Homotopy Continuation Method For Solving Polynomial Systems

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Solving polynomial system

$$P(\mathbf{x}) = (p_1(\mathbf{x}), \ldots, p_n(\mathbf{x})) = 0, \quad \mathbf{x} \in \mathbf{C}^n$$

1. Linear Homotopy (Begins in 1979)

2. Nonlinear Homotopy
   (Polyhedral Homotopy)

   (1995, The state of the art)

1. Linear Homotopy

$$\begin{cases} x^2 + y^2 = 5 \\ x - y = 1 \end{cases} \qquad \text{solutions: } (x, y) = \begin{cases} (2, 1) \\ (-1, -2) \end{cases}$$

$$\begin{cases} x^2 = 1 \\ y = 1 \end{cases} \qquad \text{solutions: } (x, y) = \begin{cases} (1, 1) \\ (-1, 1) \end{cases}$$

$$(1-t)\begin{pmatrix} x^2 - 1 \\ y - 1 \end{pmatrix} + t \begin{pmatrix} x^2 + y^2 - 5 \\ x - y - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$t = \frac{1}{4} \qquad \begin{cases} x^2 + \frac{1}{4}y^2 - 2 = 0 \\ \frac{1}{4}x + \frac{1}{2}y - 1 = 0 \end{cases}$$

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$t = 1/4$	$t = 1/2$	$t = 3/4$
$x^2 + 0.25y^2 - 2 = 0$	$x^2 + 0.5y^2 - 3 = 0$	$x^2 + 0.75y^2 - 4 = 0$
$0.25x + 0.5y - 1 = 0$	$0.5x + y - 1 = 0$	$0.75x - 0.5y - 1 = 0$

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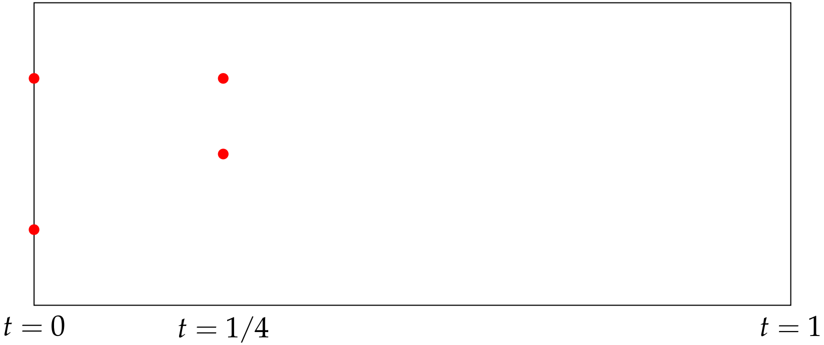

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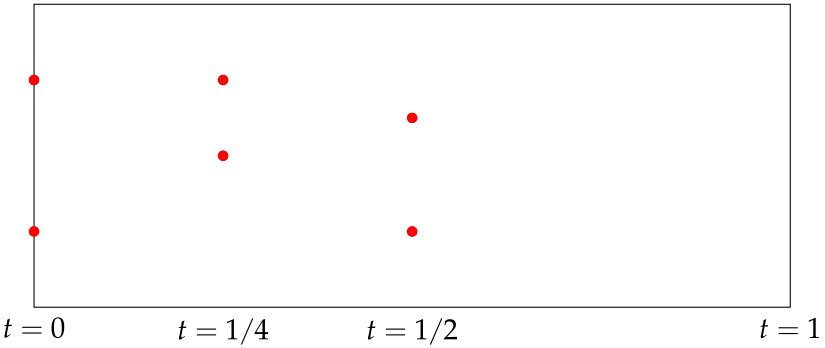

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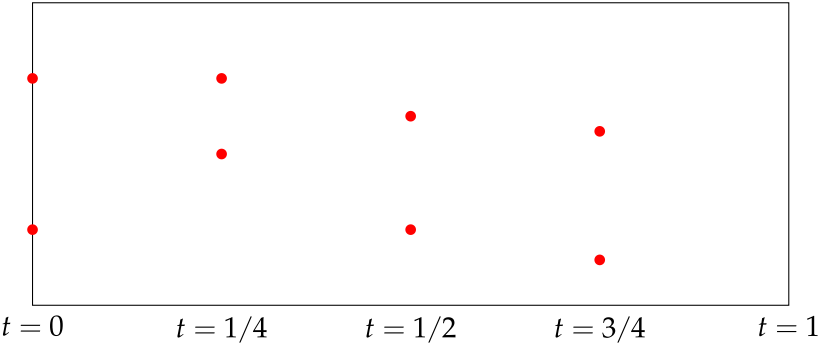

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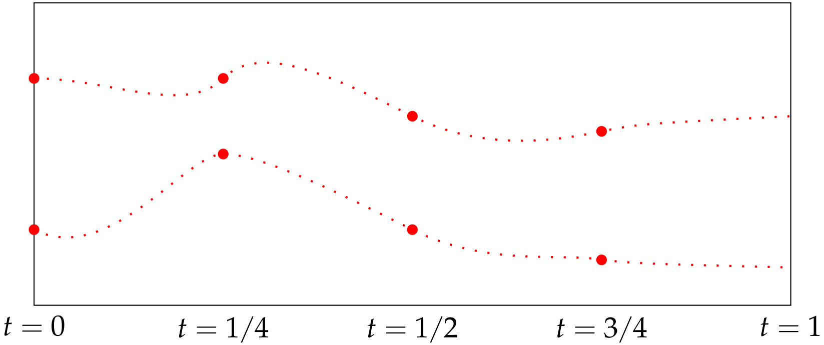

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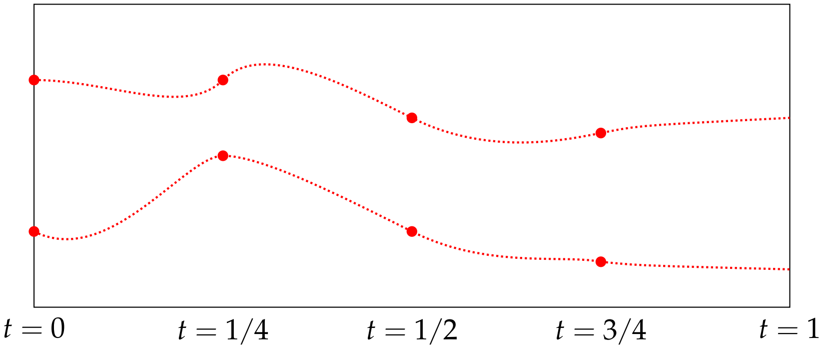

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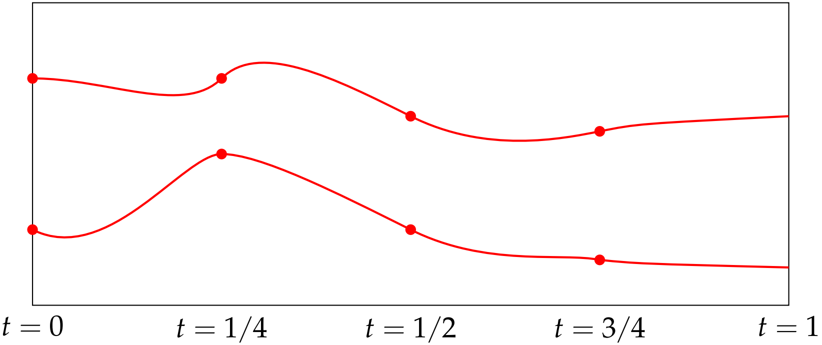

Potential Problems

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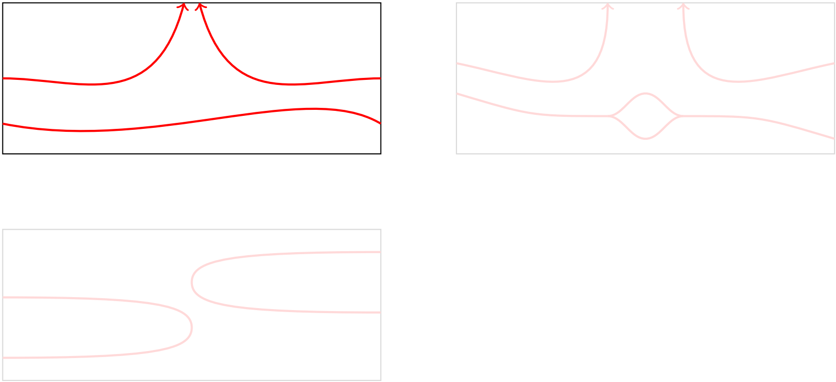

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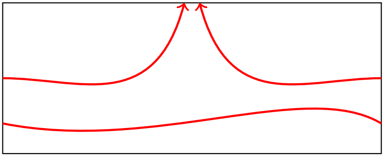

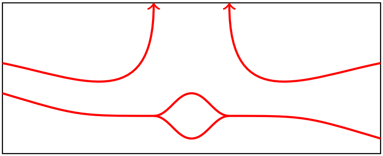

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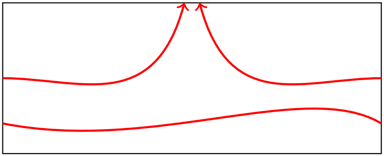

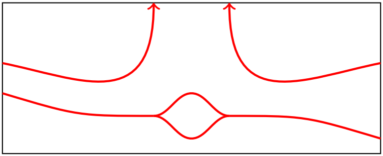

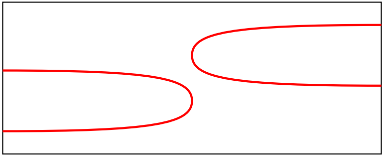

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$$(1-t)\begin{pmatrix}\alpha_1 x_1^{d_1}-\beta_1\\ \vdots\\ \alpha_n x_n^{d_n}-\beta_n\end{pmatrix} + t\begin{pmatrix}p_1(x_1,\ldots,x_n)\\ \vdots\\ p_n(x_1,\ldots,x_n)\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$$

Form the homotopy:

$$(1-t)\begin{pmatrix}a_1x^2-b_1\\ a_2y^1-b_2\end{pmatrix} + t\begin{pmatrix}x^2+y^2-5\\ x-y-1\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

Theorem

For almost all $(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n)$ *the above homotopy "works".*

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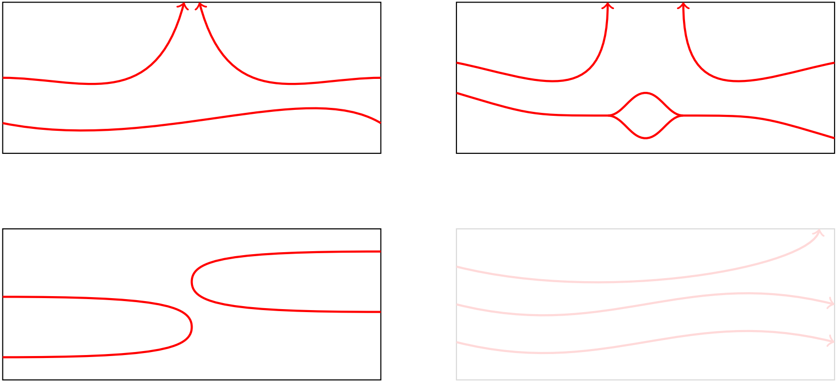

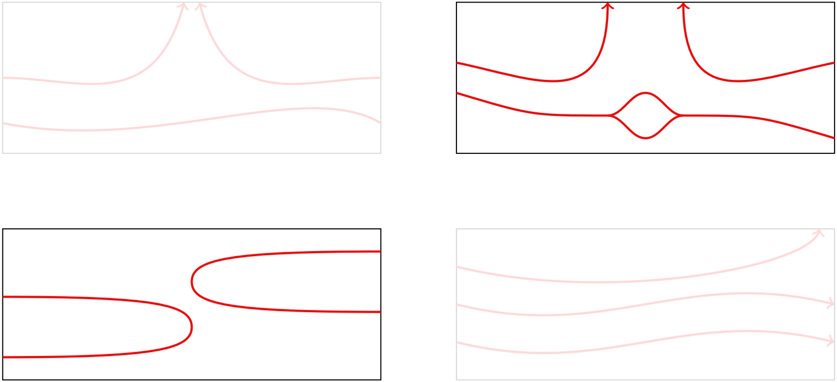

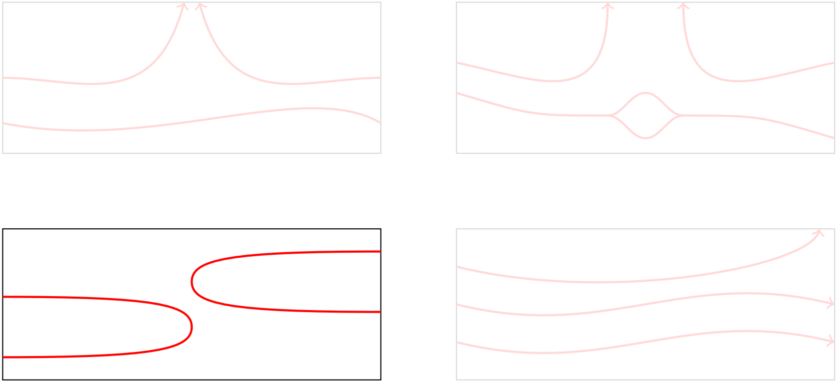

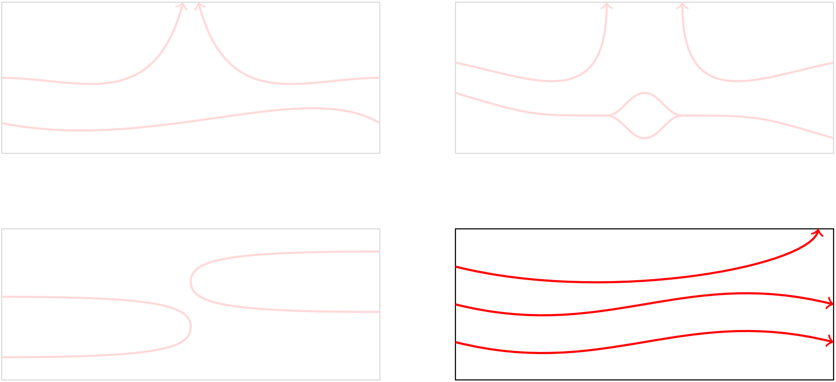

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$$\text{(total degree) } d := d_1 \times d_2 \times \cdots \times d_n$$

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$$Ax = \lambda x$$

$$\lambda x - Ax = 0$$

$$\begin{aligned} \lambda x_1 - (a_{11}x_1 + \cdots + a_{n1}x_n) &= 0 \\ &\vdots \\ \lambda x_n - (a_{n1}x_1 + \cdots + a_{nn}x_n) &= 0 \\ c_1x_1 + \cdots + c_nx_n + c_0 &= 0 \end{aligned}$$

$$(\lambda, x_1, \cdots, x_n)$$

$$(1-t)\begin{pmatrix} a_1x_1^2 - b_1 \\ \vdots \\ a_nx_n^2 - b_n \\ a_{n+1}\lambda - b_{n+1} \end{pmatrix} + t\begin{pmatrix} \lambda x_1 - (a_{11}x_1 + \cdots + a_{n1}x_n) \\ \vdots \\ \lambda x_n - (a_{n1}x_1 + \cdots + a_{nn}x_n) \\ c_1x_1 + \cdots + c_nx_n + c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

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$t = 0$
$t = 1$

Cheater's Homotopy

1. Li, Sauer & Yorke (1989)
2. Morgan & Sommese (1989)

$$p_1 = x_1^3 x_2^2 + c_1 x_1^3 x_2 + x_2^2 + c_2 x_1 + c_3 = 0,$$

$$p_2 = c_4 x_1^4 x_2^2 - x_1^2 x_2 + x_2 + c_5 = 0.$$

$$\mathbf{c} = (c_1, \ldots, c_5)$$

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$$p_1(\mathbf{c},\mathbf{x}) = x_1^3 x_2^2 + c_1 x_1^3 x_2 + x_2^2 + c_2 x_1 + c_3 = 0,$$

$$p_2(\mathbf{c},\mathbf{x}) = c_4 x_1^4 x_2^2 - x_1^2 x_2 + x_2 + c_5 = 0.$$

1. Pick $\mathbf{c}^* = (c_1^*, \ldots, c_5^*)$ at random.
2. Solve

$$p_1(\mathbf{c}^*,\mathbf{x}) = 0,$$
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$$\left(\begin{array}{lcl} \text{Say by choosing} & & \\ q_1(\mathbf{x}) = a_1 x_1^5 - b_1 & \longrightarrow & \text{might have a} \\ q_2(\mathbf{x}) = a_2 x_2^6 - b_2 & & \text{big waste.} \end{array} \right)$$

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2. Nonlinear homotopy

(Polyhedral homotopy, Huber & Sturmfels 1995)

$$\begin{aligned} 3x_1x_2 + 4x_1 - x_2 + 5 &= 0, \\ 6x_1x_2^2 - 2x_1^2x_2 + 7 &= 0. \end{aligned}$$

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The system is **in general position** unless the set of coefficients $\{c_{ij}\}$ satisfies certain polynomial system.

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The number of isolated zeros of $P(\mathbf{x})$ in $(\mathbf{C}^*)^n$ is a fixed number when $P(\mathbf{x})$ is in general position.

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This number $=$ the **mixed volume** of the system

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To solve $P(\mathbf{x}) = \mathbf{0}$,
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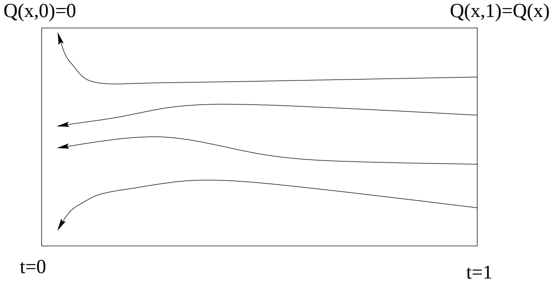

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$$Q(x) = c_1x^5 + c_2x^4 + c_3x^3 + c_4x + c_5$$

Pick random powers of t,

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$$H(x,1) \equiv Q(x) \qquad H(x,0) \equiv 0$$

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Binomial Equation

Equation of 2 terms: can be solved easily, no matter the degree.

$$\begin{aligned} 3x^{100} + 2x^{93} &= 0 \\ 3x^{100} &= -2x^{93} \\ x^{100-93} &= -2/3 \\ x^7 &= -2/3 \end{aligned}$$

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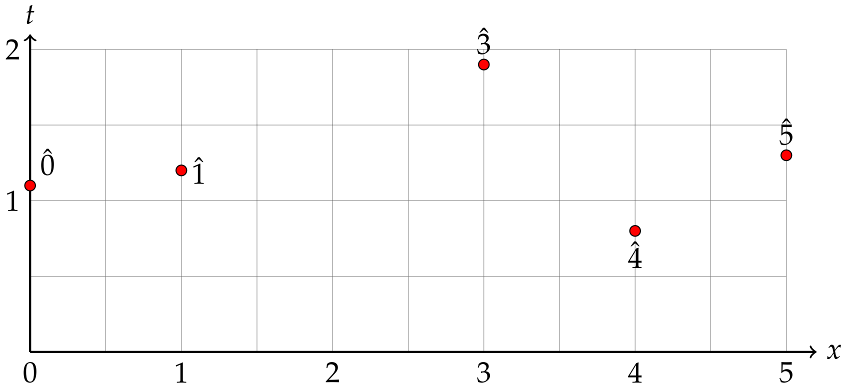

$\langle \hat{5}, \hat{\alpha} \rangle = 1.675$ $\langle \hat{4}, \hat{\alpha} \rangle = 1.1$ $\langle \hat{3}, \hat{\alpha} \rangle = 2.125$

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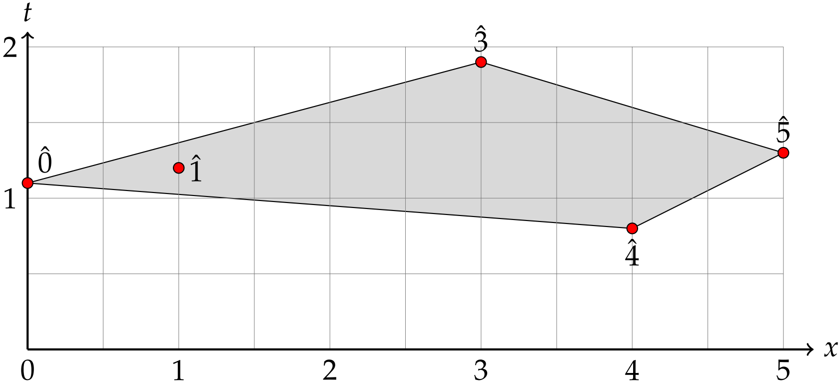

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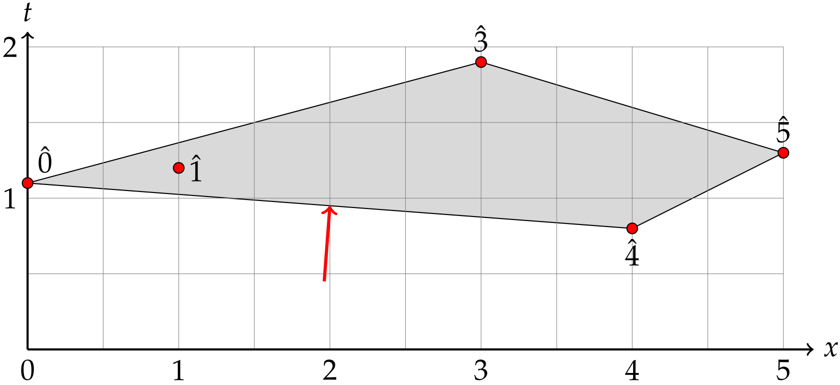

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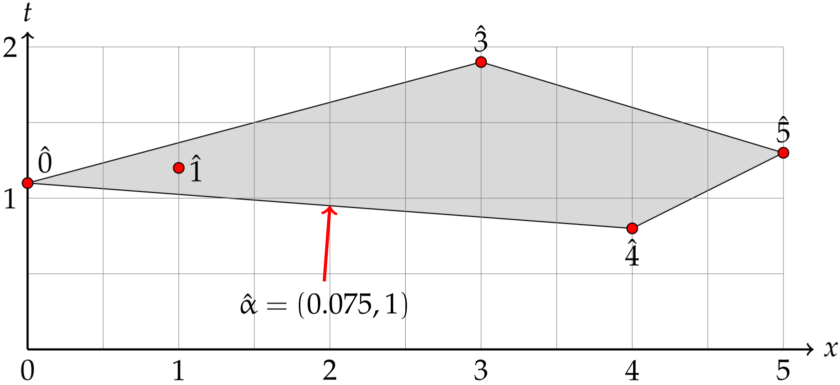

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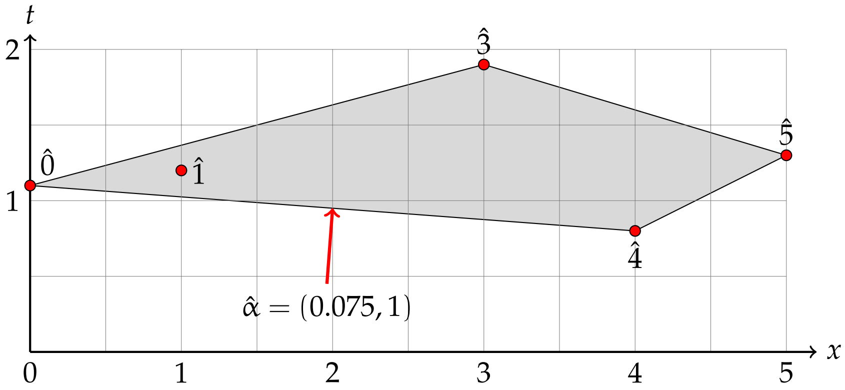

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Solution: Use change of variables with $\alpha = 0.075$

$$x = yt^{\alpha}$$

Note that

$$\text{at} \quad t = 1 \qquad\qquad x = y$$

Then

$$\begin{aligned}
H(x,t) &= c_1 x^5 t^{1.3} + c_2 x^4 t^{0.8} + c_3 x^3 t^{1.9} + c_4 x t^{1.2} + c_5 t^{1.1} \\
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can be solved and it generally has 4 solutions. Hope:

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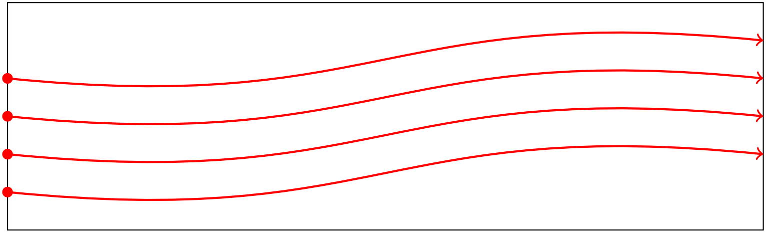

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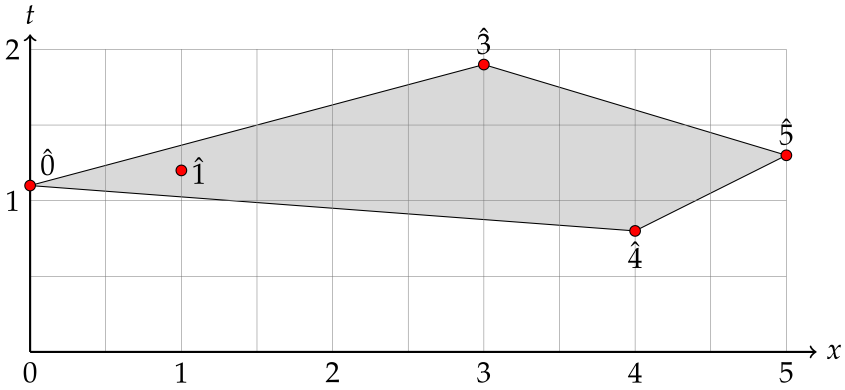

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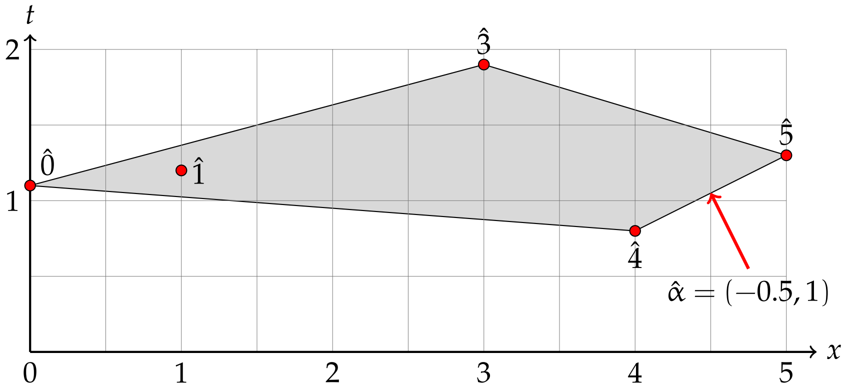

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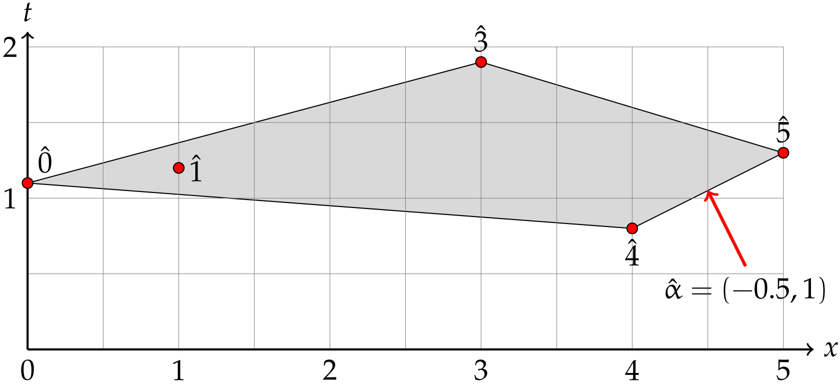

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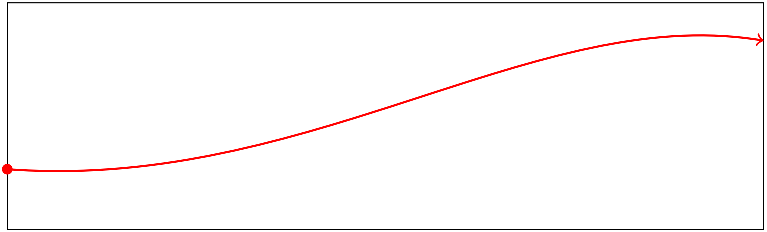

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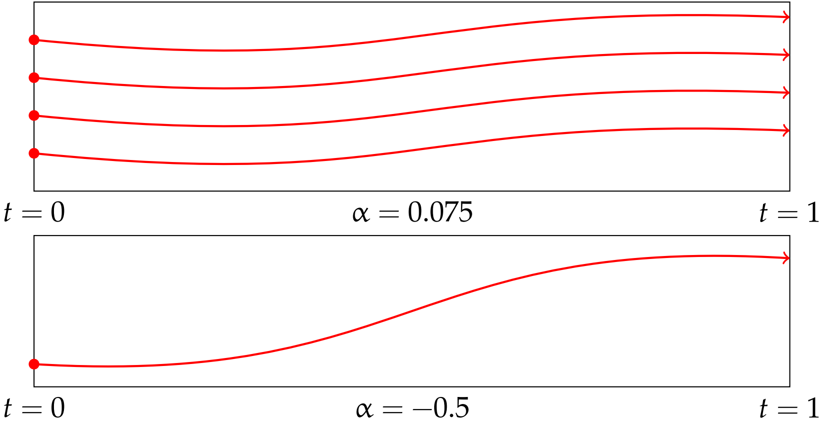
Together, we get all 5 solutions of $Q(x) = 0$.



How to find α

Recall

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To solve a system of polynomial equations $P(x) = 0$

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$$P(x) = x^5 + 2x^4 - 4x^3 + x - 5 = 0$$

$$\downarrow$$

$$Q(x) = c_1 x^5 + c_2 x^4 + c_3 x^3 + c_4 x + c_5$$

$$\downarrow$$

$$\tilde{H}(x,t) = (1-t)Q(x) + tP(x)$$

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$$H(x,t) = c_1 x^5 t^{1.3} + c_2 x^4 t^{0.8} + c_3 x^3 t^{1.9} + c_4 x t^{1.2} + c_5 t^{1.1}$$

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Binomial system

$$\begin{aligned} \bar{c}_{11}\mathbf{y}^{\mathbf{a}_{11}} + \bar{c}_{12}\mathbf{y}^{\mathbf{a}_{12}} &= 0, \\ &\vdots \\ \bar{c}_{n1}\mathbf{y}^{\mathbf{a}_{n1}} + \bar{c}_{n2}\mathbf{y}^{\mathbf{a}_{n2}} &= 0. \end{aligned}$$

1. It can be solved constructively and efficiently
2. The number of isolated zeros in $(\mathbf{C}^*)^n$

$$= \left| \det \begin{pmatrix} \mathbf{a}_{11} - \mathbf{a}_{12} \\ \vdots \\ \mathbf{a}_{n1} - \mathbf{a}_{n2} \end{pmatrix} \right|$$

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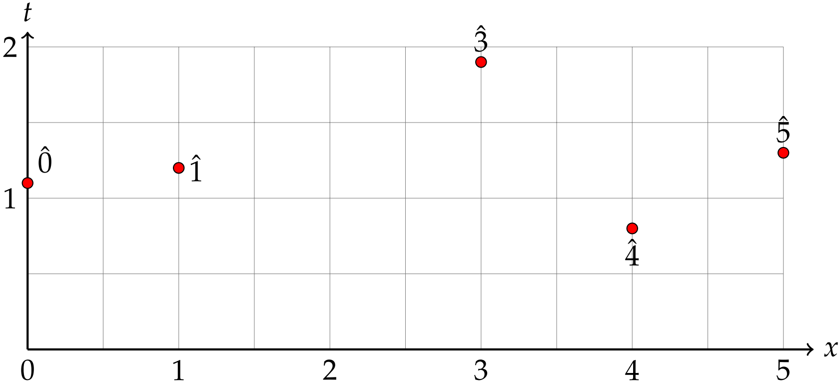
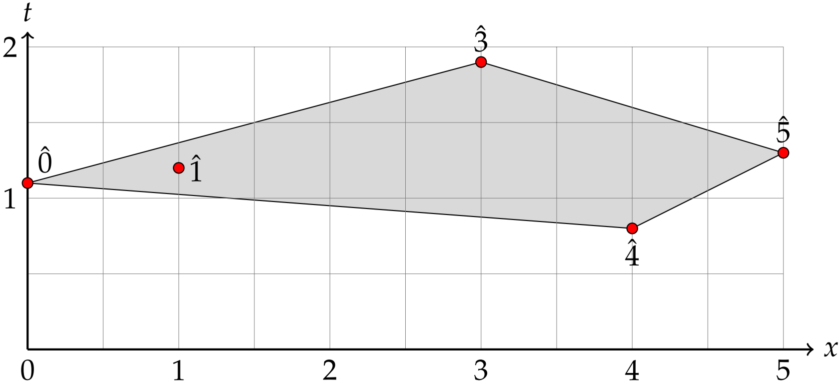


$\langle \hat{5}, \hat{\alpha} \rangle = 1.675$ $\quad \langle \hat{4}, \hat{\alpha} \rangle = 1.1$ $\quad \langle \hat{3}, \hat{\alpha} \rangle = 2.125$

$\langle \hat{1}, \hat{\alpha} \rangle = 1.275$ $\quad \langle \hat{0}, \hat{\alpha} \rangle = 1.1$



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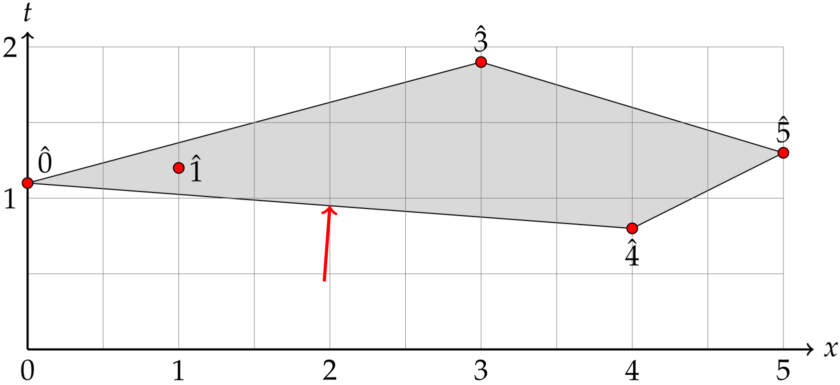

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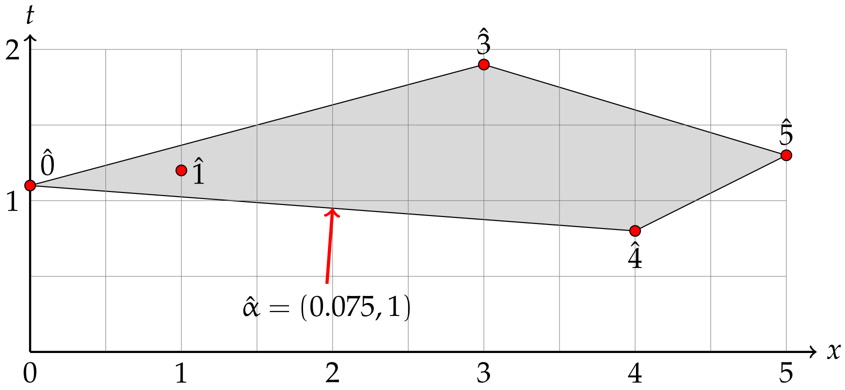

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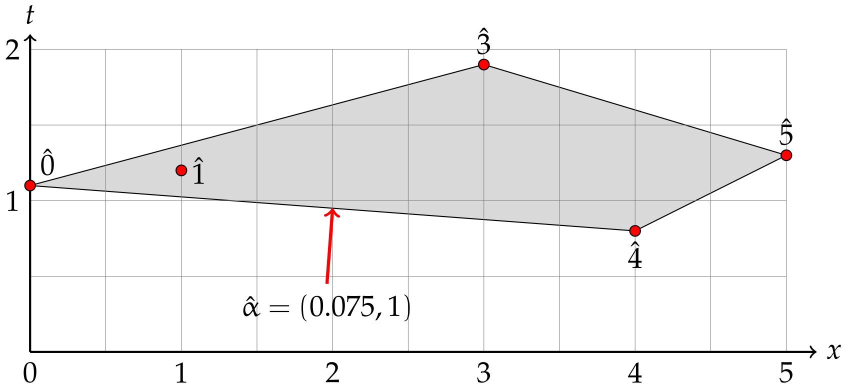

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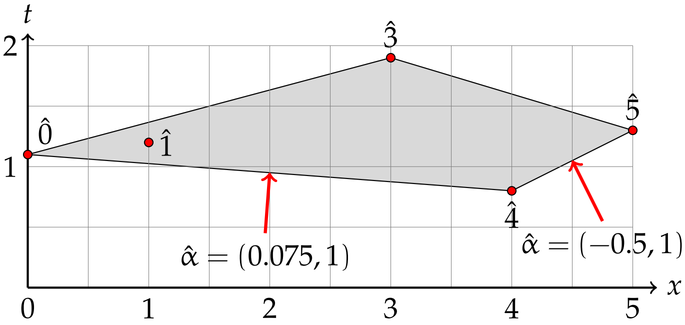
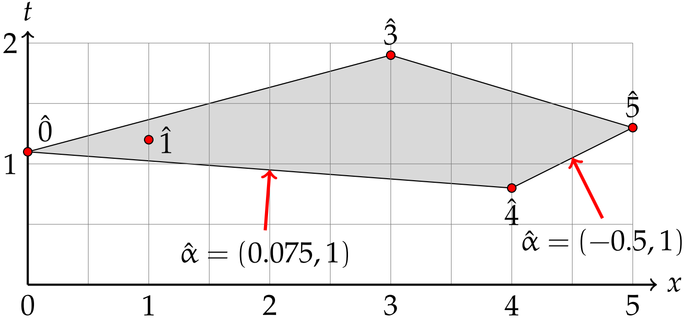


In algebraic language,

Choose $\hat{\alpha} = (\alpha, 1)$, s.t. among

$$\langle \hat{5}, \hat{\alpha} \rangle \qquad \langle \hat{4}, \hat{\alpha} \rangle \qquad \langle \hat{3}, \hat{\alpha} \rangle \qquad \langle \hat{1}, \hat{\alpha} \rangle \qquad \langle \hat{0}, \hat{\alpha} \rangle$$

The minimum is attained exactly twice



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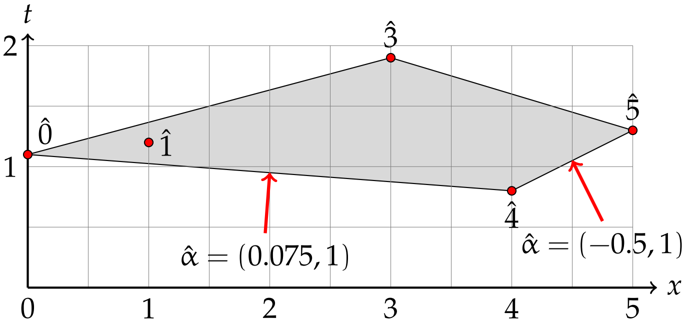

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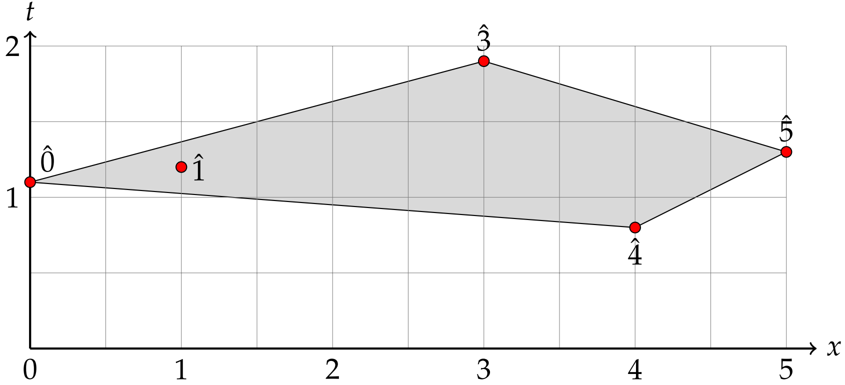
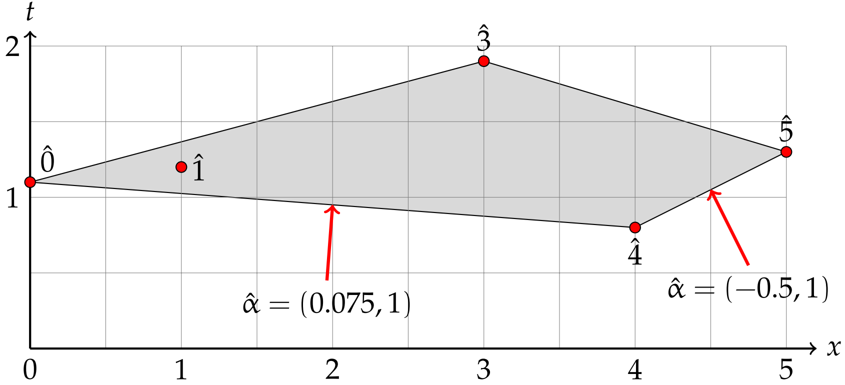
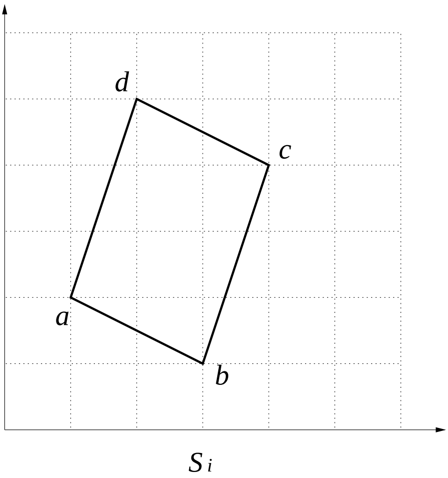

t
2
1
x
0
1
2
3
4
5
$\hat{0}$
$\hat{1}$
$\hat{3}$
$\hat{4}$
$\hat{5}$



t
x
0
1
2
3
4
5
1
2
$\hat{0}$
$\hat{1}$
$\hat{3}$
$\hat{4}$
$\hat{5}$
$\hat{\alpha} = (0.075, 1)$
$\hat{\alpha} = (-0.5, 1)$

$$S_1,\ S_2,\ \ldots\ S_n \ \subset\ \mathbb{N}_0^n$$



$$\omega_i : S_i \to \mathbb{R}, \quad i = 1, \ldots, n$$

$$\hat{S}_i = \{\hat{a} = (a, \omega_i(a)) \mid a \in S_i\}$$

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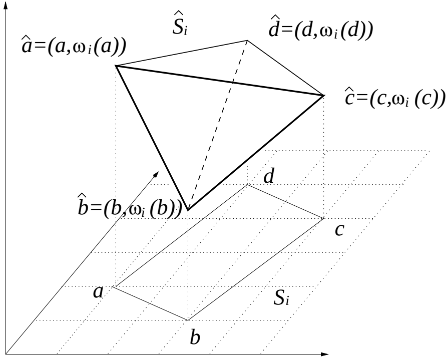
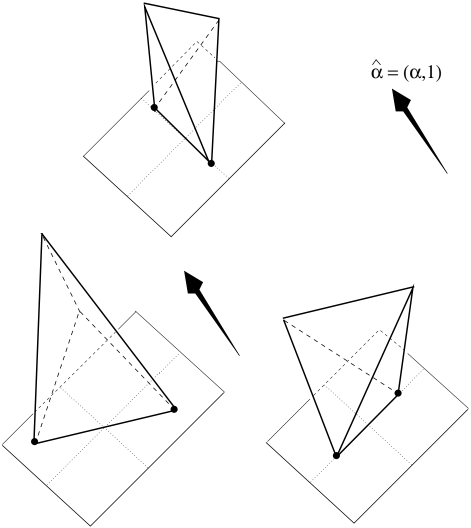


Problem: Look for hyperplane with normal $\hat{\alpha} = (\alpha, 1)$ which supports each $\hat{S}_i$ at exactly 2 points



Looking for $\alpha \in \mathbf{R}^n$, and pairs

$$\begin{aligned} \{\mathbf{a}_{11}, \mathbf{a}_{12}\} &\subset S_1, \\ &\vdots \\ \{\mathbf{a}_{n1}, \mathbf{a}_{n2}\} &\subset S_n \end{aligned}$$

such that

$$\begin{aligned} \langle \hat{\alpha}, \hat{\mathbf{a}}_{11} \rangle &= \langle \hat{\alpha}, \hat{\mathbf{a}}_{12} \rangle < \langle \hat{\alpha}, \hat{\mathbf{a}} \rangle, \forall \mathbf{a} \in S_1 \backslash \{\mathbf{a}_{11}, \mathbf{a}_{12}\}, \\ &\vdots \\ \langle \hat{\alpha}, \hat{\mathbf{a}}_{n1} \rangle &= \langle \hat{\alpha}, \hat{\mathbf{a}}_{n2} \rangle < \langle \hat{\alpha}, \hat{\mathbf{a}} \rangle, \forall \mathbf{a} \in S_n \backslash \{\mathbf{a}_{n1}, \mathbf{a}_{n2}\}. \end{aligned}$$

where $\quad \hat{\alpha} = (\alpha, 1), \quad \hat{\mathbf{a}} = (\mathbf{a}, \omega(\mathbf{a}))$

The **Mixed Volume** computation.

Change of variable

$$x = yt^{\alpha}$$

$$\downarrow$$

$$x \equiv y \quad \text{when } t = 1$$

$$\downarrow$$

$$\begin{aligned} H(x,t) &= c_1 x^5 t^{1.3} + \dots \\ &= c_1 (yt^{\alpha})^5 t^{1.3} + \dots \\ &= c_1 y^5 t^{5\alpha+1.3} + \dots \\ &= c_1 y^5 t^{\langle (5,1.3),(\alpha,1) \rangle} + \dots \\ &= c_1 y^5 t^{\langle \hat{5},\hat{\alpha} \rangle} + \cdots \end{aligned}$$

Change of variables

$$\left.\begin{aligned} x_1 &= y_1 t^{\alpha_1} \\ &\vdots \\ x_n &= y_n t^{\alpha_n} \end{aligned}\right\} \quad x = yt^{\alpha}$$

Then

$$x \equiv y \quad \text{when } t = 1$$

A typical term in h_i looks like

$$c^* x^a t^w = c^* y^a t^{\langle \hat{\alpha},\hat{a} \rangle}$$

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$$\begin{aligned} H(x,t) &= c_1 x^5 t^{1.3} + \dots \\ &= c_1 (yt^{\alpha})^5 t^{1.3} + \dots \\ &= c_1 y^5 t^{5\alpha + 1.3} + \dots \\ &= c_1 y^5 t^{\langle (5,1.3),(\alpha,1)\rangle} + \dots \\ &= c_1 y^5 t^{\langle \hat{5},\hat{\alpha}\rangle} + \cdots \end{aligned}$$

Change of variables

$$\left.\begin{aligned} x_1 &= y_1 t^{\alpha_1} \\ &\vdots \\ x_n &= y_n t^{\alpha_n} \end{aligned}\right\} \quad x = yt^{\alpha}$$

Then

$$x \equiv y \quad \text{when } t = 1$$

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$$c^* x^a t^w = c^* y^a t^{\langle \hat{\alpha}, \hat{a}\rangle}$$

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and they are each attained exactly twice.

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a binomial system,

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Theorem

For almost all choices of the (complex) coefficients, constant terms, and the (rational) powers of t, the polyhedral homotopy "works".

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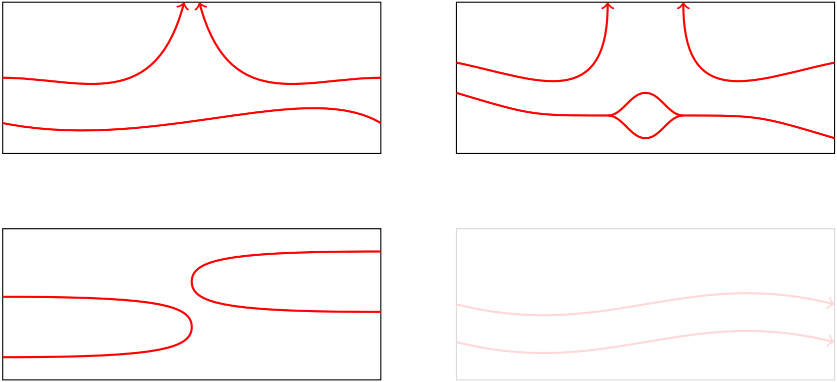

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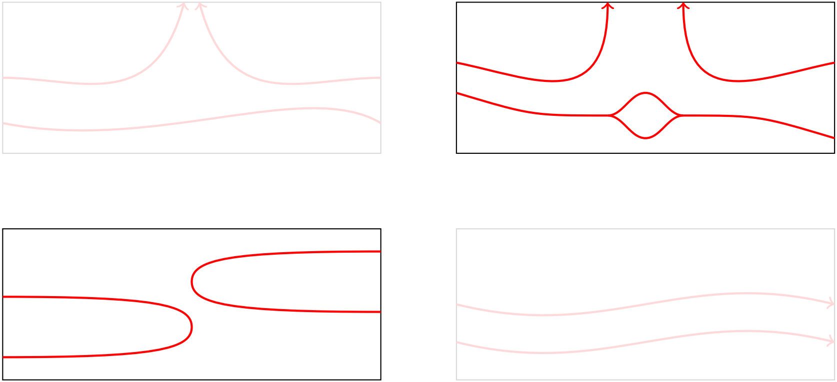

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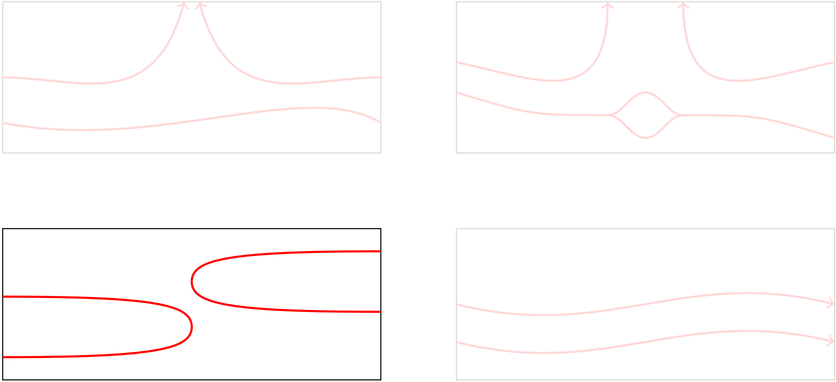

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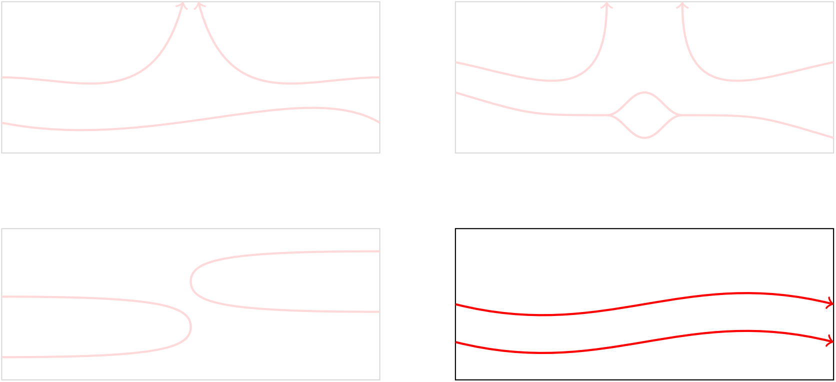

- **PHC**pack J. Verschelde (1999),

  "Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation", *ACM Trans. Math. Softw.*, **25**, 251-276.

- **PHoM** T. Gunji, S. Kim, M. Kojima, A. Takeda, K. Fujisawa and T. Mizutani (2004),

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- **HOM4PS** (1999) Tangan Gao, T.Y.Li
- **HOM4PS-2.0** (2008) T.L.Lee, T.Y.Li, C.H.Tsai
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eco-n	Total degree $= 2 \cdot 3^{n-2}$

$$(x_1 + x_1x_2 + \cdots + x_{n-2}x_{n-1})x_n - 1 = 0$$
$$(x_2 + x_1x_3 + \cdots + x_{n-3}x_{n-1})x_n - 2 = 0$$
$$\vdots$$
$$x_{n-1}x_n - (n-1) = 0$$
$$x_1 + x_2 + \cdots + x_{n-1} + 1 = 0$$

noon-n	Total degree $= 3^n$

$$x_1(x_2^2 + x_3^2 + \cdots + x_n^2 - 1.1) + 1 = 0$$
$$x_2(x_1^2 + x_3^2 + \cdots + x_n^2 - 1.1) + 1 = 0$$
$$\vdots$$
$$x_n(x_1^2 + x_2^2 + \cdots + x_{n-1}^2 - 1.1) + 1 = 0$$

cyclic-n Total degree $= n!$
$x_1 + x_2 + \cdots + x_n = 0$ $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 = 0$ $x_1x_2x_3 + x_2x_3x_4 + \cdots + x_{n-1}x_nx_1 + x_nx_1x_2 = 0$ $\vdots$ $x_1x_2\cdots x_n - 1 = 0$

katsura-n Total degree $= 2^n$
$2x_{n+1} + 2x_n + \cdots + 2x_2 + x_1 - 1 = 0$ $2x_{n+1}^2 + 2x_n^2 + \cdots + 2x_2^2 + x_1^2 - x_1 = 0$ $2x_nx_{n+1} + 2x_{n-1}x_n + \cdots + 2x_2x_3 + 2x_1x_2 - x_2 = 0$ $2x_{n-1}x_{n+1} + 2x_{n-2}x_n + \cdots + 2x_1x_3 + x_2^2 - x_3 = 0$ $\vdots$ $2x_2x_{n+1} + 2x_1x_n + 2x_2x_{n-1} + \cdots + 2x_{n/2}x_{(n+2)/2} - x_n = 0$ $2x_2x_{n+1} + 2x_1x_n + 2x_2x_{n-1} + \cdots + x_{(n+1)/2}^2 - x_n = 0$

reimer-n	Total degree $= (n+1)!$

$$\begin{aligned} 2x_1^2 - 2x_2^2 + \cdots + (-1)^{n+1}2x_n^2 - 1 &= 0 \\ 2x_1^3 - 2x_2^3 + \cdots + (-1)^{n+1}2x_n^3 - 1 &= 0 \\ &\vdots \\ 2x_1^{n+1} - 2x_2^{n+1} + \cdots + (-1)^{n+1}2x_n^{n+1} - 1 &= 0 \end{aligned}$$

Dell PC with a Pentium 4 CPU of 2.2GHz, 1GB of memory

Polynomial system	Mix Vol = # of paths	HOM4PS cpu time	HOM4PS-2.0 cpu time	Speed-up ratio
eco-16	16,384	2h55m12s	6m35s	26.6
eco-17	32,768	-	22m23s	-
noon-10	59,029	3h20m45s	5m12s	38.6
noon-11	177,125	-	23m27s	-
noon-12	531,417	-	1h28m00s	-
noon-13	1,594,297	-	7h02m10s	-
katsura-13	8,192	3h40m54s	4m56s	44.8
katsura-14	16,384	-	25m15s	-
katsura-15	32,768	-	1h50m26s	-
cyclic-9	11,016	8m37s	44s	11.8
cyclic-10	35,940	58m02s	2m47s	20.9
cyclic-11	184,756	-	19m40s	-
cyclic-12	500,352	-	1h36m40s	-
reimer-7	40,320	7m47s	2m49s	2.8
reimer-8	362,880	1h44m18s	36m43s	2.8
reimer-9	3,628,800	-	8h47m42s	-

Polynomial system	Total degree	PHoM cpu time	HOM4PS-2.0 cpu time	Speed up
eco-14	1,062,882	9h57m15s	52.9s	677.4
eco-15	3,188,646	-	2m25s	-
eco-17	28,697,814	-	22m23s	-
noon-9	19,683	5h01m06s	1m15s	240.9
noon-10	59,049	-	5m12s	-
noon-13	1,594,323	-	7h02m10s	-
katsura-11	2,048	1h21m13s	28s	174.0
katsura-12	4,096	4h00m09s	1m42s	141.3
katsura-13	8,192	-	4m56s	-
katsura-15	32,768	-	1h50m26s	-
cyclic-8	40,320	32m32s	6.8s	287.0
cyclic-9	362,880	-	44s	-
cyclic-12	479,001,600	-	1h36m40s	-
reimer-6	5,040	1h14m50s	12.1s	371.0
reimer-7	40,320	-	2m49s	-
reimer-9	3,628,800	-	8h47m42s	-

System	Total degree	CPU time		Speed-up ratio
		PHCpack	**HOM4PS-2.0**	
noon-9	19,683	33m28s	22.2s	90.5
noon-10	59,049	2h33m27s	1m27s	105.8
noon-11	177,147	-	5m32s	-
noon-13	1,594,323	-	3h7m10s	-
katsura-14	16,384	2h49m00s	2m52s	59.0
katsura-15	32,768	8h22m45s	7m03s	71.3
katsura-16	65,536	-	16m25s	-
katsura-20	1,048,576	-	8h58m00s	-
reimer-6	5,040	15m08s	9.6s	94.5
reimer-7	40,320	3h45m43s	1m58s	114.7
reimer-8	362,880	-	30m43s	-
reimer-9	3,628,800	-	7h52m40s	-

System	Total degree	CPU time		Speed-up ratio
		PHCpack	**HOM4PS-2.0**	
eco-14	1,062,882	1h26m04s	52.9s	97.6
eco-15	3,188,646	3h55m23s	2m25s	97.4
eco-17	28,697,814	-	22m23s	-
eco-18	86,093,442	-	1h51m30s	-
cyclic-9	362,880	3h50m48s	44s	314.7
cyclic-10	3,628,800	11h00m23s	2m47s	237.2
cyclic-11	39,916,800	-	19m40s	-
cyclic-12	479,001,600	-	1h36m40s	-

Use Polyhedral Homotopy

System	Maximum solvable size					
	PHoM		**PHCpack**		**HOM4PS-2.0**	
eco -	14	(1,062,882)	15	(3,188,646)	18	(86,093,442)
noon -	9	(19,683)	10	(59,049)	13	(1,594,323)
katsura -	12	(2,048)	15	(32,768)	20	(1,048,576)
cyclic -	8	(40,320)	10	(3,628,800)	12	(479,001,600)
reimer -	6	(5,040)	7	(40,320)	9	(3,628,800)

Numerical results of HOM4PS-2.0para

All the computations were carried out on a cluster 8 AMD dual 2.2 GHz cpus (**1** master and **7** workers). Again, we only list those benchmark systems that can be solved within 12 hours cpu time.

- Master-worker type of environment is used.
- Use MPI (message passing interface) to communicate between the master processor and worker processors

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- Use MPI (message passing interface) to communicate between the master processor and worker processors

System	CPU time	Total degree	Mixed Vol. (# of paths)	Curve Jumping
eco-17	2m11s	28,697,814	32,768	-
eco-18	6m30s	86,093,442	65,536	x -
eco-19	26m26s	258,280,326	131,072	-
eco-20	1h29m29s	774,840,978	262,144	-
eco-21	10h08m55s	2,324,522,934	524,288	1
cyclic-11	3m34s	39,916,800	184,756	-
cyclic-12	14m07s	479,001,600	500,352	x -
cyclic-13	1h39m10s	6,227,020,800	2,704,156	-
cyclic-14	7h32m42s	87,178,291,200	8,795,976	4

Solving systems by the polyhedral-linear homotopy with 1 master and 7 workers

System	CPU time	Total degree (=# of paths)	# curve jumping	# of isolated solutions
noon-12	2m23s	531,417+24	-	531,417
noon-13	7m48s	1,594,297+26	x -	1,594,297
noon-14	38m12s	4,782,941+28	-	4,782,941
noon-15	4h14m33s	14,348,877+30	-	14,348,877
katsura-18	9m46s	262,144	-	262,144
katsura-19	23m36s	524,288	2	524,288
katsura-20	55m10s	1,048,576	x 4	1,048,576
katsura-21	2h08m42s	2,097,152	8	2,097,152
katsura-22	4h52m01s	4,194,304	20	4,194,304
katsura-23	11h17m40s	8,388,608	52	8,388,608
reimer-8	2m36s	362,880	-	14,400
reimer-9	28m04s	3,628,800	x 8	86,400
reimer-10	8h40m46s	39,916,800	20	518,400

Solving systems by the classical linear homotopy with 1 master and 7 workers

	# of wks	Total time to solve system		Time to find mixed cells		Time to trace curve		Time to check solutions	
	k	cpu(s)	ratio	cpu(s)	ratio	cpu(s)	ratio	cpu(s)	ratio
eco	1	445.32	1.00	120.02	1.00	325.00	1.00	0.30	1.00
-16	2	223.49	1.99	60.66	1.98	162.58	2.00	0.25	1.20
	3	150.69	2.96	40.94	2.93	109.53	2.97	0.22	1.36
	5	91.31	4.88	25.22	4.76	65.89	4.93	0.20	1.59
	7	68.70	6.48	19.99	6.00	48.58	6.69	0.13	2.31
cyc	1	1475.39	1.00	38.15	1.00	1436.49	1.00	0.75	1.00
-11	2	734.96	2.00	19.10	2.00	715.41	2.00	0.45	1.67
	3	494.19	2.99	12.94	2.95	480.86	2.99	0.39	1.92
	5	295.90	4.99	8.05	4.74	287.47	5.00	0.38	1.97
	7	212.87	6.93	6.47	6.00	206.06	6.97	0.34	2.21

The scalability of solving systems by the polyhedral homotopy

System	# of workers	Total time to solve system		Time to trace curve		Time to check solutions	
	k	cpu(s)	ratio	cpu(s)	ratio	cpu(s)	ratio
noon	1	1003.32	1.00	980.72	1.00	22.50	1.00
-12	2	501.75	2.00	490.33	2.00	11.42	1.97
	3	335.18	2.99	326.68	3.00	8.50	2.65
	5	201.27	4.98	195.30	5.00	5.97	3.77
	7	143.22	7.00	138.88	7.00	4.34	5.18
reimer	1	1088.95	1.00	1087.74	1.00	1.21	1.00
-8	2	545.08	2.00	543.96	2.00	1.12	1.08
	3	363.89	2.99	362.81	3.00	1.08	1.12
	5	218.69	4.98	217.97	4.99	0.72	1.68
	7	156.54	6.96	155.86	6.98	0.68	1.78
katsura	1	1964.22	1.00	1963.12	1.00	1.10	1.00
-17	2	982.38	2.00	981.35	2.00	1.03	1.07
	3	654.17	3.00	653.22	3.00	0.95	1.16
	5	394.02	4.99	393.18	4.99	0.84	1.31
	7	280.56	7.00	279.85	7.00	0.71	1.55

The scalability of solving systems by the classical linear homotopy

Thank You!