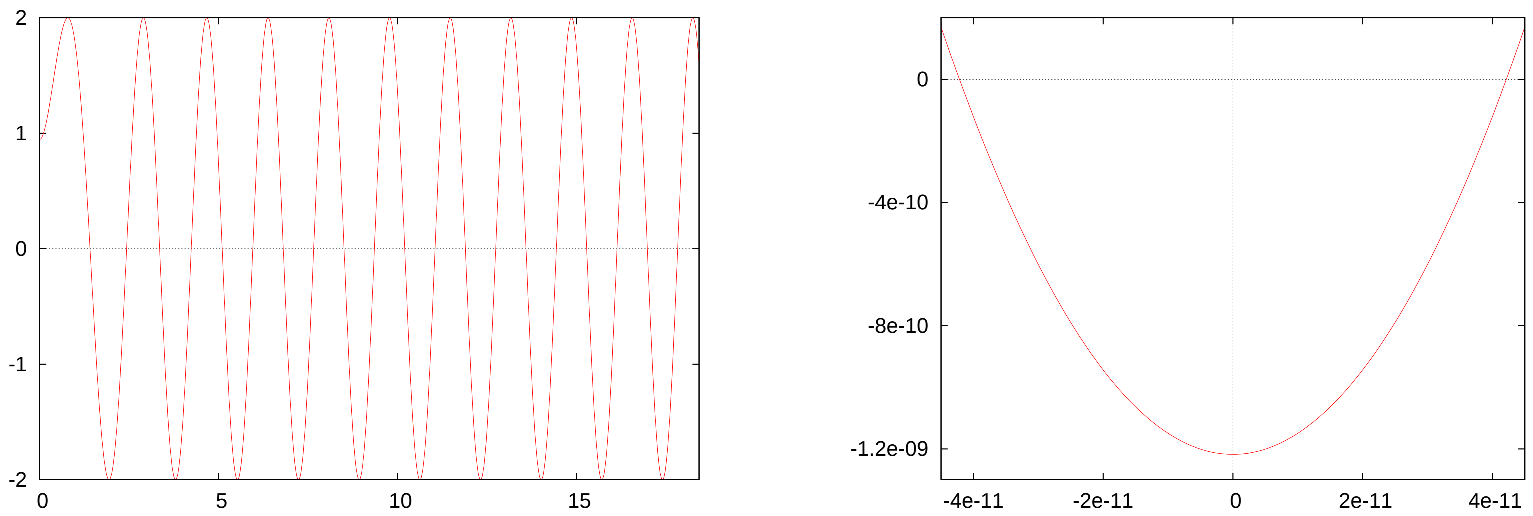
Middle: **Maximal and minimal values of the corrections** for the full range $E \in [0, 2\pi]$ as a **function of e**.

Right: The same values **scaled by e**. Note that a **linear behaviour with respect to e** (as would be the case using expansion in powers of e) is only approximated for $e << 1$.

Some further global dynamical properties

Stability of the trivial solution

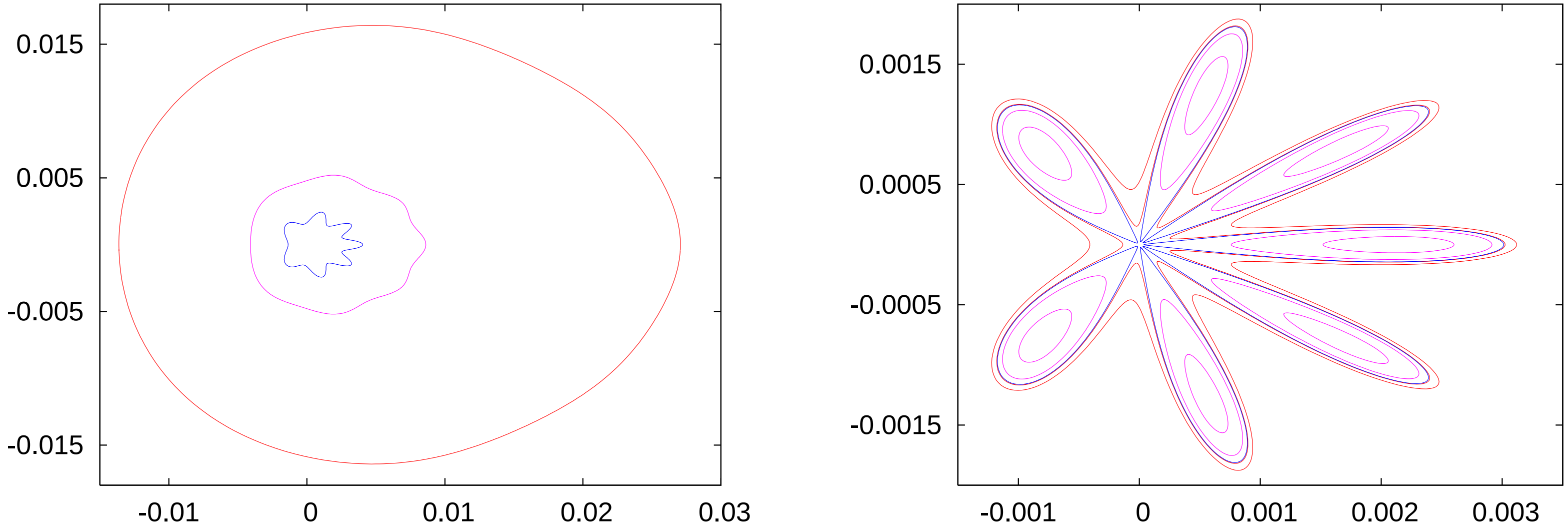
At $z = 0$ we have $d\xi/dE = (1 - e\cos(E))\eta$, $d\eta/dE = -8\xi/(1 - e\cos(E))^2$ a Hill equation of Ince's type. More general cases studied in Martínez-Samà-S for general homogeneous potentials and fully 3D.



Left: Tr vs $-\log(1-e)$. Right: a detail of open gaps below $Tr + 2 = 0$.

Proposition. All gaps at $Tr = 2$ are closed. All gaps at $Tr = -2$ are open. Exists a limit behaviour. This implies infinitely many bifurcations of periodic orbits.

The **rotation number** R at the fixed point (average angle turned by E under one Poincaré iteration) **decreases for** e **increasing**. For **fixed** e **it increases with radius**. At $e = 0, R = 1/\sqrt{8}$. It is of the form $1/m, m$ odd if Tr=2, $2/m, m$ odd if Tr=-2.



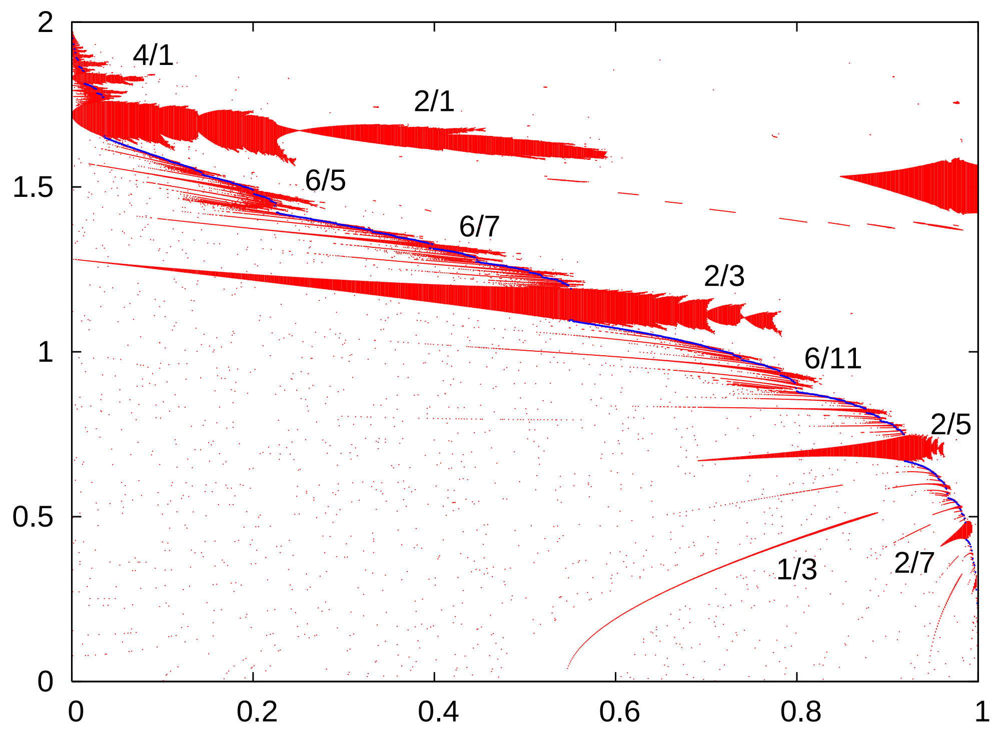
Invariant curves, away from the origin (left) and very close to the origin (right) for $e = 0.85586255$ with Tr≈ -2 and $R = 2/7$.

The **flower-like** pattern appears with $7, 9, 11, \ldots$ petals every time Tr≈ -2.

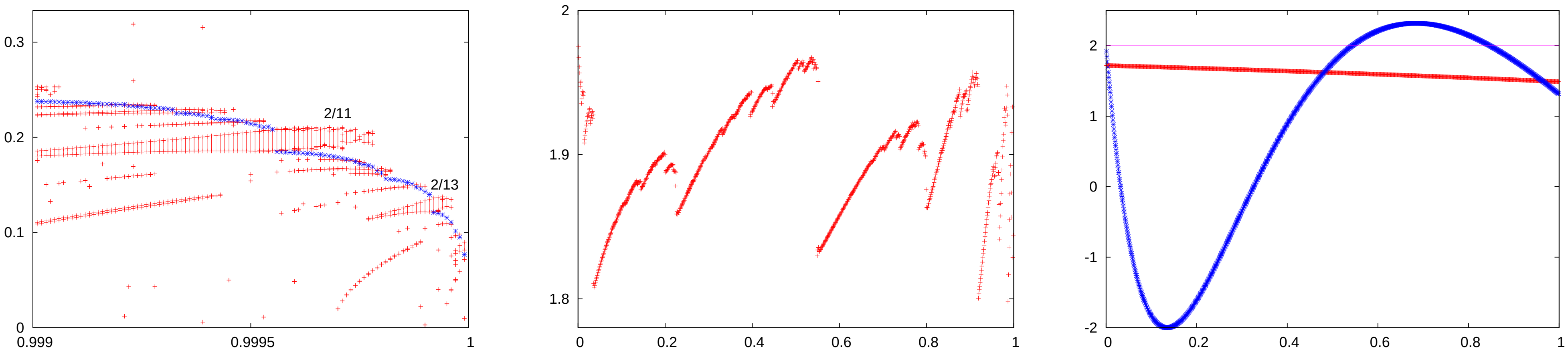
A global view on the dynamics

Computation of **rotation number, detection of islands, escape, outermost invariant rotational curve, ...** starting on $E = \pi$, 1000 values of e, 20000 values of $v \in [0, 2)$.

Statistics: in **islands** 4.6%; in rotational invariant **curves** 48.9%; in confined **chaotic** zones 0.2%; **escape** 46.3%

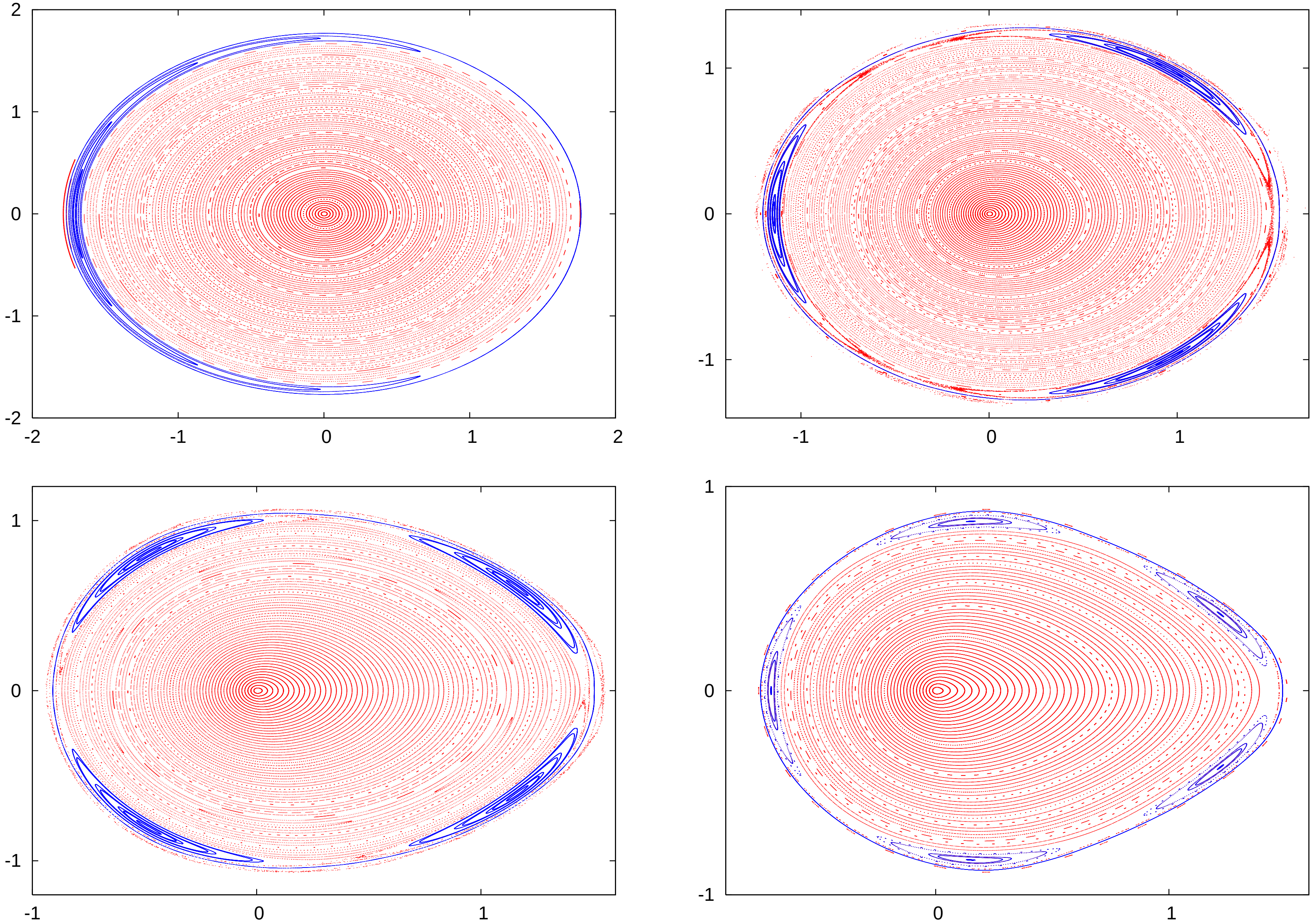


Variables: $(e, v(1+e)^{1/2})$. Red: points in islands (some islands are identified).Blue: outermost invariant curve. White below blue: rotational invariant curves. White on top of blue: escape.



Left: Same as previous plot, but for $e \in [0.999, 1)$. Middle: Location of **outermost invariant curve** modified by adding a suitable function of e, to **enhance jumps**. Right: **p.o.** of rotation number $R = 2/1$, showing initial data (red) and Tr (blue).

On next page: **Poincaré maps** for $e = 0.032, 0.540, 0.790, 0.910$, close to **breakdown of rotational i.c.** outside islands of periods $1, 3, 4, 5$, respectively.



2
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1

Conclusions

We can summarize what we have obtained and possible future work.

- It is **feasible to compute $W_{\pm}^{u,s}$ at high order**, enough to have **accurate escape/capture boundaries**.
- It is feasible to **prove the Gevrey character of the series**.
- It is feasible to obtain **rigorous and useful error estimates and optimal order**
- The **global dynamics of Sitnikov problem can be considered as fully understood for all e, in a reasonable way**.
- It **confirms the relation between Gevrey functions, asymptotic expansions, exponentially small phenomena, etc**.
- The method **opens the way to other more relevant problems, like 2DCR3BP, 3DCR3BP, 3DER3BP, general 3BP, etc**.
- The approach can allow **to produce sharp estimates on celebrated theorems by Takens (interpolation thm) and Neishtadt (averaging thm)** when a **close to the identity map** is approximated by a flow.

Additional notes

I would like to present some elementary asymptotic considerations. Assume some phenomenon is measured by a function φ depending on the **small parameter** ε and can be represented by an **asymptotic expansion**

$$\varphi(\varepsilon) \sim \sum_{m \geq 0} a_m \varepsilon^m, \qquad \text{with} \qquad \left| \sum_{m \leq n} a_m \varepsilon^m - \varphi(\varepsilon) \right| < |a_{n+1}| \varepsilon^{n+1}.$$

If we assume $|a_n|$ monotonically increasing, the **best bound** for the error is obtained for $|a_{n+1}|\varepsilon^{n+1}$ minimum. Let $b(\varepsilon)$ be the bound.

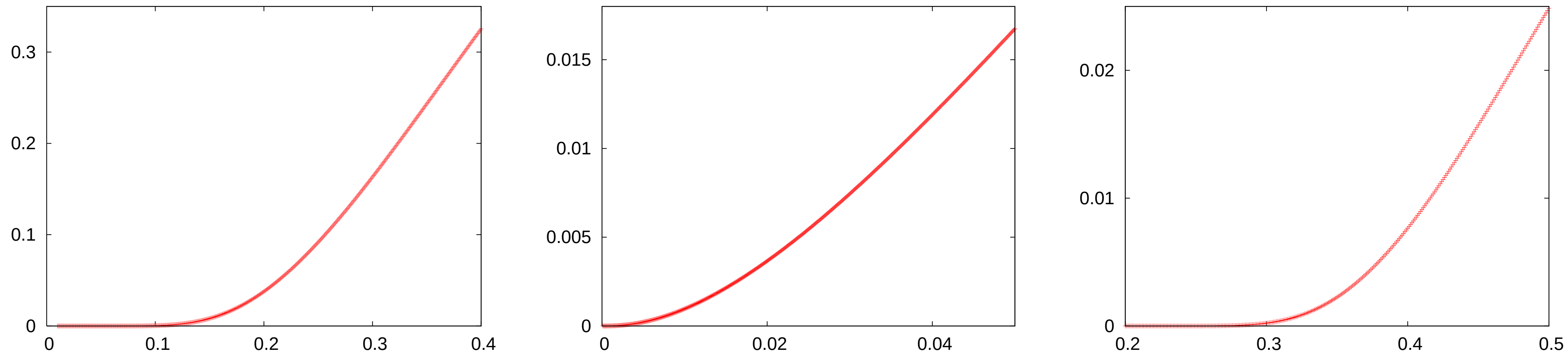
Some **examples**:

1) For $a_n = (n!)^\gamma$ (Gevrey classes) we obtain $n \simeq \varepsilon^{-1/\gamma}$ and $b(\varepsilon) \simeq (2\pi)^{\gamma/2} \varepsilon^{-1/2} \exp(-\gamma \varepsilon^{-1/\gamma})$, a typical **exponentially small** behaviour.

2) If $a_n = (n^\beta!)^\gamma$, $\beta > 1$, then n satisfies the equation $\gamma \beta^2 n^{\beta-1} \log(n) \simeq |\log(\varepsilon)|$ for $\varepsilon \to 0$ and $b(\varepsilon) \simeq \exp(K |\log(\varepsilon)|^{\beta/(\beta-1)} / \log(|\log(\varepsilon)|)^{1/(\beta-1)})$, where $K = -(1 - \beta^{-1})((\beta - 1)/\gamma\beta^2)^{1/(\beta-1)}$, for $\varepsilon \to 0$.

3) For $a_n = (\log n)^n$ we obtain $n \simeq \exp(1/\varepsilon - 1)$ and $b(\varepsilon) \simeq \exp(-\varepsilon \exp(1/\varepsilon))$. This case shows really a **quite sharp behaviour** with respect to ε:

ε	0.4	0.2	0.15	0.12	0.1
$b(\varepsilon)$	$7.6 \cdot 10^{-3}$	$1.3 \cdot 10^{-13}$	$6.5 \cdot 10^{-52}$	$1.5 \cdot 10^{-217}$	$4.0 \cdot 10^{-956}$



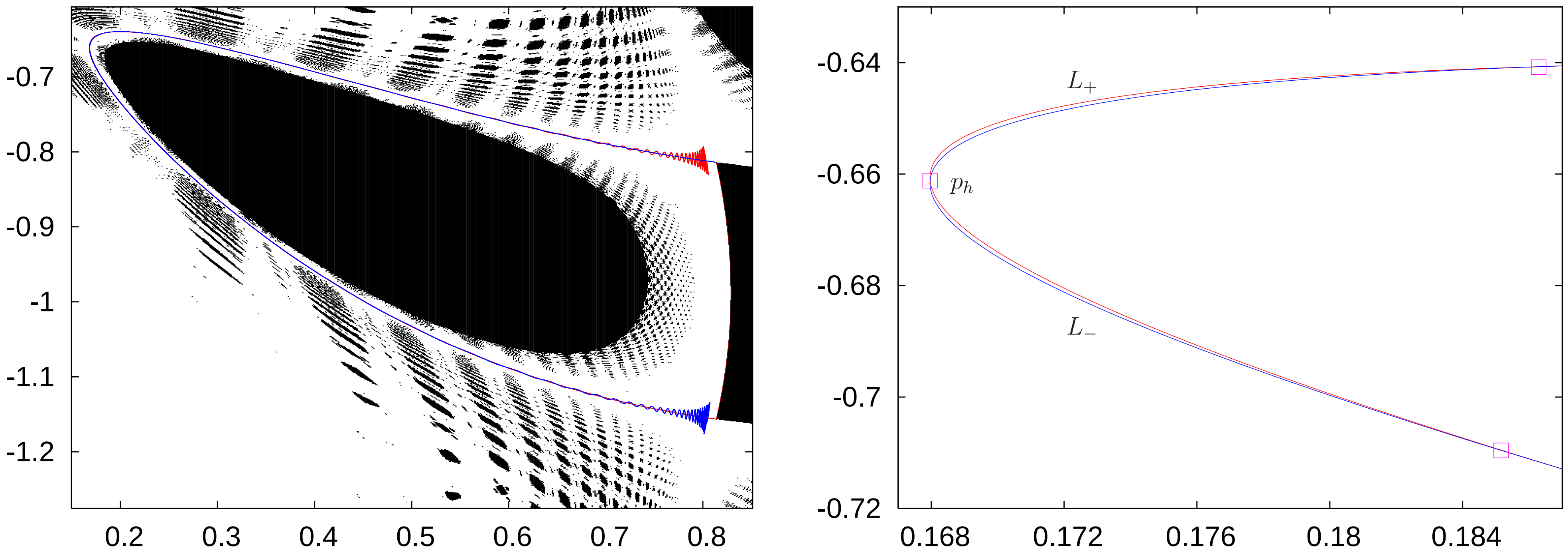
Case 1 Case 2 Case 3

All of them $\mathcal{C}^\infty$ **flat functions** but the behaviour is quite different.

The Hénon map near the 4:1 resonance

$$H_c : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} c(1-x^2) + 2x + y \\ -x \end{pmatrix}$$

A **4:1 resonance** appears for $c = 1$. We look at $c = 1.015$.



The dynamics of H_c^4 can be interpolated by the flow of a Hamiltonian

$$\mathcal{H}(x, y, \delta), \qquad \delta = (c-1)^{1/4},$$

which shows a **Gevrey character**.